11 Elementary Toposes

In which we explore the even more general notion of elementary toposes and consider what sheaves look like in this setting, after which we take up the topic of morphisms of toposes, including an extended section, full of concrete examples, that gives a glimpse into the theory of cohesive toposes.

Lawvere and Tierney introduced the definition of an elementary topos largely in order to characterize structures that behave like sets—and, in particular, those categories that apparently behave as Grothendieck toposes do—but that can be described by elementary axioms independent of set theory. The following definition is the one typically presented.

Definition 301 An *elementary topos* is a category $\mathscr{E}$ that

1. has all finite limits—equivalently, has pullbacks and a terminal object;
2. is *Cartesian closed*—that is, for each object X there is a functor, called the *exponential* $(-)^X : \mathscr{E} \to \mathscr{E}$, that is right adjoint to the functor $(-) \times X$; and
3. has a *subobject classifier* satisfying certain conditions.

A category that satisfies the first two conditions is a Cartesian closed category—that is, it has finite products and exponentials for each pair of objects—and the significance of such categories had been appreciated before the definition of an elementary topos.[211] However,

211. To give some small glimpse into the importance of Cartesian closed categories, consider that **Man** (which lacked finite inverse limits, as we already saw, since pullbacks of manifolds were not always manifolds) is not Cartesian closed either—since in particular it lacks exponentials, as the space of smooth (C^∞) maps between two smooth manifolds is not in general a manifold. In short, this means that in general for manifolds A and B and C, there is no equivalence between $A \times B \to C$ and $B \to C^A$. The significance of this is very nicely explained by Lawvere (2011), who notes that in addition to presenting some serious foundational problems for calculus of variations, this failure can be interpreted, physically, as saying that if we treat A as all possible states of time, B as some physical body, and C as some space, then the motion described by $A \times B \to C$ will not in general be equivalent to the assignment of a body to its path through the space, given by $B \to C^A$. However, even if it were cartesian closed, **Man** would still lack the resources for dealing with infinitesimal objects or structures. Incidentally, observations of these sorts of deficiencies have led to attempts to rectify such deficiencies (due to Lawvere) by constructing a category of spaces **Space**—sometimes called *smooth worlds* or *smooth toposes*, with objects *smooth spaces*—which extends or enlarges the usual category of manifolds in various ways so as to overcome the above-mentioned deficiencies. In brief, the construction usually proceeds by first embedding **Man** in the category of "formal varieties" **L**, or "loci," a category that does have finite inverse limits and also contains infinitesimal spaces; then, since in general one still cannot construct function spaces in the new category **L**, one next endows **L** with a specific Grothendieck topology J, after which the resulting smooth topos $\mathbf{Sh}(\mathbf{L}, J)$ of sheaves on **L** allows for function spaces with certain desirable properties to be constructed. One can then use the smooth category **Space**, which is in fact a topos, where each object has a differentiable structure and each morphism a derivative (and where calculus reduces to exact algebraic calculations with infinitesimals), to do

the introduction of the concept of a subobject classifier was the real conceptual advance that made possible the development of topos theory along elementary lines, for a topos may also be defined as a Cartesian closed category with a subobject classifier. In what immediately follows, we focus predominantly on constructing and familiarizing ourselves with the third object in this definition—the subobject classifier.[212]

11.1 The Subobject Classifier

We begin by recalling that since in category theory everything can be specified in terms of arrows, we do not need to consider traditional "elements" of objects. However, there is a corresponding category-theoretic notion of an element.

Definition 302 For A an object in $\mathbf{C}$, we call an arrow in $\mathbf{C}$ with codomain A an *element* (or *generalized element* or *variable element*) of A.

The domain of such an arrow is accordingly sometimes called the *stage of definition* (or *domain of variation*) of that element. For example, for an arrow $f : B \to A$, the object B is regarded as a stage or as the domain of variation in order to indicate that A is being defined over B or that it is where A is being viewed from. We can then vary the domain of variation, considering different stages of A. The identity arrow, $\mathrm{id}_A : A \to A$, for its part, is called the *generic element* of A. Elements defined over a category's terminal object—that is, generalized elements $1_{\mathbf{C}} \to A$—are special, and in the simple case of **Set** recover the usual (set theoretic) notion of elements of the set A.

Definition 303 If the category has a terminal object, which we will denote with $1_{\mathbf{C}}$ (or just 1), we can use this to define the notion of a *global element* or *point* as an arrow $1 \to A$ of A—that is, one that does not depend on any particular domain of reference or stage of variation.

By contrast, given an arrow $B \to A$, if B is not isomorphic to the terminal object, such an arrow is sometimes called the *local element* of A (at stage B).[213]

Even if a category has a terminal object, while some objects may have global elements, others may not. And in general, not all elements (in the above sense) of a given object have to be global: while the global elements of an object can be thought of as corresponding to the set-theoretic notion of points, in general there will exist elements of an object that are *not points*, which in part explains why one insists on the "domain of variation" perspective.

We will make some use of this perspective in what follows, in the development of the subobject notion, where a subobject of an object A will basically emerge as an equivalence class of certain morphisms (monomorphisms) with codomain A. First, recall that a monomorphism is the category-theoretic generalization of the notion of injective maps from sets, where it requires that whenever we have

differential geometry. In this category, analysis is called *smooth infinitesimal analysis* (SIA). See Bell (2008) for a brief and very accessible introduction to SIA; Moerdijk and Reyes (1991) and Kock (2006) are the standard references in this field (if the reader wishes to pursue these matters in more depth).

212. Detailed treatments of the three defining features of an elementary topos can be found in any standard text on topos theory, such as Johnstone (2014) or Mac Lane and Moerdijk (1994).

213. This language of "global (local) element" ultimately derives from sheaf theory, where the global elements of a sheaf on a space X are the global sections, defined on the entire space.

$$C \overset{f_1}{\underset{f_2}{\rightrightarrows}} B \xrightarrow{\ i\ } A$$

so that $i \circ f_1 = i \circ f_2$, then we have $f_1 = f_2$. In this case, we call such a morphism $i : B \to A$ a monomorphism (or monic arrow). In logical notation:

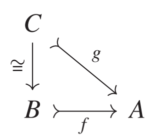
$$\forall C \forall f_1, f_2 : C \to B \qquad (i \circ f_1 = i \circ f_2) \Rightarrow f_1 = f_2.$$

In the case of sets, this "for all" reduces to C being the terminal object 1 (a singleton set $\{*\}$) in **Set** and the above recovers the notion of an injective function. Moreover, in **Set** an arrow $x : 1 \to B$ into B just corresponds to an element (in the usual sense) of the set B, and we can use such elements to examine whether two parallel arrows f, g are equal, by just checking that $f(x) = g(x)$ for every global element x of the domain. But in a number of other toposes, it need not be the case that such diagrams (of parallel arrows) commute iff they commute for every global element of the domain. The need to construe things in a more general way is part of what motivates the definition of generalized elements in the first place, but speaking of *elements* suggests that we still expect that operating with a topos will be something like operating with sets.

The notion of a monic arrow is really a way of saying that the arrow preserves distinctness in the sense that it keeps items that are seen as distinct in the domain, distinct in the codomain. It moreover allows us to define the notion of a *part* in a rather general way. The idea is that we can think of a map i with codomain A, where i is moreover a monomorphism, as supplying us with a *part of* A.

We make use of such morphisms in the following:

Definition 304 Two monomorphisms f and g satisfying



are *equivalent*, denoted $f \sim g$. Then we can define the *equivalence class* of f, denoted $[f] = \{g \mid f \sim g\}$.

Definition 305 A *subobject* of an object A is defined to be an equivalence class of monomorphisms with codomain A modulo the relation that identifies monomorphisms into A whenever one factors through the other (in the sense of definition 304). Then the *class of subobjects* of an object A will be

$$\text{Sub}(A) := \{[f] \mid \text{codom}(f) = A \text{ and } f \text{ is monic}\}.$$

There is a partial order on Sub(A), using the inclusion ordering $[f] \subseteq [g]$ (also just written $f \subseteq g$), giving us the poset (Sub(A), $\subseteq$) of subobjects of the object A. In fact, for each object A in a topos, the poset Sub(A) of subobjects of A forms a lattice.

As you might imagine, relying on the notion of monic arrows as supplying us with *parts*, in the category **Set** subobjects of a set X correspond precisely to the *subsets* of X—in this way, the notion of a subobject may be seen as a categorical generalization of the notion of a subset from set theory. But let's sit with this connection for a moment. Observe how, in the category **Set**, we generally think of arrows $X \to \{0, 1\}$ into the truth-value set

$\{0,1\}$ as *predicates* for X, or as *properties* of generalized elements of X. Moreover, we know that we can identify the subsets S of a given set X with their characteristic functions $\chi_S : X \to \{0,1\}$ into the truth-value set $\{0,1\}$, and picking out 1 from the truth-value set amounts to a function $\top : \{*\} \to \{0,1\}$. With such data, we in fact have a diagram

$$\begin{array}{ccc} S & \xrightarrow{\;!\;} & \{*\} \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ X & \xrightarrow[\chi_S]{} & \{0,1\}, \end{array}$$

where $i : S \hookrightarrow X$ is the inclusion arrow and $! : S \to \{*\}$ the unique arrow to the terminal object $\{*\}$ of **Set**. This diagram is in fact a pullback square, since pulling $\top$ back along χ_S just yields the inverse image set $\{x \mid \chi_S(x) = 1\}$, which is the same as S. This situation serves as motivation for the following more general notion.

Definition 306 Suppose **C** has finite limits (so, in particular, a terminal object $1_{\mathbf{C}}$). A *subobject classifier* (or a *generalized truth-value object*) is an object Ω in **C** together with a monomorphism (called the *truth arrow*) $\top : 1_{\mathbf{C}} \to \Omega$, such that for every object $A \in \mathbf{C}$ and every monomorphism $m : B \rightarrowtail A$, there exists a unique morphism $\chi_B = \chi_m = \chi(m) : A \to \Omega$—called the *characteristic (or classifying) arrow* of the monic m (or of the subobject B of A)—making the diagram

$$\begin{array}{ccc} B & \xrightarrow{\;!\;} & 1_{\mathbf{C}} \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ A & \xrightarrow[\chi m]{} & \Omega \end{array}$$

form a pullback square.

We often denote the composite arrow $(\top \circ !)$ by $\top_B : B \to \Omega$.

In **Set**, we already mentioned how $\Omega = \{0,1\}$, $\top(*) = 1$, and the morphism χ is the characteristic map of S in X, that is, $\chi(y) = 1$ if $y \in S$ and $\chi(y) = 0$ if $y \notin S$. The object Ω^X of arrows $X \to \Omega$ can be identified with the set of subsets of X, and the global elements $1 \to X$ just correspond to the actual elements of X.

It is straightforward to show that, in a general category with finite limits, a subobject classifier, when it exists, is unique up to isomorphism. Moreover, given $\mathscr{E}$ a topos, and two objects A, B of $\mathscr{E}$, by Cartesian closedness the global elements of B^A (i.e., the arrows $1 \to B^A$) correspond bijectively with $1 \times A \to B$ (i.e., with the arrows $A \to B$). Thus, in particular, for $\mathscr{E}$ a topos, and given an object $A \in \mathscr{E}$, the global elements of Ω^A will correspond bijectively with the subobjects of A.

Proposition 307 In a topos $\mathscr{E}$, given an object $A \in \mathscr{E}$, we have

$$\mathrm{Sub}(A) \cong \mathrm{Hom}_{\mathscr{E}}(A, \Omega).$$

Proof. By the fact that, in a topos, global elements of B^A correspond bijectively with the arrows $A \to B$, the global elements of Ω^A correspond bijectively with the arrows $\chi : A \to \Omega$. The pullback of χ with the subobject classifier $\top : 1 \to \Omega$ will yield a subobject $m : S \rightarrowtail A$. By the definition of a subobject classifier, this correspondence is bijective. $\square$

We are going to want to see how these notions apply in the setting of presheaves. In short, since the category $\mathbf{Set}^{\mathbf{C}^{op}}$ has functors as objects, we might expect the subobjects will be defined in terms of *subfunctors*. A subfunctor F of a functor $G : \mathbf{C}^{op} \to \mathbf{Set}$ amounts to an inclusion $F \to G$ that is in fact a monic map in the presheaf category $\mathbf{Set}^{\mathbf{C}^{op}}$, meaning that each subfunctor is indeed a subobject—moreover, all subobjects are supplied by subfunctors. In such a setting, we would thus be looking to isolate a *functor* Ω that can be treated as a subfunctor classifier. Since a subobject was just defined as a class of equivalent monic arrows sharing the same codomain, a subobject classifier in the presheaf category, should there be such a thing, ought to classify, in particular, the subobjects (subpresheaves) of each representable presheaf. The proposition above, together with the Yoneda lemma, implies that if the presheaf category has a subobject classifier $\Omega : \mathbf{C}^{op} \to \mathbf{Set}$, it should satisfy

$$\mathrm{Sub}(\mathrm{Hom}_{\mathbf{C}}(-,a)) \cong \mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{op}}}(\mathrm{Hom}_{\mathbf{C}}(-,a),\Omega) \equiv \mathrm{Nat}(\mathrm{Hom}_{\mathbf{C}}(-,a),\Omega) \cong \Omega(a)$$

for every $a \in \mathrm{Ob}(\mathbf{C})$. In this way, we can see that the subobject classifier Ω is defined on objects $\Omega(a)$ by taking subfunctors of the representable functor $\mathrm{Hom}_{\mathbf{C}}(-,a)$.

With such a description, the attentive reader may be reminded of the notion of a *sieve*. We already saw that there is a natural bijection between sieves over a given stage and the subfunctors of the Yoneda functor of that stage, suggesting that $\Omega(a) = \{S \mid S \text{ is a subfunctor of } \mathrm{Hom}(-,a)\}$. Thus,

$$\text{set of all sieves on } a \cong \mathrm{Sub}(\mathrm{Hom}(-,a)) \cong \mathrm{Nat}(\mathrm{Hom}(-,a),\Omega) \cong \Omega(a).$$

If we let sieves(a) denote the collection of all sieves on an object a of a category $\mathbf{C}$, the presheaf $\Omega : \mathbf{C}^{op} \to \mathbf{Set}$ is thus defined by $\Omega(a) = \mathrm{sieves}(a)$. For any arrow $f : b \to a$, we know that the action of $\Omega(f)$ will be given by pulling back along $\mathrm{Hom}(-,a)$, that is, $\Omega(f) = f^*$, where $f^* : \Omega(a) \to \Omega(b)$ maps sieves S on a to sieves on b by $S \mapsto \{c \xrightarrow{g} b \mid f \circ g \in S\}$, as in

$$\begin{array}{ccc}
\Omega(f)(S) & \longrightarrow & S \\
\downarrow & & \downarrow \\
\mathrm{Hom}(-,b) & \xrightarrow{\mathrm{Hom}(-,f)} & \mathrm{Hom}(-,a).
\end{array}$$

To finish the identification of the subobject classifier here, we need the truth arrow $\top : 1_{\mathbf{PreSh}(\mathbf{C})} \to \Omega$, and thus the terminal object. But the terminal presheaf 1 is defined by $1(a) = \{*\}$ for each $a \in \mathbf{C}$, and the truth arrow $\top : 1 \rightarrowtail \Omega$ will come from the maximal sieves, that is, by taking $\top_a(*) = \mathrm{Hom}(-,a) = M_a$ (i.e., the maximal sieve, or set of *all* arrows into a). Then, considering a subfunctor $R \rightarrowtail F$, it is fairly straightforward to construct a characteristic mapping $\chi : F \to \Omega$ for R, and then show this to be unique. The fully detailed construction of the corresponding characteristic arrows and the verification that the pair $(\Omega, \top)$ really does give a subobject classifier for the presheaf category is routine and can be found in many places in the literature. It can also be shown that the presheaf category $\mathbf{Set}^{\mathbf{C}^{op}}$ has finite limits and exponentials, which, together with the fact that this category has a subobject classifier, suffice to show that $\mathbf{Set}^{\mathbf{C}^{op}}$ meets the requirements for being an elementary topos. Instead of going through the details of this, let us now turn to make some observations regarding the subobject classifier for the presheaf category, and then consider these matters in more detail via an example.

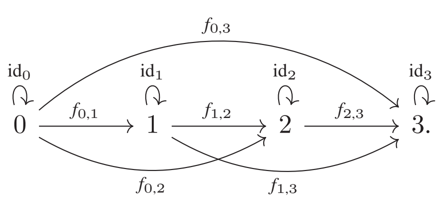
Remark 308 Recall the result from proposition 132 (chapter 5), letting us recover the poset of open sets of a topological space X from the category $\mathbf{Sh}(X)$ of sheaves on X, specifically by attending to the subobjects of the terminal sheaf 1. We now have another way of appreciating why this is the case. As any object will have a unique arrow to the terminal object, an object S of a topos $\mathscr{E}$ will be a subobject of the terminal object 1 exactly when the unique arrow $S \to 1$ is in fact monic. Those U for which $U \to 1$ is monic can be called *open* in $\mathscr{E}$, giving us subobjects of 1, and if we let $\mathscr{E} = \mathbf{Sh}(X)$ for a topological space X, such "open sheaves" just correspond to the usual open subsets of X.

Let us make another observation. The generalized elements of an object A that belong to B at all stages may be regarded as those whose characteristic arrows have truth-value 1 throughout, or which is assigned the maximal sieve at each stage, while those elements of A that do not belong to B at any stage should have truth-value 0, where the value 0 is given by the empty sieve at each stage. Of course, in the category **Set**, that is the end of the story: we have $\Omega = \{0, 1\}$, and subobjects of a set X correspond bijectively to subsets U of that set, where the subsets can be identified with their characteristic functions $\chi_U : X \to \{0, 1\}$, indicating that every element of a set either belongs to another set or does not. But in considering sets varying on categories other than the terminal category—that is, considering presheaves over an arbitrary category—those elements of A that belong to B at some stage will in general be captured by partial or intermediate truth-values, and in general Ω will not reduce to two truth-values.

The correspondence between sieves and subfunctors of Ω effectively allows us to provide an assignment of generalized truth-values to every sieve. In this way, sieves enable us to probe *when* an element or subobject "becomes a part of" another object, rather than just say *whether or not* it belongs, connecting up with the movement beyond classical logics with two truth-values. Whenever there are more than just the maximal ("true") and the empty ("false") sieves on an object, we will have the ability to deal with nonglobal elements, determining at which stage an element or subobject "falls into" an object. In general, in considering presheaves that vary on a category, elements of a presheaf set can be regarded as belonging to a subset *to a certain degree*, or *at some stage*. Different sieves in the classifier set Ω will correspond to different degrees of belonging, a partial value representing the fact that perhaps an assertion holds at one stage but not at an earlier stage.

Let us look a little closer at this way of seeing the subobject classifier concept via a particular instance, examining the structure of the topos $\mathbf{Set}^{\mathbf{[3]}^{op}}$.

Example 309 Consider $\mathbf{Set}^{\mathbf{[3]}^{op}}$, with the linear order category **[3]** (we might also call it **4**):[214]



214. This example is essentially from Spivak (2014, 416).

In the spirit of the preceding discussion, we would like to understand each of the objects of this category by looking at the set of arrows coming into that object, or equivalently, by understanding the sieves. For instance, focusing on the object 2, we can consider the hom-functor $\mathrm{Hom}(-, 2) = Y_2$. But this functor is defined, on objects, simply by plugging in the objects of the category and looking at the resulting set of arrows, that is, we have $\mathrm{Hom}(0, 2) = \{f_{0,2}\}$, $\mathrm{Hom}(1, 2) = \{f_{1,2}\}$, $\mathrm{Hom}(2, 2) = \{\mathrm{id}_2\}$, and $\mathrm{Hom}(3, 2) = \emptyset$. Joining these together into the set $\{\{\mathrm{id}_2\}, \{f_{1,2}\}, \{f_{0,2}\}\} \in \mathbf{Set}^{\mathbf{C}^{op}}$, we have an explicit construction of Y_2.

But observe that such a set is in fact none other than the maximal sieve on 2. Note, moreover, that the functor Y_2 supports the subfunctor construction. Indeed, there are four subfunctors of Y_2:

$$Y_2 \supseteq \emptyset = \text{“none”}$$

$$Y_2 \supseteq Y_2 \cong \{\{\mathrm{id}_2\}, \{f_{1,2}\}, \{f_{0,2}\}\} = \text{“all”}$$

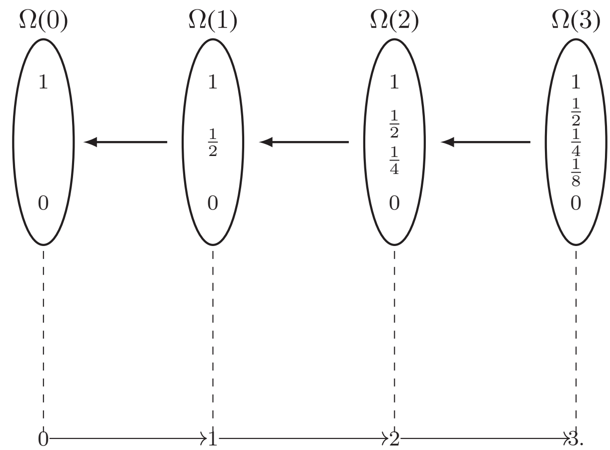
$$Y_2 \supseteq Y_2 \setminus (Y_2(2)) \cong \{\{f_{1,2}\}, \{f_{0,2}\}\} = \text{“back1”}$$

$$Y_2 \supseteq Y_2 \setminus (Y_2(1)) \cong \{f_{0,2}\} = \text{“back2”}$$

We can perform the same sort of analysis for the other three objects of **[3]**:

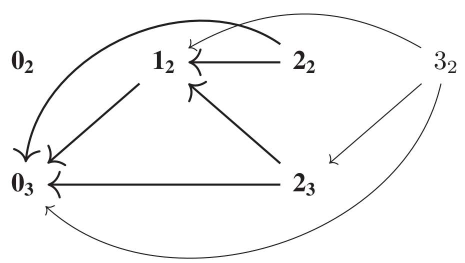
Data			
Object/Stage (n)	Sieves	Subfunctors $\Omega_{\mathbf{C}}(n)$	Truth-values
3	$\emptyset$ $\{f_{0,3}\}$ $\{\{f_{1,3}\}, \{f_{0,3}\}\}$ $\{\{f_{2,3}\}, \{f_{1,3}\}, \{f_{0,3}\}\}$ $\{\{\mathrm{id}_3\}, \{f_{2,3}\}, \{f_{1,3}\}, \{f_{0,3}\}\}$	$\ulcorner\emptyset\urcorner$ Back3 Back2 Back1 All $\cong Y_3$	0 $\frac{1}{8}$ $\frac{1}{4}$ $\frac{1}{2}$ 1
2	$\emptyset$ $\{f_{0,2}\}$ $\{\{f_{1,2}\}, \{f_{0,2}\}\}$ $\{\{\mathrm{id}_2\}, \{f_{1,2}\}, \{f_{0,2}\}\}$	$\ulcorner\emptyset\urcorner$ Back2 Back1 All $\cong Y_2$	0 $\frac{1}{4}$ $\frac{1}{2}$ 1
1	$\emptyset$ $\{f_{0,1}\}$ $\{\{\mathrm{id}_1\}, \{f_{0,1}\}\}$	$\ulcorner\emptyset\urcorner$ Back1 All $\cong Y_1$	0 $\frac{1}{2}$ 1
0	$\emptyset$ $\{\mathrm{id}_0\}$	$\ulcorner\emptyset\urcorner$ All $\cong Y_0$	0 1

Because of how the terminal object of $\mathbf{Set}^{\mathbf{C}^{op}}$ is defined, the morphism $\mathbf{1} \to \Omega_{\mathbf{[3]}}$ will pick out the functor corresponding to the maximal sieve from each row. In terms of truth-values and the contravariant Ω functor, the general idea could be pictured thus:



We see that under the action of Ω (regarded as a functor), the degrees of truth are reduced as we move backwards to earlier stages. On the other hand, one might interpret this to mean that as time goes on, the notion of truth obtains more "shades" or nuances.

Now that we have $\Omega_{\mathbf{C}} \in \mathrm{Ob}(\mathbf{Set}^{\mathbf{C}^{op}})$, given any other functor $X : \mathbf{C}^{op} \to \mathbf{Set}$, we know that the subfunctors of X will be in bijective correspondence with the morphisms $X \to \Omega_{\mathbf{C}}$. Suppose given the following instance $X : \mathbf{[3]}^{op} \to \mathbf{Set}$:



The portion of the graph in bold corresponds to a particular subfunctor $A \subseteq X$. This subfunctor in fact is just a natural transformation $\chi(A) : X \to \Omega_{\mathbf{[3]}}$. For example, $\chi(A)(3)$ sends $3_2 \in X(3)$ to *back1* $\in \Omega(3)$, but sends 3_1 to *none*; while $\chi(A)(1)$ sends $1_1 \in X(1)$ to *none* $\in \Omega(1)$, and sends 1_2 to *all*. As for the sieve $\{f_{0,3}\}$, which only belongs to Ω at stage 3, this can be thought of as capturing the element of the set A which only finally falls into some other subset at the zero-th stage, meaning that it takes three stages for it to arrive in the given subset—thus it could be thought of as taking on a truth-value of $\frac{1}{2^3}$.

There is of course no reason to restrict ourselves to linear categories like the $\mathbf{C} = \mathbf{[3]}$ given above. We can consider poset categories or more complicated categories, and the concept of sieves and truth-values will apply just as one would expect.

11.2 Examples of Elementary Toposes

What follows are some notable examples of elementary toposes, where only a very abridged sketch is given in most cases. In subsequent sections, we will look at particular toposes in more detail.

Example 310 **Set** is an elementary topos (and an important one at that, since properties found there largely motivate the three requirements in the definition of an elementary topos). It has finite limits and given any two sets X and Y, we can always form the set Y^X, where this is just the set of functions from X to Y—and this set will enjoy the universal property that the bijection $\mathrm{Hom}_{\mathbf{Set}}(Z, Y^X) \cong \mathrm{Hom}_{\mathbf{Set}}(Z \times X, Y)$ will be natural (in both Y and Z), thus giving rise to an adjunction between (the left adjoint functor) $- \times X : \mathbf{Set} \to \mathbf{Set}$ and (right adjoint) $(-)^X : \mathbf{Set} \to \mathbf{Set}$. Moreover, as already discussed, $\Omega = \{0, 1\}$, and Ω^X can be shown to be isomorphic to the set $\mathbb{P}(X)$ of *subsets* of X, the subobject classifier being defined as we saw.

Example 311 The category $\mathbf{Set}^{\mathbf{C}^{op}}$ of presheaves on a small category **C** is an elementary topos.[215] If **C** is a small category, we know that the category of presheaves on **C** is complete. As for exponentials: supposing we have two presheaves F, G, the Yoneda lemma demands that the exponential G^F is defined as

$$G^F(c) \cong \mathrm{Nat}(\mathrm{Hom}(-, c), G^F) \cong \mathrm{Nat}(\mathrm{Hom}(-, c) \times F, G)$$

for every object $c \in \mathbf{C}$. By taking the right-hand side as a definition, we still have to verify that the adjunction

$$\mathrm{Hom}(H, G^F) \cong \mathrm{Hom}(H \times F, G)$$

holds for all presheaves H, not just the representable ones. However, by proposition 162 we know that we can express any presheaf as a colimit of representable functors, colim Y_c. Thus, given a third functor H, and writing it as a colimit of representable functors,

$$\begin{aligned}
\mathrm{Hom}(H, G^F) &\cong \mathrm{Hom}(\mathrm{colim}(\mathrm{Hom}(-, c_i)), G^F) \\
&\cong \lim \mathrm{Hom}(\mathrm{Hom}(-, c_i), G^F) \text{ by hom-functor transforms colimits into limits} \\
&\cong \lim G^F(c_i) \text{ by the Yoneda lemma} \\
&\cong \lim \mathrm{Hom}(\mathrm{Hom}(-, c_i) \times F, G) \text{ by definition of } G^F \\
&\cong \mathrm{Hom}(\mathrm{colim}\ (\mathrm{Hom}(-, c_i) \times F), G) \\
&\cong \mathrm{Hom}((\mathrm{colim}\ \mathrm{Hom}(-, c_i)) \times F, G) \text{ as } (-) \times F \text{ preserves colimits in } \mathbf{Set} \\
&\cong \mathrm{Hom}(H \times F, G).
\end{aligned}$$

In the previous section, we saw a sketch of how the subobject classifier Ω looks for a presheaf category, where for $c \in \mathbf{C}$, we will have $\Omega(c) = \{S \mid S \text{ is a subfunctor of } Y_c\}$.

The terminal presheaf 1 is defined as we have already seen, namely by $1(c) = \{*\}$ precisely when $c \in \mathbf{C}$, and the monic map $\top : 1 \rightarrowtail \Omega$ is given by $\top_c(*) = Y_c$.

215. Henceforth, I will stop adding "elementary"; topos will mean "elementary topos." Going forward, if Grothendieck topos is ever meant, this will be explicitly indicated.

Because of the powerful fact that so many categories of interest can be constructed as a presheaf category, this last example will supply us with a large store of examples. We will explore a few of them in more detail in coming sections, but for now note that all the presheaf categories introduced so far will thus fit into this mold. To take one more or less at random, the category of G-sets (for a fixed group G), as nothing other than the category of presheaves on (the categorified version of G) $\mathcal{G}$, is a topos.

Example 312 The category **FinSet** of finite sets *is* a topos.

If X, Y are finite sets, then Y^X will be finite; moreover, $\Omega = \{0, 1\}$ is finite, and a finite limit of finite sets is finite as well. Thus, the topos structure of the category of sets effectively carries over to the category of finite sets.

But notice that the category of finite sets lacks infinite products. There are precisely two morphisms from the terminal 1 to Ω, so if $\Omega \times \Omega \times \cdots$ were to exist, with the α factors infinite, then there would be precisely 2^α morphisms from 1 to $\Omega \times \Omega \times \cdots$. But the assumption was that $\Omega \times \Omega \times \cdots$ will be a finite set, so this cannot be. Incidentally, this helps account for why we said in example 284 (chapter 10) that the topos of finite sets is *not* a Grothendieck topos.

Similarly, *the category of finite presheaves on a finite category is a topos* (but, again, *not* a Grothendieck topos, due to a lack of completeness). By a finite presheaf on **C** finite, we of course just mean a functor $\mathbf{C}^{op} \to \mathbf{FinSet}$ that is valued in the category of *finite* sets.[216]

Example 313 Let us call the (large) category consisting of the ordinal numbers, partially ordered in the natural way, **On**.[217] Then $\mathbf{Set}^{\mathbf{On}}$ is not a topos, since in particular any representable functor will have a nonsmall class of subfunctors, so it cannot have a subobject classifier. $\mathbf{Set}^{\mathbf{On}^{op}}$, by contrast, *is* a topos. Note that in $\mathbf{Set}^{\mathbf{On}^{op}}$ there will still be a proper class of subfunctors of the terminal constant functor (with value 1)—thus, it is not locally small, as there will be a proper class of morphisms $1 \to \Omega$.

Incidentally, readers who know about surreal numbers (a construction that includes both the real numbers and the infinite ordinal numbers) might wish to consider how this all unfolds using (the appropriate category of) such numbers.[218]

Example 314 The category **Ring** of rings has rings (with identity) for objects and ring homomorphisms (preserving the identity) for morphisms.[219] This category is both complete and cocomplete, with the zero (trivial) ring **0** for terminal object. However, there are no nontrivial morphisms in **Ring** out of **0**, that is, from the terminal object to any nonzero

216. The constructions basically work as they do for general presheaf toposes, provided **C** itself is finite. In general, when **C** is an infinite category, one will run into difficulties in trying to use the usual presheaf topos constructions for *finite* presheaves. An interesting exception to this is with the category of finite G-sets (on an arbitrary group G), which *is* a topos, even without imposing a finiteness condition on G.

217. This differs from the category of finite ordinals, or the category of natural numbers, by generalizing to numbers of possibly infinite magnitudes.

218. For the proper class **No** of surreal numbers, built on top of a broader combinatorial theory of games, see Conway (2000). Joyal (1977) later gave a category theoretic description of the situation, defining a category of games where objects are sets of games, morphisms are given by winning strategies (for the "Left" player playing second in a particular game), the identity morphism is the "copycat" strategy, and composites are a little more complex to describe. See Joyal (1977) and Cockett, Cruttwell, and Saff (2010) for more details.

219. This category differs from **Rng**, the category of rings (without unit) for objects and ring homomorphisms for morphisms.

ring. This is enough to show that even though this category has a terminal object, it cannot have a subobject classifier. Thus, **Ring** is not a topos.

We have seen a few examples of toposes that are not Grothendieck toposes. However, it can be shown that a Grothendieck topos $\mathbf{Sh}(\mathbf{C}, J)$ is always a topos.

Example 315 Every Grothendieck topos (sheaf on a site) is an example of a topos.

In the last chapter, we saw how the inclusion functor $i : \mathbf{Sh}(\mathbf{C}, J) \to \mathbf{PreSh}(\mathbf{C})$ has a left adjoint $\mathbf{a}$, which preserves finite limits and (small) colimits. In fact, a Grothendieck topos is complete since $\mathbf{PreSh}(\mathbf{C})$ is complete, and limits in $\mathbf{Sh}(\mathbf{C}, J)$ are computed as in the presheaf category. As such, it actually has all (small) limits, which are preserved by the inclusion functor i; (small) limits are computed pointwise and the terminal object $1_{\mathbf{Sh}(\mathbf{C},J)}$ of $\mathbf{Sh}(\mathbf{C}, J)$ is the functor $t : \mathbf{C}^{op} \to \mathbf{Set}$ that sends each object $c \in \mathbf{C}$ to the singleton set $\{*\}$. Moreover, since $\mathbf{PreSh}(\mathbf{C})$ is cocomplete, so too is $\mathbf{Sh}(\mathbf{C}, J)$, where colimits in $\mathbf{Sh}(\mathbf{C}, J)$ are got by applying the associated sheaf functor $\mathbf{a}$ to the colimits in the presheaf category.

$\mathbf{Sh}(\mathbf{C}, J)$ also has exponentials, where these are formed just as in the presheaf topos $\mathbf{PreSh}(\mathbf{C})$. Moreover, it can be shown that if G is a sheaf for a topology J on $\mathbf{C}$, then for any presheaf F the presheaf G^F will itself be a sheaf.

Finally, the category $\mathbf{Sh}(\mathbf{C}, J)$ has a subobject classifier. One way of describing the subobjects is as follows: given a sieve S on c, we say that S is *J-closed* provided, for any arrow $f : c' \to c$, the pullback $f^*(S)$ being in $J(c')$ implies that $f \in S$. Then, we define $\Omega : \mathbf{C}^{op} \to \mathbf{Set}$ on objects $c \in \mathbf{C}$ as the set of all J-closed sieves on c, and on morphisms by the pullback $f^*(-)$ by a sieve. Then, the arrow $\top : 1_{\mathbf{Sh}(\mathbf{C},J)} \to \Omega$ is defined by $\top(*)(c) =$ the maximal sieve on c, and this is a subobject classifier. The classifying arrow $\chi_{F'} : F \to \Omega$ of a subobject $F' \subseteq F$ in $\mathbf{Sh}(\mathbf{C}, J)$ is then given by

$$\chi_{F'}(c)(x) = \{f : c' \to c \mid F(f)(x) \in F'(c')\}$$

for any $c \in \mathbf{C}$ and $x \in F(c)$.[220]

So while every Grothendieck topos is an elementary topos, the converse is not necessarily true: an elementary topos need not be a Grothendieck topos. Elementary toposes accordingly supply a proper generalization of Grothendieck toposes. Some of these non-Grothendieck toposes are of particular use or relevance in logic, particularly in the study of higher-order intuitionistic type theory. In practice, one important general difference between the two notions seems to be that the presence of sites in Grothendieck toposes provides a setting where one can more readily rely on or make use of geometric intuitions. In the next section, we explore a little more about elementary toposes, consider some translations of the relevant notions back to the realm of Grothendieck toposes, and then look closely at a particular example.

11.3 Lawvere-Tierney Topologies and Their Sheaves

Constructing a Grothendieck topos can generally be thought of as breaking down into two steps:

220. The reader can consult Mac Lane and Moerdijk (1994, III.6–7) for further details on how a Grothendieck topos satisfies the properties of an elementary topos.

1. given **Set** and some (small) category **C**, we pass to the category $\mathbf{Set}^{\mathbf{C}^{op}}$ of presheaves on **C**; then
2. we pass to the category $\mathbf{Sh}(\mathbf{C}, J)$ of sheaves for a particular Grothendieck topology J.

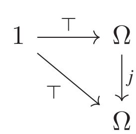
Via the notion of an elementary topos, we are essentially able to replace **Set** in step (1) by any topos $\mathscr{E}$. The purpose of the notions introduced in the present section, in particular that of a *Lawvere-Tierney topology*, is to similarly generalize the second step, ultimately enabling us to provide an even more general account of sheaves. We can moreover translate between earlier notions: in particular, a Grothendieck topology J on a category **C** can be shown to correspond to a Lawvere-Tierney topology (or local operator), defined as an invariant in the context of the topos $\mathbf{Set}^{\mathbf{C}^{op}}$.

To motivate this, suppose we are working in the context of a topological space. Suppose S is any collection of open sets U_i of the space X. Then, consider the collection of all the open sets *covered by* the U_i, and denote this for now by a generic symbol $\blacksquare(S)$. But then $\blacksquare$ of the entire space X should of course be everything, as taking the collection of all the open sets covered by the entire space should of course return the entire space itself. Moreover, we would expect that this operation of taking such collections will be idempotent, in the sense that $\blacksquare(\blacksquare(S)) = \blacksquare(S)$. What about intersections? It is also reasonable to expect that $\blacksquare(S \cap S') \subseteq \blacksquare(S) \cap \blacksquare(S')$—and, whenever both S and S' are sieves, this would become an equality.

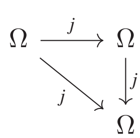
But what if the collection S is in fact a *sieve* on an open set $U \subseteq X$? In that case, $\blacksquare(S)$ will also be a sieve and S will just be an element in $\Omega(U)$, where Ω is the subobject classifier for the topos $\mathscr{E} = \mathbf{Set}^{\mathscr{O}(X)^{op}}$ of presheaves on a topological space X. But altogether, this amounts to saying that the $\blacksquare$ just described is essentially an endomap $\blacksquare : \Omega \to \Omega$ satisfying certain equations, where the map itself effectively specifies what each sieve covers. This line of reasoning can serve as motivation for the following definition for toposes more generally (where we now denote the abstract operator $\blacksquare$ by j).

Definition 316 Let $\mathscr{E}$ be a topos, and let Ω together with the arrow $\top : \mathbf{1} \to \Omega$ be its subobject classifier. A *Lawvere-Tierney topology* (or sometimes *local operator*) on $\mathscr{E}$ is a map $j : \Omega \to \Omega$ in $\mathscr{E}$ such that the following three properties are satisfied:

1. $j \circ \top = \top$, that is,



2. $j \circ j = j$, that is,



3. $j \circ \wedge = \wedge \circ (j \times j)$, that is,[221]

221. The $\wedge : \Omega \times \Omega \to \Omega$ here is just the meet operation (in the internal Heyting algebra formed by Ω). With $\leq$ the internal partial order on Ω, note that the first condition could also be expressed by saying $\mathrm{id}_\Omega \leq j$.

$$\begin{array}{ccc} \Omega\times\Omega & \xrightarrow{\wedge} & \Omega \\ {\scriptstyle j\times j}\downarrow & & \downarrow{\scriptstyle j} \\ \Omega\times\Omega & \xrightarrow[\wedge]{} & \Omega. \end{array}$$

One then sometimes refers to the pair $\mathscr{E}_j = (\mathscr{E}, j)$ as an *elementary site*.

It may help to think of j in a different way. We can effectively transfer the *algebraic* structure supplied by j and $\wedge$ on the special object Ω to a corresponding *functorial* structure on hom-sets, via the fact that in any topos, given an object E, we will have $\mathrm{Sub}(E) \cong \mathrm{Hom}(E, \Omega)$. Moreover, for each object E in a topos, it can be shown that the poset $\mathrm{Sub}(E)$ of subobjects of E forms a lattice (actually, it is a Heyting algebra!). In this way, j can equivalently be displayed as a particular *closure operator* on the Heyting algebra of subobjects.

Definition 317 A closure operator on a topos $\mathscr{E}$ will be an operator on the subobjects $A \rightarrowtail E$ of each object E of $\mathscr{E}$

$$\overline{(-)} : \mathrm{Sub}_{\mathscr{E}}(E) \to \mathrm{Sub}_{\mathscr{E}}(E)$$

$$A \mapsto \overline{A}$$

natural in E. It can be shown that j is a Lawvere-Tierney topology if and only if this operator has, for all $A, B \in \mathrm{Sub}(E)$, the following ("universal closure operator") properties

1. $A \subseteq \overline{A}$,
2. $\overline{\overline{A}} = \overline{A}$,
3. $\overline{A \cap B} = \overline{A} \cap \overline{B}$.[222]

The converse is also true: an operator defined on all of $\mathrm{Sub}(E)$, natural in E, with those three properties—that is, a closure operator on the subobjects of each object of $\mathscr{E}$—always arises from a unique Lawvere-Tierney topology j.

The last two equalities are basically direct translations of the last two conditions on a Lawvere-Tierney topology. For the first, the operator j determines a closure operator on the subobjects $A \rightarrowtail E$ of each object E by

$$\begin{array}{ccc} \mathrm{Hom}(E,\Omega) & \xrightarrow{\cong} & \mathrm{Sub}(E) \ni A \\ {\scriptstyle \mathrm{Hom}(\mathrm{id},j)}\downarrow & & \downarrow \\ \mathrm{Hom}(E,\Omega) & \xrightarrow{\cong} & \mathrm{Sub}(E) \ni \overline{A}. \end{array}$$

This means that, given a subobject $s : A \rightarrowtail E$ with the classifier $\chi_s : E \to \Omega$, then, referring to the diagram (with both squares formed by pullbacks along the truth arrow $\top$),

222. Another condition would be that closure commutes with pullback along morphisms of $\mathscr{E}$, in the sense that given $f : Y \to X$ and a subobject $X' \rightarrowtail X$ (the closure of which is denoted $\overline{X'} \rightarrowtail X$), we will have $f^*(\overline{X'}) \cong \overline{f^*(X')}$ as subobjects of Y.

$$\begin{array}{ccccccc} \overline{A} & & & \longrightarrow & & & 1 \\ \Big\downarrow{\scriptstyle \overline{s}} & & A & \longrightarrow & 1 & & \Big\downarrow{\scriptstyle \top} \\ & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \top} & & \\ E & = & E & \xrightarrow[\chi_s]{} & \Omega & \xrightarrow{j} & \Omega, \end{array}$$

the composition $j \circ \chi_s$ defines another subobject $\overline{s} : \overline{A} \rightarrowtail E$, the one we call the j-closure $\overline{A}$, such that s is a subobject of $\overline{s}$. Conversely, a closure operator $\overline{(-)}$ on a topos will give a unique topology j via the following pullback diagram:

$$\begin{array}{ccc} \overline{1} & \longrightarrow & 1 \\ {\scriptstyle \top}\downarrow & & \downarrow \\ \Omega & \xrightarrow{j} & \Omega. \end{array}$$

This description can be used to translate back and forth between the statement $j(\top) = \top$ and the property $A \subseteq \overline{A}$. Because of this equivalence between the two perspectives, we can freely refer to Lawvere-Tierney topologies in terms of *universal closure operators*.

Observe how, from property (3) of definition 317, and since $A \subseteq B$ iff $A \wedge B = A$, we can further obtain that $A \subseteq B$ implies $\overline{A} \subseteq \overline{B}$, which informs us that such an operator is also a closure operator in the earlier sense of definition 182—where this meant an operator that was monotone, extensive, and idempotent.[223] Recall also how we have seen that the modalities of modal logic could be described as closure operators on the poset of propositions (with their truth-valuations). But this, together with the fact that the subobject Ω can be interpreted as a collection of generalized "truth-values," suggests that we may regard j as a kind of modal operator! Let us roughly sketch the idea.

In the usual setting of topological spaces, when we say that a property holds *locally* at a point x of a space X, we mean that it holds at all points "nearby," or throughout some neighborhood V of x. Another way of saying this is that a property holds locally for an object U provided *it can be covered by open sets that all have the property as well*. A natural suggestion, due ultimately to Lawvere, is thus to regard j as a sort of modal operator that says "it is *locally* the case that."

For $E \in \mathscr{E}$, observe that we can view a morphism $p : E \to \Omega$ as a predicate, where in particular if $E = \mathbf{1}$, then the elements $p : \mathbf{1} \to \Omega$ may be seen as propositions. We will also have a function $p(U) : E(U) \to \Omega(U)$ for any object U. For any object $E \in \mathscr{E}$, there will be a poset formed out of the predicates $p, q : E \to \Omega$ on E, where $p \leq q$ provided p implies q. In this way, we might then read the three conditions of definition 316 as saying

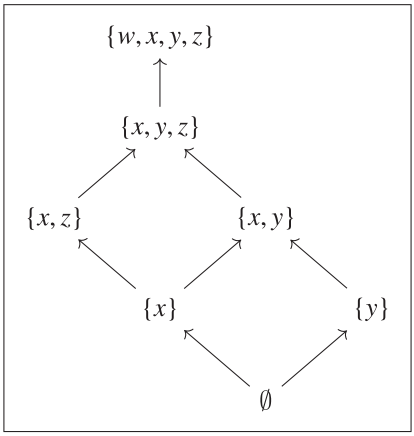
1. if p is true, then p is locally true;
2. p locally is locally true iff p is locally true;

223. However, a universal closure operator in the above sense should not be confused with the notion of a Kuratowski closure on the lattice of all subsets of a topological space. In particular, note that Kuratowskian closure always commutes with finite unions, but in general not with intersections. On the other hand, a universal closure operator commutes with intersections, but does not typically commute with unions. As such, while such a closure operator in a sense combines various of the properties of the interior and closure operators from standard topological spaces, the reader should be careful to note that the closure operators discussed here are distinct from "closed subsets" from topological spaces.

3. $p \wedge q$ is locally true iff p and q are each locally true.

In this way, the modality j can be seen as generalizing what it means to be *local* to the setting of toposes.[224]

Example 318 Recall the particular downset completion poset of the poset discussed in example 252:



j will range over the elements of this downset poset **Down**($\mathscr{P}$). A particular Lawvere-Tierney topology may then be given by

$$\begin{aligned}
&j(\emptyset) = \emptyset \\
&j(\{x\}) = \{x, z\} \\
&j(\{y\}) = \{y\} \\
&j(\{x, y\}) = \{w, x, y, z\} \\
&j(\{x, z\}) = \{x, z\} \\
&j(\{x, y, z\}) = \{w, x, y, z\} \\
&j(\{w, x, y, z\}) = \{w, x, y, z\}.
\end{aligned}$$

You can verify that this constitutes a Lawvere-Tierney topology on the poset of downsets.

Stepping back, the reader should by now fully appreciate that to define sheaves, what is really important is what gets "covered" and how the data of the sheaf behaves with respect to (what formally acts as) covers. With Grothendieck toposes, when defining a sheaf, we did so in terms of coverages or coverings by sieves. Ultimately, our new "topology" j will similarly track such information, functioning to specify what each sieve covers, but now in the more general setting of elementary toposes. Accordingly, this notion of a Lawvere-Tierney topology will enable us to define (more general) sheaves for any topos. We can

224. An interesting question we leave to the reader: following this idea of j being regarded as a modal operator of sorts, since (following the ideas explored in chapter 7) we are used to modal operators coming in dual pairs, what (if anything) might the dual pair of "it is locally valid" be? For more details on j as a modal operator, see Goldblatt (2006, chap.14).

already appreciate the greater generality of this notion, though, by considering the following proposition concerning how Lawvere-Tierney topologies include the Grothendieck topologies.

Proposition 319 Each Grothendieck topology J on a small category $\mathbf{C}$ determines a Lawvere-Tierney topology j on the corresponding presheaf topos $\mathbf{Set}^{\mathbf{C}^{op}}$.

Without giving a proper proof of this fact, the idea here is this: remember that the subobject classifier Ω for $\mathbf{Set}^{\mathbf{C}^{op}}$ is the functor that takes an object c to the collection of sieves on c in $\mathbf{C}$, and the action of morphisms of $\mathbf{C}$ on Ω is by "pulling back," in the sense that for $f: c' \to c$, we take

$$f^*: \Omega(c) \to \Omega(c'), \quad f^*(S) = \{g \mid f \circ g \in S\}.$$

Recall also that a Grothendieck topology J is just an assignment, for each object c, of a collection of sieves that are "covering" and, as such, meet certain properties. A subobject $J \hookrightarrow \Omega$ is thus just a selection from the collection of sieves, which is moreover closed under pullback. Given a Grothendieck topology J, we accordingly define $j: \Omega \to \Omega$ by taking

$$\begin{aligned} j_c(S) &= \{g \mid S \text{ covers } g: d \to c\} \\ &= \{g \mid g^*(S) \in J(\mathrm{dom} g)\}. \end{aligned}$$

Skimming over the details of how this works in the general case, let us look at our running example and see how to convert from a particular Grothendieck topology on $\mathcal{P}$ to a Lawvere-Tierney topology on the associated presheaf topos.

Example 320 Recall the dense (Grothendieck) topology on $\mathcal{P}$ from example 252:

$$\begin{aligned} &J_{dense}(x) = \{\{x\}\} \\ &J_{dense}(y) = \{\{y\}\} \\ &J_{dense}(z) = \{\{x\}, \{x,z\}\} \\ &J_{dense}(w) = \{\{x,y\}, \{x,y,z\}, \{w,x,y,z\}\}. \end{aligned}$$

This induces a particular Lawvere-Tierney topology on the associated downset completion $\mathbf{Down}(\mathcal{P})$. In particular, let us look at $j(\{x\})$. Consider that $\downarrow x \cap \{x\} = \{x\} \in J_{dense}(\{x\})$ and also $\downarrow z \cap \{x\} = \{x\} \in J_{dense}(\{z\})$; yet for no other p do we have $\downarrow p \cap \{x\} \in J_{dense}(p)$. Accordingly, we take $j(\{x\}) = \{x, z\}$. Similarly for the other elements. Altogether, the J_{dense} Grothendieck topology on $\mathcal{P}$ induces a corresponding Lawvere-Tierney topology j given by that already described in example 318.

The general result informs us, incidentally, that any Grothendieck topology J can be given a description in terms of a closure operator.[225]

In fact, for presheaf toposes, we can go the other way as well, recovering a Grothendieck topology on $\mathbf{C}$ from a Lawvere-Tierney topology on $\mathbf{Set}^{\mathbf{C}^{op}}$. In other words, it can be shown that every Lawvere-Tierney topology j on a presheaf topos arises (as above) from a Grothendieck topology.[226]

225. For further discussion of the explicit description of this, see Mac Lane and Moerdijk (1994).
226. Though, for more general toposes, there can be other Lawvere-Tierney topologies.

Proposition 321 If **C** is a small category, then the Grothendieck topologies J on **C** correspond precisely to Lawvere-Tierney topologies on the presheaf topos $\mathbf{Set}^{\mathbf{C}^{op}}$.

Again skimming over the details of a proof of this fact, any Lawvere-Tierney topology $j : \Omega \to \Omega$ on the presheaf topos classifies the subobject $J \hookrightarrow \Omega$ by taking $S \in J(c)$ iff $j_c(S) = M_c$, the maximal sieve. Such a J can be shown to be a Grothendieck topology.[227]

In particular, given a poset $\mathcal{P}$ and a Lawvere-Tierney topology j on $\mathcal{D}(\mathcal{P})$, the family

$$\{J(p) \mid J(p) = \{\downarrow p \cap X \mid p \in j(X) \text{ and } p \in \mathcal{P}\}\}$$

will form a Grothendieck topology. Again, let's look at how this works in our running example.

Example 322 Suppose given the Lawvere-Tierney topology (on the poset of downsets) from example 318. Then, for instance, $J(z)$ will be computed as follows:

$$\begin{aligned} J(z) &= \{\downarrow z \cap \{x\}, \downarrow z \cap \{x,z\}, \downarrow z \cap \{x,y\}, \downarrow z \cap \{x,y,z\}, \downarrow z \cap \{w,x,y,z\}\} \\ &= \{\{x,z\} \cap \{x\}, \{x,z\} \cap \{x,z\}, \{x,z\} \cap \{x,y\}, \{x,z\} \cap \{x,y,z\}, \{x,z\} \cap \{w,x,y,z\}\} \\ &= \{\{x\}, \{x,z\}\}. \end{aligned}$$

We can similarly compute the rest of $J(p)$ for all $p \in \mathcal{P}$.

The following extended example gives us a chance to delve into some of these ideas in more depth, illustrating how Lawvere-Tierney topologies work more explicitly, in the context of graphs. This example will moreover help motivate the still more general definition of a sheaf with respect to a Lawvere-Tierney topology toward which we are building.

Example 323 Recall that if we take

$$\mathcal{G} := \boxed{V \underset{t}{\overset{s}{\rightrightarrows}} A}$$

for our indexing category, then presheaves on $\mathcal{G}$ recover directed graphs (or quivers), so that we have the equivalence of $\mathbf{Set}^{\mathcal{G}^{op}}$ and the category **Quiv**. As a presheaf category (on $\mathcal{G}$, a small category), we know that $\mathbf{Set}^{\mathcal{G}^{op}}$ will have the requisite properties of a topos: (1) it has finite limits (terminal object 1 and pullbacks); (2) it has exponentials (i.e., the functor $- \times X$ has a right adjoint for each X); (3) it has a "truth-values" object Ω which classifies subobjects. While the first two properties of a topos are more or less standard constructions, the third can again use some explanation, and it will be instructive to see how this works in the particular case of graphs. This will further allow us to construct various explicit Lawvere-Tierney topologies.

The first thing to recall is the fact that since every object of a presheaf topos is built by gluing a suitable set of representables, and the representable functors provide us with a set of generators for the topos (since two arrows differ iff they differ over some representable), we can build the classifier Ω by considering subobjects of the representables. Moreover, we know that the subobject classifier in a presheaf topos $\mathbf{PreSh}(\mathbf{C})$ is the presheaf that sends

227. See Mac Lane and Moerdijk (1994, V.4) for a proof of this fact—which, together with proposition 319, lets us verify that $j \mapsto J$ and $J \mapsto j$ are inverse, proving the main result.

each object $c \in \mathbf{C}$ to the set of sieves on it,

$$\Omega : c \mapsto \{\text{sieves}(c)\},$$

and the sieves on c may equivalently be regarded as the set of subobjects of the representable presheaf Y_c. The classifier morphism $\top : 1 \to \Omega$ is then the natural transformation that selects, for each object $c \in \mathbf{C}$, the maximal sieve. Accordingly, the first step is to address the question: What are the representable functors?

In general, we know that there is one representable $Y_c = \text{Hom}_{\mathbf{C}}(-, c)$ functor for each object c of $\mathbf{C}$, where this of course associates to c the set of all morphisms into c (and the definition on arrows supplied by composition). So in our present case, the representable functors giving us the "generic figures" that can be found in directed graphs will be

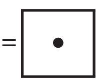
$$\text{Hom}_{\mathcal{G}}(-, V) \text{ and } \text{Hom}_{\mathcal{G}}(-, A),$$

which you might think of as supplying us with the "generic vertex" and "generic arrow" (and that is all we need to consider, as the only two objects of $\mathcal{G}$ are V and A). Now we need only range over the objects of $\mathcal{G}$, plugging them in to the above hom-sets, seeing what pops out. Let's do this first for $\text{Hom}_{\mathcal{G}}(-, V)$:

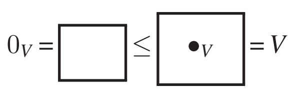
$$\text{Hom}_{\mathcal{G}}(V, V) = \{\text{id}_V\} = \{V\}$$

$$\text{Hom}_{\mathcal{G}}(A, V) = \emptyset$$

So the representable $\text{Hom}_{\mathcal{G}}(-, V)$ will just consist of a single vertex or node (with no arrows)—where this can be thought of as the generic vertex. This will clearly have just two subobjects: V, for the vertex itself; and 0_V, for the empty vertex. These two subobjects will ultimately give us the two vertices of Ω, seen as a directed graph in its own right, that is, $\Omega(V) = \{V, 0_V\}$. These two subobjects of

$\text{Hom}_{\mathcal{G}}(-, V) =$ 

are clearly ordered among themselves, in the sense that

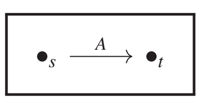


Now for the representable $\text{Hom}_{\mathcal{G}}(-, A)$, we get

$$\text{Hom}_{\mathcal{G}}(V, A) = \{s, t\}$$

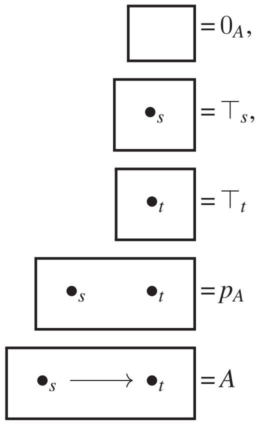
$$\text{Hom}_{\mathcal{G}}(A, A) = \{\text{id}_A\} = \{A\}.$$

This tells us that s and t furnish us with two vertices, and id_A (denoted A) for a single arrow. Altogether, the representable $\text{Hom}_{\mathcal{G}}(-, A)$ can then be pictured as

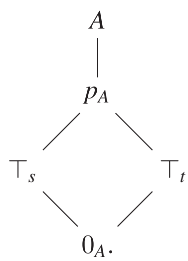


Conceptually, that the representable associated to V is given by a single vertex, while the representable associated to A is given by a single arrow, tells us that arbitrary figures built from such an indexing category will be a matter of building with directed arrows between vertices—and that's as it should be, since we are dealing with directed graphs!

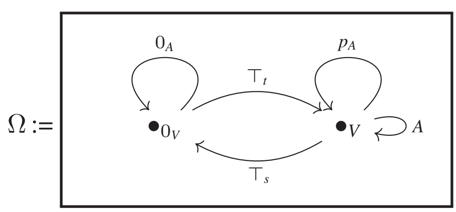
The representable $\mathrm{Hom}_{\mathscr{G}}(-, A)$, for its part, will have five subobjects (natural names given on the right):



These will supply Ω with its arrow set, that is, $\Omega(A)$. These subgraphs of $\mathrm{Hom}_{\mathscr{G}}(-, A)$ are again clearly ordered by inclusion, an order that assembles into the Hasse diagram



As Ω is itself a presheaf, the last thing to do, before assembling all this into Ω, is spell out how the right action works. As 0_A is the empty graph, we must have that $0_A s = 0_V$ and $0_A t = 0_V$. Moreover, $As = At = V$ and likewise $p_A t = p_A s = V$. Finally, $\top_s s = V$ and $\top_s t = 0_V$ (as t sends the only vertex of $\mathrm{Hom}_{\mathscr{G}}(-, V)$ into $\top_t$, not $\top_s$), and similarly $\top_t t = V$ and $\top_t s = 0_V$. Altogether, this data of Ω is supplied by the graph



The terminal graph 1 is just a single vertex together with a single loop. Clearly the truth classifier arrow (graph homomorphism) $\top : 1 \to \Omega$ will then send the (only) vertex of 1 to V and the only arrow of 1 to A.

Now, suppose we are given a graph G and a subgraph $u : H \hookrightarrow G$. How does the classifying map $\chi_u : G \to \Omega$ work?

$$\begin{array}{ccc} H & \xrightarrow{!} & 1 \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ G & \xrightarrow[\chi_u]{} & \Omega \end{array}$$

First off, on vertices: χ_u will map vertices of G that are not in H to 0_V, while vertices of G that are in H are mapped to $\bullet = V$. In other words, for vertices, the situation is simple: given a subgraph of a graph G, either a vertex of the graph G is or is not to be found in a certain subgraph. Ω has precisely two vertices to "represent" such a binary choice on vertices.

On arrows: if an arrow is in H, it will get mapped to A; if not, then the situation is more involved. In particular, there are a few alternatives:

- arrows whose source and target are not in H get mapped by χ_u to 0_A;
- arrows that have just their source (but not target) in H, get mapped by χ_u to $\top_s$;
- arrows whose target (but not source) is in H, get mapped by χ_u to $\top_t$; while
- arrows that have both their source and target in H get mapped by χ_u to p_A.

These four cases—plus the first taking an arrow that is indeed in the subgraph H to A—give us the five total possibilities, which five maps are represented by the five arrow-objects of Ω. In this way, the "classification" with arrows emerges as more refined or complicated than it is for vertices. In the end, the underlying idea here is that we are using Ω to tell us, for each part of a graph G, how much of it happens to be in a subgraph H. That there are more classifying maps for arrows than there are for vertices informs as that, as far as graphs are concerned, vertices are a comparatively simpler sort of structure than arrows.

In thinking of Ω as the "object of truth-values" for its topos, a natural question is how logical operators on such truth-values are to be defined, enabling us to reason within the topos. We can define the conjunction operator $\wedge : \Omega \times \Omega \to \Omega$ as the characteristic map of the subobject $\langle \top, \top \rangle : 1 \to \Omega \times \Omega$, and the negation operator $\neg : \Omega \to \Omega$ as the characteristic map of $\bot : 1 \to \Omega$. On vertices, since these have only two truth-values (0_V and V), $\wedge$ and $\neg$ behave just as you would expect—that is, just as they do in **Set**. However, on arrows, $\wedge$ acts as meet with respect to the order on the arrows of Ω, for instance, $p_A \wedge t = t$, $s \wedge t = 0_A$; while $\neg$ takes $\neg 0_A = A$ and $\neg A = 0_A$, but also $\neg p_A = 0_A$ (which means basically that this operator acts to ignore whether or not an arrow was in a subgraph, as long as both its source and target were in the subgraph).

Using these notions, we can take the *complement* of a subgraph $H \hookrightarrow G$ by taking the subobject $\neg H$ of G classified by $\neg \circ \chi_u$, and this will be the same as taking all the vertices of G that are not in H, together with all the arrows in G between those particular G-vertices. Note that this is the same as what we discussed back in example 197 (chapter 7), where $\neg X$ was treated as the largest subgraph disjoint from X. More explicitly, and in the general case of presheaf toposes,

Definition 324 For any subobject $A \rightarrowtail E$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ and any object c in $\mathbf{C}$,

$$(\neg A)(c) = \{x \mid x \in E(c) \text{ and for all } f : b \to c, A(f)(x) \notin A(b)\}.$$

It is clear that, in general, $\neg A \vee A = E$ need not hold.

Returning back to the special case of graphs: suppose we apply $\neg$ again to the result, yielding $\neg\neg H$. This will be the same as adding to H all the arrows of G that have their source and target in H. More explicitly, and again in general,

$$(\neg\neg A)(c) = \{x \mid x \in E(c) \text{ and } \forall f : b \to c, \exists g : a \to b \text{ such that } A(g)(A(f)(x)) \in A(a)\}.$$

This composite map $\neg \circ \neg$ acting on subobjects in fact forms an example of a Lawvere-Tierney topology on a topos, specifically the *double negation topology* $\neg\neg : \Omega \to \Omega$. As such, the map $\neg\neg$ acts as a *closure operator*. In our particular case of subgraphs $H \hookrightarrow G$, this amounts to adding to the arrow set of H all the arrows of G between those vertices that are in H. Thus, $\neg\neg H$ will add to H all the arrows of G that have source and target in H.[228]

In terms of other Lawvere-Tierney topologies that exist for our graphs, in addition to the double negation topology just described, there are also always two trivial topologies available, namely that given by the identity on Ω and that given by

$$\begin{array}{ccc} \Omega & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \top} \\ 1 & \xrightarrow[\chi_!]{} & \Omega \end{array}$$

the composite diagonal map $\top \circ ! : \Omega \to \Omega$. Finally, there is one last, nontrivial, topology on **Quiv**, specified as follows. First, note that condition (1) on a topology, that is, preservation of truth, requires specifically that $j(V) = V$ and $j(A) = A$. Considering the other objects, if we then set $j(0_V)$ to 0_V, we would just recover either the identity topology or the double negation topology (after making suitably constrained choices for the rest of the object assignments). On the other hand, if $j(0_V) = V$, then setting $j(0_A) = A$ generates the other trivial topology $j = \top \circ !$. Therefore, we will instead set $j(0_A) = p_A$; then condition (3) on a Lawvere-Tierney topology implies that $j(\top_s) = j(\top_t) = p_A$, since $j(\top_s) \wedge j(\top_t) = j(\top_s \wedge \top_t) = j(0_A) = p_A$. Thus, altogether we have the following assignment:

$$j(V) = V = j(0_V),$$

$$j(A) = A, j(0_A) = p_A = j(\top_s) = j(\top_t) = j(p_A).$$

This assignment describes a nontrivial topology distinct from the $\neg\neg$ topology. It is sometimes called the *closed topology* on the global element p_A, defined as $(- \vee p_A)$. In total, there are four topologies or local operators for the Ω we have been describing for quivers.

Let us now take the opportunity to introduce some important terminology. This is entirely general, but we will continue to refer to quivers to keep things concrete.

Definition 325 Let $u : H \hookrightarrow G$ be a subobject (subgraph), with characteristic map $\chi_u : G \to \Omega$. Then, given a topology j, we call the subobject $\overline{H}$ classified by $j \circ \chi_u$ ("composition with j") the *closure* of H in G (with respect to the topology j).

We also say that H is *dense* provided its closure is equal to G, that is, $\overline{H} = G$.

For instance, the closure with respect to the closed topology above adds to a subgraph H of G all the vertices of the graph. Composition of a characteristic map with the closed topology j adds all the nodes of G to the subgraph H. The dense subobjects (with respect to the closed topology) of G are those subgraphs that include all the arrows of G (and

228. In this connection, it turns out that for a presheaf topos $\mathbf{Set}^{\mathbf{C}^{op}}$, the double negation (Lawvere-Tierney) topology coincides with the dense (Grothendieck) topology we explored in chapter 10. An illuminating proof of this general fact can be found in Mac Lane and Moerdijk (1994, 273). This $\neg\neg$ topology is rather important, and we return briefly to it below.

note that there is a *minimal* such dense subobject, namely that given by the arrow set of G itself). This should be evident since composition of the characteristic map with j does not add arrows to H—since besides A itself, no arrow of Ω is sent to A by j. In short: to be dense with respect to this closed topology, a subgraph H needs to already include precisely all the arrows of G.

What about the $\neg\neg$-topology? Here, closure amounts to adding to a subgraph H all the arrows of G that have source and target in H; this should also make some sense, since the characteristic map sends arrows to p_A and then $\neg\neg$ must take the arrow p_A to A. The dense subobjects of G are given by the subgraphs that include all the vertices of G (and there is a minimal such dense subobject, given by the vertex set of G itself). As before, to be dense, a subgraph H must include precisely all the vertices of G.

We have almost arrived at the point of being able to define a general sheaf for such topologies. One last set of definitions is needed.

Definition 326 An object X of a topos $\mathscr{E}$ with topology j is called *j-separated* (or just *separated*, when j is understood) provided for every object Y, every j-dense subobject $u : S \hookrightarrow Y$, and every morphism $f : S \to X$, there exists at most one $g : Y \to X$ "factoring" f through u:

$$\begin{array}{ccc} S & & \\ {\scriptstyle u}\big\downarrow & \searrow^{f} & \\ Y & \xrightarrow[g]{} & X, \end{array}$$

that is, making the above diagram commute.

We will then call an object *j-complete* (or just *complete*, j understood) provided such a unique factorization always exists.

Equipped with these notions, we can at last offer a rather succinct and general definition of a sheaf with respect to a (Lawvere-Tierney) topology, that is, for an elementary site.

Definition 327 (*Definition of j-sheaf*) A *j-sheaf* is an object F of a topos $\mathscr{E}$ that is both j-separated and j-complete.

To unpack this a little: we could have also said that F is a sheaf provided, given a map from a dense subobject of E into F, as in

$$\begin{array}{ccc} A & \longrightarrow & F \\ {\scriptstyle dense}\big\downarrow & & \\ E, & & \end{array}$$

this can be uniquely extended to a map on all of E, as in

$$\begin{array}{ccc} A & \longrightarrow & F \\ {\scriptstyle dense}\big\downarrow & \nearrow_{!} & \\ E. & & \end{array}$$

F is thus a j-sheaf provided for every dense monomorphism $m : A \rightarrowtail E$, composition with m induces the isomorphism $\mathrm{Hom}_{\mathscr{E}}(E, F) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{E}}(A, F)$.

In terms of this map, we could also have defined an object G as *separated* if for each dense $A \rightarrowtail E$, $\text{Hom}_{\mathscr{E}}(E, G) \to \text{Hom}_{\mathscr{E}}(A, G)$ is a monomorphism (so that the extension above need not exist, but will be unique if it does).

We use $\mathbf{Sh}_j(\mathscr{E})$ to denote the (full) subcategory (of $\mathscr{E}$) consisting of j-sheaves of $\mathscr{E}$, and $\mathbf{Sep}_j(\mathscr{E})$ for the (full) subcategory consisting of the separated objects.

Given the correspondence between Grothendieck topologies on a category $\mathbf{C}$ and Lawvere-Tierney topologies on the associated presheaf topos $\mathbf{Set}^{\mathbf{C}^{op}}$, presented in proposition 321, the following result illustrating how all this relates to the earlier notions of Grothendieck toposes may have been expected. Given $\mathbf{C}$ a small category and j a Lawvere-Tierney topology on the presheaf topos $\mathbf{Set}^{\mathbf{C}^{op}}$, we can form the corresponding Grothendieck topology J on $\mathbf{C}$ (as indicated by 321), and with such a setup, we have the following result:

Proposition 328 A presheaf $P \in \mathbf{Set}^{\mathbf{C}^{op}}$ will be a sheaf (in the sense of definition 327) for the Lawvere-Tierney topology j iff P is a J-sheaf (in the sense of definition 256).[229]

Thus, the definition of a sheaf for a Grothendieck topology J emerges as a special case of the elementary axiomatic approach to sheaves for a Lawvere-Tierney topology. While this sort of result helps reinforce the importance of the presheaf category construction, there are toposes that are not Grothendieck toposes (as we have seen), so the Lawvere-Tierney notions are accordingly a proper generalization of the Grothendieck notions.

Example 329 Before moving on, let us briefly look a little closer at this rather abstract definition of a j-sheaf via our particular case of **Quiv**. For the closed topology, j-sheaves will just be graphs with a single vertex, their topos being equivalent to **Set**. Here, the only j-separated object (not also j-complete) is the empty graph.

With the $\neg\neg$-topology, however, the story unfolds differently. The objects of **Quiv** that are $\neg\neg$-separated will be precisely the graphs without parallel arrows, that is, graphs with at most one arrow between each pair of vertices; the $\neg\neg$-complete graphs, for their part, have at least one arrow between each pair of vertices, that is, they are the complete graphs. Finally, then, $\neg\neg$-sheaves are precisely the complete graphs equipped with self-loops (which also forms a topos equivalent to **Set**), that is, a quiver or directed graph X is such a sheaf precisely when there exists exactly one arrow between any pair of vertices. If we denote the full subcategory of $\neg\neg$-separated quivers by $\mathbf{Sep}_{\neg\neg}(\mathbf{Quiv})$, and the further full subcategory consisting of $\neg\neg$-sheaves by $\mathbf{Sh}_{\neg\neg}(\mathbf{Quiv})$, the category of *sheaves* here forms a topos, and a *Boolean* one at that; however, note that $\mathbf{Sep}_{\neg\neg}(\mathbf{Quiv})$ is not quite a topos![230]

In general, for any topos $\mathscr{E}$, the topos of sheaves for the double-negation topology is Boolean, where this means that the law of the excluded middle $\models \phi \vee \neg\phi$ holds. The law of the excluded middle does not hold in general in a topos; it may indeed hold for certain individual formulas ϕ (and even for arbitrary individuals), without this allowing us to infer

229. See Mac Lane and Moerdijk (1994, V.4) for a proof.

230. In fact, it is what is called a *quasitopos*. Instead of having the ordinary subobject classifier of a topos, a quasitopos has a classifer only for *certain* subobjects. Moreover, as one might anticipate given that the separated objects are such a quasitopos, in general a quasitopos satisfies the uniqueness part of the sheaf axioms, but not the existence part. nLab Authors (2017) has a nice discussion of some of the topos features of quivers that we have been exploring, as well as the quasitopos subcategories mentioned above.

that the corresponding universal generalization holds. Those toposes for which it does hold are called "Boolean toposes."[231]

11.3.1 A Glimpse into Topos Logic

Lawvere and Tierney first found that to each topos, one can associate operations that are analogous to the usual logical operations of conjunction, disjunction, negation, implication, and (universal and existential) quantification, defined in a topos $\mathscr{E}$ on its object of truth-values Ω, where we think of the global elements $1 \to \Omega$ as the truth-values, and where the morphisms analogous to the usual logical operations satisfy categorical versions of the laws of a Heyting algebra. Those morphisms induce operations on the collection $\mathrm{Hom}_{\mathscr{E}}(1, \Omega)$ of truth-values, making it into a Heyting algebra. A topos can accordingly be regarded as a category that "models" intuitionistic higher-order logic (or set theory).

Lawvere and Tierney's elementary axiomatization, whereby we may reinterpret the specification of formal covers in terms of what is fundamentally a modal operator on an object of generalized truth-values, further suggests a number of fascinating connections to logic, just a few of which are indicated very sketchily in this section (some taken up more earnestly in the final section of this chapter). To better contextualize these remarks, notice that, in particular, the subobject classifier object in any presheaf topos will support a bi-Heyting algebra structure. This further implies that any such topos will support an infinite hierarchy of intermediate modalities.[232] Moreover, for any sheaf F on a site $(\mathbf{C}, J)$, the lattice formed by all the subsheaves of F also forms a complete Heyting algebra. Another way of thinking about this is that a **cHa** can be realized as a subobject lattice in a Grothendieck topos.[233] These sorts of facts, and many others that we have not explored, suggest that a number of strong connections to logic should be lurking here.

One particular feature of interest is that the laws satisfied by the logical operations admitted in a topos in general—and in particular the general logical laws satisfied by all sheaves—correspond not to classical logic, but to intuitionistic logic.[234] While we can provide a topos-theoretic interpretation of the logical connectives, there are a few subtleties.

231. A nice illustration of the interest in such a distinction can be found in the context of smooth infinitesimal analysis (SIA), mentioned at the beginning of the chapter. SIA is essentially a topos-theoretic approach to analysis based on a new construction of the continuum and infinitesimals via a "smooth real line" object R on which various axioms are placed. For such an R, equipped with a notion of location and a relation "=" of identity or coincidence of locations (i.e., $a \neq b$ is read as "a and b are distinguishable as locations or points"), it is not assumed that the identity relation is *decidable*, where this means that for any a, b, either $a = b$ or $a \neq b$. In some models of SIA, such as *basic smooth infinitesimal analysis* (BSIA), the law of excluded middle is positively refutable. However, in most models of SIA more generally, the law of excluded middle will be true in the restricted sense, namely whenever α is a *closed* sentence (having no free variables), then indeed $\alpha \vee \neg\alpha$ will hold. As Bell notes: "Thus, in smooth infinitesimal analysis, the law of excluded middle fails 'just enough' for variables so as to ensure that all maps on R are continuous, but not so much as to affect the propositional logic of closed sentences" (Bell 2008, 106). This is rather interesting, since the so-called well-adapted models of SIA still do not allow us to go on to infer from the fact that the law of excluded middle applies to arbitrary individuals/points that the corresponding universal generalization—that is, *for all* x and *for all* y in R either $x = y$ or $x \neq y$—holds. Certain elements of R simply cannot always be distinguished, though it does also contain points that can be distinguished; however, on account of the existence of the former, as Bell notes, R *cannot* be thought of as the sum total of its elements.

232. See Reyes and Zolfaghari (1996) and Zalamea (2012).

233. See Mac Lane and Moerdijk (1994, III.8) for details on this.

234. This does not mean that all toposes have an internal logic than is nonclassical, as evidenced by the rather conspicuous case of the topos $\mathscr{E} = \mathbf{Set}$; also, the topos of monoid actions is classical whenever M is a group.

For instance, in order to fully incorporate set-theoretic constructions into the formal language, we need to specify the topos semantics for symbols like $\in$. In this same vein, recall that in topos theory in general there will be a difference between $x \in_{\mathbf{1}} A$ and $x \in_c A$. One basic solution is to use intuitionistic type theory, where different types are provided by the distinct objects of a topos.[235] Complementing the Mitchell-Bénabou syntactic language, Kripke-Joyal provide a well-known semantics for toposes, translating the logician's "forcing" statements about a topos $\mathscr{E}$ into ordinary or naive assertions about $\mathscr{E}$. It turns out that via this semantics, it can be seen that the general logical laws satisfied by all sheaves coincide with those of intuitionistic logic—that is, where laws like the excluded middle, $\phi \vee \neg\phi$, and the law of double negation, $\neg\neg\phi \to \phi$, fail.

The basic idea of the Kripke-Joyal semantics is roughly this: (1) an individual element b (a constant) of a given type X is just understood as a morphism $b : 1 \to X$; (2) a k-argument property P of type $X_1 \times \cdots \times X_k$ is to be understood as a subobject P of $X_1 \times \cdots \times X_k$; (3) a k-argument function f from $X_1 \times \cdots \times X_k$ to Y is understood to be a morphism f from $X_1 \times \cdots \times X_k$ to Y; and (4) a term $t(x_1, \ldots, x_k)$ with free variables in the $x_1, \ldots, x_k$ (ranging over the sorts $X_1 \times \cdots \times X_k$) is to be understood as a morphism from $X_1 \times \cdots \times X_k$ to some codomain object, enabling the *interpretation* of this morphism to be defined. In the special case where the topos is the topos of presheaves $\mathbf{Set}^{\mathbf{C}^{op}}$—that is, the category of sheaves on $\mathbf{C}$ for the trivial (Grothendieck) topology—the semantics can be described more evocatively, and in this context it can be shown that the logic of sheaves includes classical logic as a special case. When applied to Grothendieck toposes, the Kripke-Joyal semantics emerges as none other than a sheaf semantics. Moreover, all the *forced formulas* in all the sheaves about a fixed space X will constitute a logic intermediate between intuitionistic and classical, which depends exclusively on the topology of X. The intermediate logic associated to a space X can be seen as a multivalued logic with values in the Heyting algebra on X. It is possible to enhance the logic of sheaves, and thus the intuitionistic logic, with new connectives (including modal operators) not expressible in terms of the usual ones.

There are many fascinating connections to logic that we might naturally further explore at this stage. But a proper development of the formal details would take further chapters of their own. Instead, we will just highlight certain high-level features of this story, leaving the reader to track down logical matters on their own.[236] Philosophically, that intuitionistic logic may be regarded as the "default" logic of sheaves in general can be seen as inviting perspectives that better attune with more continuous logics adequate to *extended things*, where whenever properties hold at a point of its domain of extension, they are expected to hold nearby that point. In this setting, what emerges is a strong emphasis on neighborhood relations over points or punctual properties, in which context classical logic can be seen as a limit or extreme case in a much vaster universe of more "relaxed" logics.[237]

Related to this is the appreciation of the fact that points are simply ideal limits in a more fundamental logic of neighborhoods. The intimate connection between intuitionistic logic and sheaves thus provides another important perspective on the subtle connections

235. For details on the Mitchell-Bénabou internal language of a topos, see Mac Lane and Moerdijk (1994).

236. There is already plenty to work with in Mac Lane and Moerdijk (1994) and Goldblatt (2006).

237. Caicedo (1995) takes up some of the ideas vaguely alluded to in this paragraph.

between continuity, generality, and the failure of the excluded middle, connections already suggested many years ago by Charles Peirce in other contexts:

> If we are to accept the common sense idea of continuity (after correcting its vagueness and fixing it to mean something) we must either say that a continuous line contains no points or we must say that the principle of excluded middle does not hold of these points. The principle of excluded middle only applies to an individual (for it is not true that "Any man is wise" nor that "Any man is not wise"). But places, being mere possibles without actual existence, are not individuals. Hence a point or indivisible place really does not exist unless there actually be something there to mark it, which, if there is, interrupts the continuity. (Peirce 1997, 6.168, marginal note)
> The *general* might be defined as that to which the principle of excluded middle does not apply. A triangle in general is not isosceles nor equilateral; nor is a triangle in general scalene. (Peirce 1997, 5.505)

Peirce's intuition that what makes something general is ultimately due to the failure of applicability of the law of the excluded middle, together with his repeated insistence that "continuity" (in some very broad but demanding philosophico-mathematical sense) is ultimately a form of generality, is sharpened in the close connection that emerges between sheaf logic (as the continuous logic of local validity) and intuitionistic logic (in which the law of the excluded middle is dropped).

11.4 Morphisms of Toposes

We have been exploring toposes—both elementary toposes and Grothendieck toposes—and we have begun to see some of the power of these concepts. However, we have not yet defined an appropriate notion of *morphisms of toposes*, so we have not really begun to tap the full potential of the topos construction. In these final two sections, we look closely at these notions and consider a few interesting examples.

11.4.1 Geometric Morphisms Defined

One might suspect that the notion of a morphism of toposes could be captured by a functor that preserves finite limits, exponentials, and the subobject classifier. This indeed defines a legitimate functor between toposes, namely a so-called *logical functor*. Such functors can play an important role in the theory—in particular, from the perspective of an elementary topos as the syntactic category of a higher-order (intuitionistic) type theory. Logical morphisms are those functors that preserve all the structure of the topos (that is, the finite limits and colimits, exponentials and subobject classifier; also "inherited" structure like the Heyting algebra structure of Ω, the validity of formulas, and so on). However, there is another natural type of morphism to consider between toposes: *geometric morphisms*. Viewed in a certain light, the notion of a geometric morphism can even be regarded as the more pertinent of the two sorts of morphisms. In the context of the overall perspective that regards toposes as "generalized spaces," geometric morphisms can be regarded as the corresponding "generalized continuous maps." As such, this sort of morphism can initially be thought of as preserving the geometric structure of toposes (compared to how logical morphisms can be thought of as preserving the elementary logical structure).

Formally, the definition of geometric morphisms between toposes just uses the concept of adjunctions and is rooted in the example of sheaves on topological spaces. We saw earlier how a continuous function $f : X \to Y$ between topological spaces induces a pair of functors—the inverse image functor f^* and the direct image functor f_*. These are such that, in fact, $f^* \dashv f_*$

$$\mathbf{Sh}(X) \underset{f_*}{\overset{f^*}{\underset{\bot}{\rightleftarrows}}} \mathbf{Sh}(Y).$$

As a left adjoint, f^* clearly preserves (finite) colimits. But, actually, consideration of the definition of the inverse image functor f^* reveals that f^*, while not generally being a logical morphism of toposes (preserving *all* structure), does also preserve *finite limits* (pullbacks and the terminal object), that is, it is *left exact*.[238] This particular situation—where a morphism of toposes is not quite logical (preserving all structure), and yet, in addition to preserving colimits (by virtue of being a left adjoint), certain (finite) limits are preserved—motivates the following definition of a different sort of morphism.

Definition 330 A *geometric morphism* $f : \mathscr{F} \to \mathscr{E}$ between toposes consists of a pair of functors $f^* : \mathscr{E} \to \mathscr{F}$ and $f_* : \mathscr{F} \to \mathscr{E}$ such that $f^* \dashv f_*$ and also f^* is left exact. Modeled on the particular situation described above, f_* is generally referred to as the *direct image* part of f, and f^* the *inverse image* part of the geometric morphism.

If we have two geometric morphisms $f, g : \mathscr{F} \to \mathscr{E}$, a natural transformation

$$\mathscr{F} \underset{g}{\overset{f}{\rightrightarrows}} \mathscr{E} \qquad \Downarrow \eta$$

is given by a natural transformation $f^* \Rightarrow g^*$ between the inverse image parts (or equivalently, by adjunction, we could define this in terms of a map $g_* \Rightarrow f_*$ between direct image parts). Toposes together with geometric morphisms and the natural transformations between them forms a 2-category, where the objects are toposes, the 1-cells are the geometric morphisms, and the 2-cells are natural transformations as specified above; but we will not make much use of this formulation.

Finally, a geometric morphism $f : \mathscr{F} \to \mathscr{E}$ is said to be *essential* if the inverse image part f^* also has a left adjoint (in addition to having a right adjoint in f_*, which it does have by virtue of being a geometric morphism), usually denoted $f_!$.

$$\mathscr{F} \xleftarrow{\;f^*\;} \mathscr{E}, \qquad f_! \dashv f^* \dashv f_* \quad (f_!, f_* : \mathscr{F} \to \mathscr{E})$$

Note that the exactness property of f^* will automatically be satisfied whenever there is such a further left adjoint, since this makes f^* a right adjoint, and we know from RAPL (proposition 175) that right adjoints preserve limits.[239]

238. A proper proof of this fact can be found in Borceux (1994).

239. Following Grothendieck, the asterisk notation is meant to suggest functors that exist for every f, while the exclamation point is meant to suggest functors that exist only for special sorts of f. Moreover, for either of these, the subscript position is meant to indicate functors having the same direction as f, while the superscript position denotes functors going in the opposite direction of f.

We first look at a number of important but rather abstract examples of this notion. In the section that follows, we will look at some more concrete illustrations.

Example 331 We already saw, in motivating the definition of geometric morphisms, how any continuous map $f : X \to Y$ of topological spaces induces a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ between the sheaves on the spaces. The direct image $\mathbf{Sh}(f)_*$ (or just f_*) is given, for each sheaf F on X, by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for each $V \in \mathscr{O}(Y)$. The inverse image $\mathbf{Sh}(f)^*$ (or just f^*), for its part, will act on étalé bundles over Y, sending such a bundle $p : E \to Y$ to the étalé bundle over X (by "pulling back" p along $f : X \to Y$).[240]

Example 332 Generalizing the previous example, any morphism of sites $f : (\mathbf{C}, J) \to (\mathbf{D}, J')$ induces a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(\mathbf{D}, J') \to \mathbf{Sh}(\mathbf{C}, J)$, with direct image the functor

$$- \circ f^{op} : \mathbf{Sh}(\mathbf{D}, J') \to \mathbf{Sh}(\mathbf{C}, J).$$

Example 333 Any functor $F : \mathbf{C} \to \mathbf{D}$ between small categories induces a geometric morphism (actually, an *essential* geometric morphism) between the associated presheaf categories

$$\mathbf{Set}^{\mathbf{C}^{op}} \underset{F_*}{\overset{F_!}{\underset{\perp}{\overset{\perp}{\xleftarrow{\;F^*\;}}}}} \mathbf{Set}^{\mathbf{D}^{op}}.$$

If $\mathbf{C}$ and $\mathbf{D}$ are both categories with finite limits, and $F : \mathbf{C} \to \mathbf{D}$ is left exact, then in the same diagram of adjoint functors, $F_!$ will also be left exact.

Example 334 For $(\mathbf{C}, J)$ a site, the inclusion functor ι of $\mathbf{Sh}(\mathbf{C}, J)$ into $\mathbf{Set}^{\mathbf{C}^{op}}$ is the direct image of a geometric morphism, with the inverse image given by the associated sheafification functor

$$\mathbf{Sh}(\mathbf{C}, J) \underset{\iota}{\overset{\mathbf{a}}{\rightleftarrows}} \mathbf{PreSh}(\mathbf{C}),$$

as described in theorem 291 (chapter 10).

Example 335 Another important example of a geometric morphism, one that we will explore in greater detail in the coming section, is given by the pair (Δ, Γ)

$$\mathbf{Set}^{\mathbf{C}^{op}} \underset{\Gamma}{\overset{\Delta}{\rightleftarrows}} \mathbf{Set},$$

where Δ is the *constant* (or *discrete*) functor, and Γ is the *points* (or *global sections*) functor.

In more detail: recall the *constant* presheaf functor $\Delta : \mathbf{Set} \to \mathbf{Set}^{\mathbf{C}^{op}}$ that, in the above case, will send each set $S \in \mathbf{Set}$ to the constant S itself, and each arrow to the identity morphism on that object, that is, $(\Delta S)(c) = S$ and $(\Delta S)(f) = \mathrm{id}_S$. We can then of course

240. Actually, with a fairly weak assumption on the space Y (namely, that the space is "sober"), via this construction we can actually show a *bijection* between the continuous maps from X to Y and the isomorphism classes of geometric morphisms from the topos $\mathbf{Sh}(X)$ to the topos $\mathbf{Sh}(Y)$. See Johnstone (1986).

consider morphisms between such a constant presheaf and some other presheaf, that is,

$$\mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{op}}}(\Delta S, P).$$

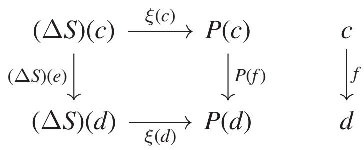
But recall that a natural transformation $\Delta S \to P$ will just be a *cone* from the set S to the functor P. In more detail: if we consider an arbitrary presheaf $P : \mathbf{C}^{op} \to \mathbf{Set}$ and for $c \in \mathbf{C}$, the arrows (which are in fact natural transformations)

$$\Delta S \longrightarrow P,$$

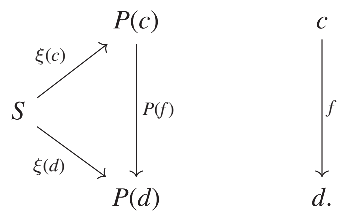
we get that a typical arrow in $\mathbf{Set}^{\mathbf{C}^{op}}$ corresponding to these arrows is just a natural transformation, that is, a family of arrows of $\mathbf{C}$,

$$(\Delta S)(c) \xrightarrow{\xi(c)} P(c),$$

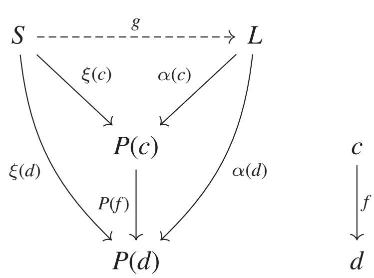
indexed by the various objects or nodes of $\mathbf{C}$ and such that



commutes for each such edge $f : c \to d$ in $\mathbf{C}$. But when we apply the functor Δ, this commutative square collapses to the commutative triangle



The definition guarantees that whenever the indexing category has composable edges, the corresponding composite triangles commute. The natural transformation $\xi : S \to P$ represented by the triangle gives a cone over P with summit vertex S. Recall also that the *limit* of P is then defined in terms of a universal cone, where a cone $\alpha : L \to P$ with vertex L is universal with respect to P when for every cone $S \to P$, there is a unique map $g : S \to L$ making



commute. In such a case, one usually refers to the universal cone by just the vertex $L = \varprojlim P$, and calls this the limit of P. Moreover, if every P has a limit in this sense, then the functor Δ has a right adjoint given by the limit.

Suppose, then, that there exists a functor Γ right adjoint to Δ, that is, such that $\Delta \dashv \Gamma$. Then, provided it exists, for each S and P, there will be a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{op}}}(\Delta(S), P) \cong \mathrm{Hom}_{\mathbf{Set}}(S, \Gamma(P)).$$

In particular, for $S = \mathbf{1}$, the terminal presheaf (using bold font here to disambiguate from the terminal object 1 of **Set**), we would need

$$\mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{op}}}(\mathbf{1}, P) \cong \mathrm{Hom}_{\mathbf{Set}}(S, \Gamma(P))$$

associating to each $\mathbf{1} \xrightarrow{\xi} P$ in $\mathbf{Set}^{\mathbf{C}^{op}}$ an element ξ of $\Gamma(P)$ in **Set**. But, on objects, this forces the definition of Γ to be: $\Gamma(P) = \mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, P)$; it is also forced on morphisms. Namely, it will be a function γ that assigns to each object c of $\mathbf{C}$ an element $\gamma_c \in P(c)$ in such a way that $\gamma_c|f = \gamma_d$ for $f : d \to c$ holds for every f in $\mathbf{C}$. This makes γ a natural transformation $\gamma : \mathbf{1} \to P$, where $\mathbf{1}$ is just the constant functor on $\mathbf{C}^{op}$! By taking the set $\Gamma(P)$ to consist of all such γ for P, we have constructed the functor $\Gamma : \mathbf{Set}^{\mathbf{C}^{op}} \to \mathbf{Set}$. In this case, for reasons that should be evident, such γ are called *global sections* or *points*, and Γ is the *global sections* or *points* functor. (Moreover, $\Gamma(P)$ is just the limit $\varprojlim P$.)

It is easy to verify that indeed, under such definitions, $\Delta \dashv \Gamma$. Let us sketch how we can appreciate that Δ is, moreover, left exact. To do this, we need to show that Δ preserves pullbacks and the terminal object. But given a pullback

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ Q & \longrightarrow & R \end{array}$$

in **Set**, we need to show that

$$\begin{array}{ccc} \Delta S & \longrightarrow & \Delta T \\ \downarrow & & \downarrow \\ \Delta Q & \longrightarrow & \Delta R \end{array}$$

is a pullback in $\mathbf{Set}^{\mathbf{C}^{op}}$. But this is the same as showing that the diagram on objects is a pullback in **Set**; and such a diagram reduces back to the original diagram, on account of how Δ is defined. That Δ preserves the terminal object $\Delta 1 = \mathbf{1}$ is basically immediate.

This sketch suffices to show that the pair (Δ, Γ) forms a geometric morphism.

In fact, we can show that this is really the *only* geometric morphism from $\mathbf{Set}^{\mathbf{C}^{op}}$ to **Set**, revealing **Set** to be a sort of terminal object in the category of presheaf toposes with geometric morphisms between them. Suppose we have an arbitrary geometric morphism (p^*, p_*). Take S a set. Any set can be written $S = \coprod_{s \in S}\{s\}$. As a left adjoint, p^* preserves colimits, so

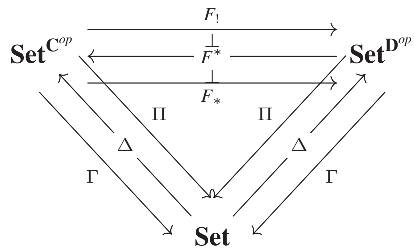
$$p^*(S) = p^* \coprod_{s \in S}\{s\}$$

and the entity on the right is isomorphic to $p^* \coprod_{s \in S} 1$. As a geometric morphism, p^* must in particular preserve the terminal object, so

$$\coprod_{s \in S} p^*(1) = \coprod_{s \in S} \mathbf{1} = \Delta(S).$$

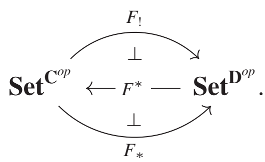
Thus, we have shown that p^* is isomorphic to Δ. And since adjoints are unique, we must likewise have that $p_* \cong \Gamma$.

In the next section, we will see some particular examples of this geometric morphism, in the context of presheaf toposes that we are already familiar with. We will also see that there is actually a further functor Π left adjoint to Δ, making (Δ, Γ) an essential geometric morphism. In fact, (Δ, Γ) such that $\Pi \dashv \Delta \dashv \Gamma$ is a special case of example 333, taking $\mathbf{D} = \mathbf{1}$. In this connection, it is moreover straightforward to show how from a functor $F : \mathbf{C} \to \mathbf{D}$, the induced diagram of geometric morphisms

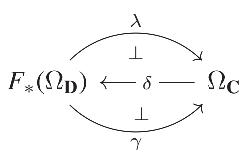


in fact commutes, where this means $F^* \circ \Delta = \Delta$, $\Gamma \circ F_* = \Gamma$, and $\Pi \circ F_! = \Pi$, in effect showing that the pair $(F^* \circ \Delta, \Gamma \circ F_*)$ is itself a geometric morphism.[241]

Example 336 Recall from example 333 how, given a functor $F : \mathbf{C} \to \mathbf{D}$ between two small categories, this functor gives rise to a diagram



But as presheaf toposes, we can then ask whether and how the subobject objects Ω in these presheaf toposes compare. In fact, the functors $F_! \dashv F^* \dashv F_*$ will induce a further diagram of "internally adjoint" morphisms



addressing this question.[242]

Moreover, we can then define two further operators $\Diamond, \Box : F_*(\Omega_{\mathbf{C}}) \to F_*(\Omega_{\mathbf{C}})$ by

$$\Diamond = \delta \circ \lambda, \Box = \delta \circ \gamma.$$

It can be shown that these satisfy that $\Diamond \dashv \Box$, and

1. $\Box \leq \text{id} \leq \Diamond$,
2. $\Box\Box = \Box, \Diamond\Diamond = \Diamond$,

241. Reyes, Reyes, and Zolfaghari (2008) contains a demonstration and discussion of this.
242. See Reyes, Reyes, and Zolfaghari (2008, chap.14) for details on the definitions of the adjoints in question.

revealing how we can construct modal operators within topos theory. Defined in this way, it has been suggested that to get the theory of modalities off the ground one might start by thinking of $\mathbf{Set}^{\mathbf{D}^{op}}$ as providing the background standard of constancy, taking $\mathbf{Set}^{\mathbf{C}^{op}}$ as the locus of change and modal qualifications.[243]

11.5 Toward Cohesive Toposes

In physics and other fields like materials science and even architecture, one will occasionally hear the tendency or force holding certain types of matter together referred to as their *cohesion*. The study of the various properties or qualitative features of material bodies, in their mutual interactions, reveals that bodies can have different sorts or amounts of resilience, elasticity, brittleness, bounciness, stiffness, flappiness, twistability, capability of creasing, "creep" (e.g., the distortion of a bow under prolonged loading; creep in textiles is why the knees of pants get baggy), and so on.

In a different context, the philosopher Hegel sought to understand, in a very general fashion, what makes determinate things *determinate*. In his "philosophy of nature," this amounted to inquiries into the main "general forms" describing the distinctive ways material bodies in particular can be seen to become determinate. Among such general forms was what Hegel called their *cohesion*. Hegel thought that, over and above the usual mechanisms of dynamic forces of attraction and repulsion, gravitational attraction, and chemical reactions, there are various ways that certain material bodies can be found to achieve determinacy essentially by "holding together" in a distinctive way. Hegel accordingly sought to advance a more refined concept of cohesion, as opposed to what he called "the common understanding of cohesion," which just refers to a particular snapshot of the "quantitative strength of connection between the parts of a body."[244] For Hegel, cohesion is a more fundamental phenomenon, even more fundamental than shape or density of a body, in that it does not even dictate any particular shape—and even if cohesiveness is fundamentally about the relations between the parts of a spatially-extended body, this is not something the analysis of which could be reduced to uniform geometrical considerations of the "mere outer shape" of a body. While in certain instances, cohesion will involve how a body retains a specific spatial configuration or shape, in general cohesiveness can be considered independently of shape. On this approach, cohesiveness is something like the most general form of a number of more particular processes and phenomena found throughout nature, all involving some "inward determination" of a material body that reveals itself only in physical interactions with other material bodies, in which interactions one can find a distinctive way of resisting being broken apart by forces of another body.

In exploring sheaves from various angles, and at various levels of generality, we have in a sense been exploring various sophisticated modifications of our notions of space, and making use of various "generalized spaces." In the (pre)sheaf context, the domain category from which we map our shapes of a certain figure has been seen to supply the form or blueprint according to which certain parts of a certain type are to hold together. How do the parts or points of any space hang together or cohere? Intuitively, certain toposes can

243. Details on this, and the perspective just mentioned, can be found in Reyes and Zawadowski (1993) and Reyes and Zolfaghari (1991).

244. Georg Wilhelm Fredrich Hegel (1991, 241).

appear to have a rather involved cohesiveness, while others (like **Set**) can appear to be almost devoid of cohesiveness. Informally, what different "spatial" categories seem to have in common is something like this feature of *cohesiveness*, which they may have in different ways and to differing degrees. Certain maps between toposes may then function to reveal comparisons in different degrees or types of cohesiveness.

The notion of a *cohesive topos* (or, sometimes, a *category of cohesion*)—first explored by Lawvere—basically emerged in an attempt to axiomatize those intuitively "cohesive" properties of a topos that would make it a setting in which some sort of generalized geometry (wherein familiar notions of geometry resurface in subsumed form) could take place. While this effort is still somewhat in its infancy, the reader may enjoy exploring some of these otherwise advanced notions through a slew of examples. In terms of the formalism, all one really needs to approach this is a good working sense of adjunctions and a basic understanding of toposes. With this, some accessible motivating examples should be enough to give the reader a small glimpse into this fascinating and admittedly somewhat mysterious new area of research.

11.5.1 Foray into Cohesive Toposes

We have been using the word "geometry" a lot recently, while suggesting that we intend this in a rather general way. But what sorts of things do we meet in our usual geometries? Among other things, a very primitive notion is that of a *point*. Euclid, for instance, thought it was so important that he included a definition of it as the very first sentence of the first part of his *Elements*. In category theory, we see points in terms of the trivial *terminal* category (single object and single morphism), and functors from such a category are used to pick out the objects or "elements" of any category.

Suppose you have a certain topos $\mathscr{M}$ that supports some degree of cohesion or activity (e.g., dynamical systems, i.e., a particular category of presheaves).[245] Suppose you have another topos $\mathscr{K}$ seemingly devoid of any internal cohesion and variation (e.g., the usual category of sets). We can similarly define a *point* of $\mathcal{M}$ as a geometric morphism

$$\mathcal{K} \xrightarrow{p} \mathcal{M}.$$

This terminology is entirely sensible, especially if we regard $\mathcal{M}$ as some presheaf topos, and $\mathcal{K}$ as **Set**, for **Set** acts as a sort of terminal object in the category of presheaves. Recall from the definition of a geometric morphism that such a p is then really a pair $p = (p^*, p_*)$

$$\mathbf{Set} \underset{p_*}{\overset{p^*}{\underset{\perp}{\leftrightarrows}}} \mathbf{Set}^{\mathbf{C}^{op}}$$

with $p^* \dashv p_*$ and p^* left exact (i.e., preserving finite limits: preserving the terminal object **1** and pullbacks). Recall also that given two points p and q, a morphism from p to q will just be given by a natural transformation $p^* \Rightarrow q^*$ (and these will be bijective with those transformations between q_* and p_*). Thus, together with this notion of morphism, the points of $\mathbf{Set}^{\mathbf{C}^{op}}$ actually form a category $\mathbf{Pts}(\mathbf{Set}^{\mathbf{C}^{op}})$.

245. The ideas discussed here began with Lawvere. See, for instance, the engaging papers Lawvere (1994a, 1994b, 1996).

One can show that, for instance, **Pts**(**Set**) is just **1**, while **Pts**(**Grph**) is just the diagram with two vertices and the two nontrivial morphisms between them. One can also show that, for instance, **Pts**($\mathbf{Set}^{\mathbf{FinSet}^{op}}$) is isomorphic to **BoolAlg**, the category of Boolean algebras.

Example 337 Let's look closer at a rather simple example of this, namely **Pts**(**Set**) = **1**. Then

$$\mathbf{Set} \underset{p_*}{\overset{p^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Set}^{\mathbf{1}^{op}} \simeq \mathbf{Set}.$$

As a geometric morphism, p^* must be exact. So we will have

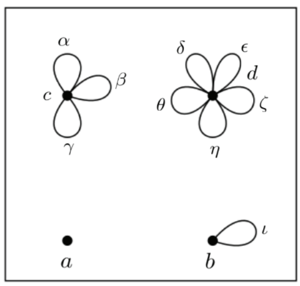
$$p^*(A) = p^*\left(\coprod_{a\in A} 1\right) = \coprod_{a\in A} p^*(1) = \coprod_{a\in A} \mathbf{1} = A.$$

But then the adjoint relation $p^* \dashv p_*$ entails that $p^* = \mathrm{id} = p_*$, thus showing that the points of **Set** are just **1** itself.

Moreover, since we saw in the previous section that (Δ, Γ) was the only geometric morphism from **Set** to the presheaf category on **C**, the above just tells us that for $\mathbf{Set}^{\mathbf{1}^{op}} \simeq \mathbf{Set}$, $\Delta = \mathrm{id} = \Gamma$.

For concreteness, let us look at how Δ and Γ work in more particular cases.

Suppose you have the bouquet X



This is, of course, an object in $\mathbf{Set}^{\mathbb{B}^{op}}$ (which is the same as **Bouq**). What is ΓX? This will be the set whose elements are given by the loops of X, that is, $\Gamma(X) = \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota\}$. If we had instead taken Y a trivial sort of bouquet, consisting of just vertices and no loops, then $\Gamma(Y)$ would just be the empty set. What about Δ? This functor goes from **Set** to $\mathbf{Set}^{\mathbb{B}^{op}} \simeq \mathbf{Bouq}$, so we have to consider how it acts on sets. Suppose S is a four-element set. Δ preserves the terminal object and colimits, so $\Delta(1+1+1+1) = \Delta(1) + \Delta(1) + \Delta(1) + \Delta(1) = \mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$, with $\mathbf{1}$ the terminal object of bouquets (the trivial bouquet). But this means that $\Delta(S)$ should just be the bouquet that consists of four vertices, each with a single loop. The adjunction (Δ, Γ) then tells us that it will be the same to look at the functions from a set S to ΓX as it will be to look at the bouquet homomorphisms between $\Delta(S)$ and a bouquet X.

If we instead took **Grph** or $\mathbf{Set}^{\mathscr{G}^{op}}$, Δ will take a set to the graph with as many directed loops (arrows with source and target the same) as there are elements in the set, while Γ will take a graph to the set having for elements the directed loops of the graph.

Example 338 Recall our earlier discussion of the *qua* category from examples 38 (chapter 2) and 196 (chapter 7). There, an interpretation X of $(\mathbf{A}, \mathscr{P})$ was just an object of the presheaf topos $\mathbf{Set}^{\mathbf{A}^{op}}$ together with a set of subobjects that correspond to the predicates in $\mathscr{P}$. One can see that the "global aspect" G is just the terminal object of $\mathbf{A}$ and we obtain the restriction of the functor X to this aspect G by applying the *global sections* functor Γ of the geometric morphism given by the pair (Δ, Γ):

$$\mathbf{Set}^{\mathbf{A}^{op}} \underset{\Gamma}{\overset{\Delta}{\underset{\perp}{\leftrightarrows}}} \mathbf{Set}.$$

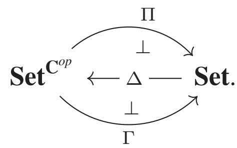
Because $\mathbf{A}$ has a terminal object, it follows that $\Gamma\Delta = \text{id}$, and that Δ is full and faithful. Moreover, since an interpretation assigns the same set to all the aspects, in the above light, this is just to say that the interpretation X is a constant presheaf. Putting these two things together, we can see that every morphism between two constant presheaves (different interpretations) will be a constant morphism. The functor Γ above basically *forgets* the aspects of X, returning just $\Gamma(X) = X(G)$ or the set of individuals (of the relevant kind). Moreover, for a predicate ϕ of X, the functor Γ, acting on ϕ, returns just a set-theoretic predicate, or subset, of $\Gamma(X)$, where this corresponds to the restriction of ϕ to the level of the global aspect. Altogether, interpretations X appear as constant objects in the presheaf topos $\mathbf{Set}^{\mathbf{A}^{op}}$ or as sets of individuals living in **Set**. These basically amount to two different points of view on X, where the constant objects of $\mathbf{Set}^{\mathbf{A}^{op}}$ have a rich structure, while the logic of **Set** is Boolean. The global judgment is carried out in **Set**, on the right of the above diagram, where Boolean logic is the rule. On the left, we have the richer logic supplied by the category that interprets the *qua* subcategory **A**.[246]

We have been exploring a little of the presheaf topos $\mathbf{Set}^{\mathbf{A}^{op}}$, for which every object X in this topos has a bi-Heyting structure. We also saw earlier how to interpret possibility and necessity operators in this context. But the logic of global judgments occurs in the Boolean setting of **Set**. Γ acts to take an X to **Set** where the predicates of $\Gamma(X)$ have the structure $(\mathscr{P}(\Gamma(X)), 0, 1, \wedge, \vee, c)$, where $c = \sim = \neg$ (c being the complement in **Set**), and where necessity and possibility operators reduce to the identity. While Γ preserves the action of $0, 1, \wedge, \vee, \sim$, and $\Box$, it does not preserve $\neg$ or $\Diamond$. In particular, $\Gamma\neg \neq c$, yielding a new operation $\Gamma\neg$ on $\mathscr{P}(\Gamma(X))$ which is not Boolean, and which acts as a kind of strong negation. As for $\Diamond$: a predicate may hold under some aspect, without being true at the global level. Thus, $\Gamma\Diamond$ also supplies yet another new operation on $\mathscr{P}(\Gamma(X))$. Both $\Gamma\neg$ and $\Gamma\Diamond$, mirroring their corresponding operations in $\mathbf{Set}^{\mathbf{A}^{op}}$, enrich the structure discussed, to give $(\mathscr{P}(\Gamma(X)), 0, 1, \wedge, \vee, c, \Gamma\neg, \Gamma\Diamond)$, a structure that, Reyes and Zolfaghari (1996) has suggested, provides a useful setting for global judgments in such situations. On the other hand, the rich general setting of $\mathbf{Set}^{\mathbf{A}^{op}}$ further suggests how we might model the patching together of local judgments regarding the applicability of certain predicates to a person *qua* the different hats they wear, or roles they play in life.

More generally, one can think of the result of applying the constant functor Δ as producing a subcategory of discrete spaces. In other words, Δ shows us, within the presheaf category (characterized by some sort of cohesiveness), what a space of *no cohesion* looks like. How do we be more formal about this?

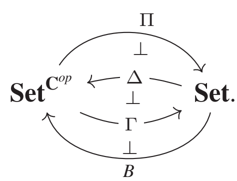
246. The topos-theoretic features of this example are explored in further detail in Reyes and Zolfaghari (1996).

Recall that if there is a further left adjoint to p^*, then we call the geometric morphism p *essential*. The geometric morphism (Δ, Γ) will be essential provided there exists a further functor Π such that $\Pi \dashv \Delta \dashv \Gamma$. But for presheaf toposes, as we have been exploring, there is precisely such a functor Π, sometimes called the *connected components* functor. For any presheaf topos, we have



Recall that Γ was actually the *limit* of a functor. The above is in fact a restatement of a rather elementary result which has $\Pi = \text{colim}$ and $\Gamma = \lim$. The definition of Π is forced by the fact that, as a left adjoint, it must preserve colimits (the gluings). By considering particular categories, such as the category of graphs, it is easy to see how Π just picks out the connected components. In other contexts, this functor can also be thought of as assigning to each object in the cohesive topos $\mathscr{M}$ the cardinal representing the number of maps it supports into discrete sets.

We mentioned a moment ago that Δ in such a setup shows us what an extreme space of no cohesion looks like. This suggests that perhaps there exists a further functor from $\mathscr{K}$ back to $\mathscr{M}$ that—at the other extreme of the result of applying the discrete functor—would yield a space of total (or infinite or trivial) cohesion. As it turns out, in certain cases, there does indeed exist such a functor, where this is right adjoint to Γ, as in



In these settings, a functor such as $B : \mathscr{K} \to \mathscr{M}$ is sometimes called the *chaotic* or *codiscrete* functor. The chaotic space produced by applying such a functor can be regarded as so "extremely cohesive" that, in moving a point to any other point, one need not concern oneself with the constraints put on the category by how the motion is parameterized (or the cohesion determined).[247] Following a suggestion of Lawvere, one might say that points in a discrete space are *distinct*, but points in a chaotic space are *indistinguishable* provided the chaotic spaces are connected.[248]

One can also show that if such a B exists, then the unit $\text{id} \to \Gamma\Delta$ of the adjunction $\Delta \dashv \Gamma$ is an isomorphism, and the counit $\Gamma B \to \text{id}$ of the adjunction $\Gamma \dashv B$ is also an isomorphism.[249]

For concreteness, when $B : \mathbf{Set} \to \mathbf{rGrph}$, landing in the category of reflexive graphs, $B(S)$ will yield a vertex for each of the elements of S and for each pair (x, y) of elements of

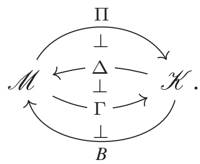
247. It is a fact that the functor Γ has a right adjoint B iff every representable (or generic figure) has a point.

248. See Lawvere (1994a).

249. A proof can be found in Reyes, Reyes, and Zolfaghari (2008). This reference also discusses further important consequences of this, namely the fact that the functor B will exist when the presheaf category satisfies a certain condition (that every nonempty presheaf P has a point $\mathbf{1} \xrightarrow{p} P$).

the set S, the functor will yield a unique arrow with source x and target y. In other words, $B(S)$ just describes the (directed) *complete graph* on the elements of the set S. In particular, suppose we have the set S consisting of four elements, which we can represent as a bag of four dots. Then $B(S)$ will be the graph with two arrows between each pair of vertices (one coming in and one going out), plus a single directed loop stationed at each of the four vertices.

Thus, more generally—that is, for certain special toposes $\mathscr{M}$ and $\mathscr{K}$—we can construct the adjoint quadruple diagram:



For an $\mathscr{M}$ where variation/dynamics is more relevant than cohesion, such as for the presheaf topos of dynamical systems (evolutive sets), the same adjunctions are sometimes called orbits $\dashv$ stationary points $\dashv$ equilibria $\dashv$ chaotic.

In the examination of such situations and what happens when we try to further compose such adjunctions, Lawvere noticed some curious things, which we look at more closely in the next example.

Example 339 Restricting our attention just to the adjoint functors $\Delta \dashv \Gamma \dashv B$ (defined as in the preceding example), we note that we can consider maps from the composite $\Delta\Gamma(M)$ to some M in $\mathscr{M}$ as well as maps from M to the composite $B\Gamma(M)$, that is,

$$\Delta\Gamma(M) \longrightarrow M \longrightarrow B\Gamma(M).$$

The space $\Delta\Gamma(M)$ on the left can be thought of as the closest approximation to M from the left given only its cardinality (points), while the space $B\Gamma(M)$ can similarly be thought of as its approximation from the right. For each object in the domain of the points functor, these maps provide an interval between which the object must lie, the endpoints being the two opposite subcategories, an interval that is in some sense relative to what the specific points functor does to the object on which it acts. Now, if we apply the points functor Γ again, we get a sequence of isomorphisms (in $\mathscr{K}$)

$$\Gamma\Delta\Gamma(M) \xrightarrow{\cong} \Gamma(M) \xrightarrow{\cong} \Gamma B\Gamma(M).$$

But this just says that even though the two composite maps are in general not isomorphisms in $\mathscr{M}$, applying the points functor yields an isomorphism of cardinals (in $\mathscr{K}$). The cardinal $\Gamma(M)$ or points(M) associated to a given M is at once isomorphic to the cardinal associated to the space $\Delta\Gamma(M)$ and to the cardinal associated to the space $B\Gamma(M)$. However, in the case of $\Delta\Gamma(M)$, all the points will be distinct, while in $B\Gamma(M)$ all points will be indistinguishable. In other words, we may have a definite number of points; however, via the unifying isomorphism such points will be *indistinguishable by any property*.

Lawvere (1994a) remarks how this curious situation appears to precisely capture the apparent paradox (first isolated by Cantor) that in an abstract set, all elements are distinct yet indistinguishable.[250]

The basic idea, then, is that $\mathscr{M}$ contains two opposed subcategories (the discrete and codiscrete objects) that, though inherently rather distinct, are rendered identical through the category $\mathscr{K}$. In more detail, the "unity of opposites" is precisely expressed through

$$\Gamma\Delta = \mathrm{id}_K = \Gamma B.$$

Lawvere interprets this "productive inconsistency" of having a definite number of points without these points being distinguishable by any property, and the underlying "unity of opposites," in terms of Hegelian dialectics. The basic idea is captured by the following diagram of natural transformations between the composite counit and unit functors:

$$\text{opposite}_1 \longrightarrow \text{unity} \longrightarrow \text{opposite}_2.$$

This situation is most clear in the particular case of the topos of reflexive graphs. In this context, Lawvere (1996) observed that while the notions of discrete and codiscrete are dual there, the full subcategories of discrete graphs and codiscrete graphs are each equivalent to the category of sets, and moreover, both the discrete and codiscrete graphs are identical when regarded in the category of all graphs. In such a situation, it seems entirely natural for Lawvere to have spoken of such adjunction pairs as embodying Hegel's idea of the unity and identity of opposites. In the case of such configurations in general, Lawvere speaks of *adjoint cylinders*, where the three functors involved are adjoint and the two composites are isomorphic to the identity in $\mathscr{K}$. Put otherwise: we have such an adjoint triple where there are two parallel functors that are adjointly opposite in that they are full and faithful, and moreover there exists a third functor that is left adjoint to one of them and right adjoint to the other functor; as subcategories included in the ambient category, they are opposite, but by neglecting the inclusions they are identical. In more detail, in the particular case of graphs, the category of sets gets embedded in the category of graphs through the action of the functor Δ (producing discrete graphs) and the action of the functor B (producing codiscrete graphs). These notions are dual; however, the resulting full subcategories of discrete graphs and codiscrete graphs are further equivalent to the category of sets, thereby yielding the relevant identity. From the perspective of $\mathscr{M}$, the discrete and the codiscrete are united; looked at from the other end, $\mathscr{K}$ is identified with $\mathscr{M}$ via inclusion of subcategories in two opposite ways. One might also think of this in terms of Hegel's discussion of *quantity* as the dynamic unity of the "moments" of discreteness and continuity.[251]

To make these ideas a little more concrete, consider the following.[252] If we set both $\mathscr{M}$ and $\mathscr{K}$ as the poset of natural numbers $\mathbb{N}$ (viewed as a category),[253] we can construct the two parallel functors $E, O : \mathbb{N} \to \mathbb{N}$, defined by $E(n) := 2n$, $O(n) := 2n + 1$, that is, the "even"

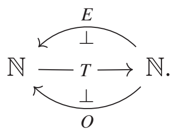
250. While there exist toposes with few connected objects such that for all $K <$ measurable cardinal we have that $\Delta(K) = B(K)$, in general these discrete and codiscrete maps are not equivalent.

251. See Georg Wilhelm Friedrich Hegel (2010, bk. 1, sec. II, chap. 1: "Quantity").

252. This very simple, but pedagogically useful, example of the adjoint cylinder construction is inspired by Lawvere (2000). The following exposition also follows nLab Authors (2018a, 2018b).

253. Technically such categories are not even toposes. The reader who cannot see why is invited to revisit the earlier section introducing toposes and try to see what makes them not qualify as toposes.

and "odd" functions. These functors obviously act to produce the two subcategories of $\mathbb{N}$, $\mathbb{N}_{even}$ and $\mathbb{N}_{odd}$, which is another way of saying that both functors are full and faithful. Such subcategories clearly are "opposed" to one another, at least in the sense that $\mathbb{N}_{even} \neq \mathbb{N}_{odd}$; however, performing the same sort of simple composite applications as above, we can produce the bijection $\mathbb{N}_{even} \xrightarrow{\cong} \mathbb{N}_{odd}$, through which they can be viewed as "identical." As subcategories, both can be seen to be united as the opposing parts of the containing category $\mathbb{N}$, in relation to which, by virtue of each being isomorphic to one another through their isomorphic maps to $\mathbb{N}$ itself, they are rendered identical. There indeed exists a third functor $T : \mathbb{N} \to \mathbb{N}$ that, together with E and O, will form the appropriate adjoint triple:



By definition of adjunctions, and given that we are working with posets, $E \dashv T$ and $T \dashv O$ just means that $E(n) \leq m$ iff $n \leq T(m)$ and $T(n) \leq m$ iff $n \leq O(m)$. Moreover, as long as T exists, we will further have that $TE = \text{id} = TO$, which just means in our particular case that $T(2n) = n$ and $T(2n+1) = n$, which forces the following piecewise definition of T as

$$T(k) = \begin{cases} \frac{k}{2} & \text{if } k \in \mathbb{N}_{even} \\ \frac{(k-1)}{2} & \text{if } k \in \mathbb{N}_{odd}. \end{cases}$$

The idea here is that the adjoint triple $E \dashv T \dashv O$ at the same time embraces the *identity* by the natural isomorphism $TE \xrightarrow{\cong} TO$, the *opposition* by the induced adjunction $ET \dashv OT$, and the *unity* by the idempotent relations $(E \circ T) \circ (E \circ T) = E \circ T$ and $(R \circ T) \circ (R \circ T) = R \circ T$. In this way, the middle functor T can be thought of as simultaneously identifying, opposing, and uniting E and O.

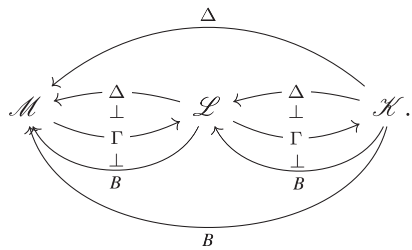
In the case of the category of presheaves on **C**, Δ yields the discrete presheaves on **C** while B yields the codiscrete presheaves on **C**, and Γ is their common projection. With such a setup, it is always the case that $\Gamma\Delta \cong \text{id} \cong \Gamma B$. In the particular context of presheaves, Lawvere takes the Hegelian notion of *Aufhebung* still further to yield a theory of dimension or "levels."[254] Roughly, a level is a functor from a given category into one that is "smaller," that moreover has both left and right adjoint sections which produce subcategories that in themselves are identical (in the smaller category) but that include themselves as subcategories in opposite ways (and which, moreover, give rise to the two composite idempotent functors on the given "larger" category). More specifically, given an adjoint cylinder situation between toposes, a *level* of a topos is defined as the inclusion of the right adjoint in this setup. In Lawvere's approach, again taking inspiration from Hegel, the *Aufhebung* of a level will be the smallest level that acts to resolve the component opposites (the opposing functors).[255] It is not the case that such an "*Aufhebung*-like" level always

254. See Lawvere (2002).

255. For more details on this theory of levels, see Lawvere (1989, 1996), and Kennett et al. (2011). In the toposes of simplicial sets, cubical sets, and reflexive globular sets—each definable in terms of presheaf categories—levels coincide with the notion of dimension.

exists for any given level, however in particular cases of special relevance to us (such as presheaf toposes over graphic categories), it does exist.[256]

This discussion could easily be extended in a number of directions, leading to a more thorough exposition of the notions and applications of cohesive toposes, in which context further adjoint cylinders arise, including situations relating to infinitesimally generated spaces; however, we leave curious readers to pursue these more advanced matters on their own.[257] For now, we content ourselves with observing that one sometimes finds further adjoint cylinders embedded in between $\mathscr{M}$ and $\mathscr{K}$ via intermediate categories $\mathscr{L}$ that are less "abstract" than **Set** but with a simpler sort of cohesion than $\mathscr{M}$:



The basic idea here is that various toposes such as $\mathscr{M}$ or $\mathscr{L}$ are contrasted with the extreme case of $\mathscr{K}$ via geometric morphisms; but the diagram above suggests that we extend this to consider intermediate toposes and chains of maps between such adjoint triples, with the effect that the various determinations of cohesivity or variation in toposes themselves can be compared.

11.5.2 Philosophical Pass: A New Dialectical Science?

Box 11.1
A New Dialectical Science?

An object that arises in a "spatial" category (a category with some cohesion or variation) can be examined via *levels*, constructions that provide a precise formulation of the unity and identity of opposites so characteristic of the (originally vaguely formulated) philosophical concept of dialectics, making higher-order relations between general forms of cohesiveness amenable to more exact solution. In the above exposition, we even saw how, from one perspective, the moments of discreteness (in the form of zero cohesion) and continuity (in the form of total cohesion) could be unified. However, this was a rather extreme case. By considering intermediate cylinders and passages between adjoint triples (or quadruples) of various toposes, we move beyond the case of relating a single category to the extreme case of discreteness or constancy (as in $\mathscr{K}$). We can now examine sequences of intermediate categories that are interlocked via cylinder maps of their own, opening onto a more refined dialectical science of cohesion, by which ultimately one could systematically characterize and compare the differing properties of cohesion and variation that emerge in certain universes or models

256. See Lawvere (2002) for details. Informally, "graphic categories" can just be thought of as certain simple enough categories that allow for finite graphic display or presentation once one constructs their corresponding presheaf category.

257. See, for instance, Lawvere (2007).

for mathematical theories that treat of objects with some dynamics. Using such intermediate adjoint cylinders, the quality of dimension or level in spaces can be compared. (See Lawvere [1994b] for more details.)

At a fundamental or more philosophical level, it could be argued that what is going on here is that the question of what is more variable, more cohesive—or how various models for certain mathematical theories dealing with dynamical phenomena or settings of cohesiveness are differently variable or cohesive—should involve functorial comparisons. The further suggestion could be made that, in addition to examining such sequences of adjoint cylinders as shown above, through the discovery and study of left-exact functors between two toposes (which may not have adjoints) we should be able to make even more precise the notion of greater or lesser discreteness (noncohesion or constancy) versus continuity (cohesion or variability). In a sense, this latter suggestion is a natural extension of the perspective of the categorical notion of continuity via (co)continuous functors (defined as preserving limits, and not just those that are finite). Going somewhat further, we could perhaps make use of such functors to begin to construct general metrics measuring something like "what it would cost" to make an "almost-adjunction" an honest adjunction; another use of the resulting metric might be to begin to more precisely analyze how far we are from the more extreme or trivial settings of infinite cohesion (continuity) on the one hand and zero cohesion (discreteness) on the other. This sort of approach should open onto a much richer terrain of dialectical subtleties, and it has the potential to provide a powerful weapon in the ever-frustrating yet beguiling dialectic of the continuous and the discrete at the heart of so much of mathematics and human thought.

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