APPENDIX C

Proof of Theorem 5.2

In this appendix we prove

Theorem 5.2 *Let S be the standard unit* $(n-1)$*-simplex, let* $\mathbf{p}$ *and* $\mathbf{p}^*$ *range over vectors in S, and let* $m: S \longrightarrow S$ *be defined as follows:*

$$m(\mathbf{p}) \ := \ \operatorname*{arg\,min}_{\mathbf{p}^*} \sum_{i=1}^{n} \sum_{j=1}^{n} p_i w_{ij} (\delta_{ij} - p_j^*)^2$$

with δ_{ij} *the Kronecker delta, and with* $w_{ij} > 0$ *for all* i, j*, and* $\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} = 1$*. Then (1) there is a* $\mathbf{p}^+ \in S$ *such that* $m(\mathbf{p}^+) = \mathbf{p}^+$*; and furthermore (2) there is only one such* $\mathbf{p}^+$*, and (3) it depends only on the* w_{ij}*.*

Proof: Clause 1 follows from Brouwer's (1911) fixed-point theorem, which (in one version) states that every continuous function from a simplex onto itself has a fixed point. It does not follow from Brouwer's theorem that the fixed point is unique.

To prove clause 2, then, one first verifies that the function that is being minimized at each step on the way to the fixed point has the Hessian

$$\begin{bmatrix} 2(p_1 w_{11} + \cdots + p_n w_{1n}) & 0 & \cdots & 0 \\ 0 & 2(p_1 w_{21} + \cdots p_n w_{2n}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2(p_1 w_{n1} + \cdots + p_n w_{nn}) \end{bmatrix}$$

This is a diagonal matrix, so its eigenvalues are the diagonal elements, which given the constraints on the p_i and w_{ij}, can be seen to be all necessarily positive. Therefore, the Hessian is positive definite everywhere, and given that a simplex is a convex set, it follows that the function that is minimized is strictly convex, and hence the minimum it reaches is unique. So, at each step toward the fixed point, a unique minimum is reached. As a result, the minimum reached at the fixed point is unique as well.

For clause 3, finally, note that at the fixed point the function that is being minimized is of the form

$$m^+(\mathbf{p}) \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} p_i w_{ij}(\delta_{ij} - p_j)^2.$$

Because the fixed point $\mathbf{p}^+$ is a minimum, it holds that $\nabla m^+(\mathbf{p}^+) = \mathbf{0}$. We obtain a system of n polynomial equations with n variables and with the w_{ij} as coefficients by setting $(\partial/\partial p_i)m^+(\mathbf{p}^+) = 0$, for all $i \leqslant n$. This system has a unique solution (in virtue of the first two clauses), which is bound to be strictly in terms of the coefficients. ■