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A New Method for Solving Physical Problems With Nonlinear Phoneme Within Fractional Derivatives With Singular Kernel

In this paper, we present a novel numerical approach for solving nonlinear problems with a singular kernel. We prove the existence and uniqueness of the solution for these models as well as the uniform convergence of the function sequence produced by our novel approach to the unique solution. Additionally, we offer a closed form and prove these results for a specific class of these problems where the free term is a fractional polynomial, an exponential, or a trigonometric function. These findings are new to the best of our knowledge. To demonstrate the effectiveness of our numerical method and how to apply our theoretical findings, we solved a number of physical problems. Comparisons with various researchers are reported. Findings demonstrate that our approach is more effective and accurate. In addition, compared to methods that address this type of problems, our approach is simple to implement and has lower computing costs. [DOI: 10.1115/1.4064719]

Keywords: nonlinear dynamical problem, uniformly convergent, singular kernel

1 Introduction

Fractional calculus (FC) is a branch of mathematics that deals with derivatives and integrals of noninteger order. Although the foundations of FC can be found in the work of mathematicians like Liouville et al., it has recently attracted new attention and found use in a number of other areas, including physics, engineering, finance, biology, and signal processing, among others. Many phenomena, including anomalous diffusion, viscoelastic materials, fractals, and fractional-order control systems, have been modeled using it [1–3].

There are several different types of fractional derivatives (FDs) that have been proposed and studied in the field of fractional calculus. Some of the commonly used types of fractional derivatives include Riemann–Liouville derivative, Caputo derivative (CD), and Grünwald–Letnikov derivative [4–6]. In this article, we use CD with a singular kernel since it is a powerful tool in fractional calculus that allows for modeling and analyzing systems with memory-dependent behavior. Its unique properties and applications make it a valuable tool in the problem under consideration.

Several scientists strive for numerical techniques to tackle fractional initial value problems (FIVP) because finding the exact solution is difficult and occasionally impossible. It can be difficult to create a suitable numerical method that is efficient. For specific types of problems, direct generalization to the numerical methods for the integer case will work, but not in the general case. It is obvious why that is. The traditional methods are made for local derivative and do not take function memory into account. But nevertheless, for the fractional case, the memory effects make it crucial to think in fractions rather than assuming the common methods will work always. The numerical techniques currently in use fall into two main categories: The first class is an analytical class, including HPM [7] and residual method [8]. The second class is a numerical class, including the Adams Bashforth method [9] and spline methods [10].

In our paper, we consider the following FIVP:

$$D^{\xi}\phi(x) = \Xi(x, \phi(x)), \phi(0) = \gamma, 0 < x \le 1, 0 < \xi \le 1 \qquad (1.1)$$

The above problem is covered by numerous researchers. In Ref. [9], the Adams Bashforth method (ABM) is used to solve this problem based on the predictor corrector idea. Their accuracy does not exceed 10^{-5} in most of the cases and usually this technique needs a huge calculations. For this reason, we observe that they sometimes take $N = 1000$. Also, In Ref. [11], a special case of this problem is

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solved when $\Xi(x, \phi(x))$ is a quadratic polynomial of ϕ. The variational iteration method (VIM) is used. The problem of this type of methods is when x is far from the initial condition, the approximation will be inaccurate. This is clear from the graphs provided in his paper. This type of method depends on the initial condition so either they give the exact solution or the solution will not be accurate when x is far from the starting point. In Ref. [7], the HPM is used for the same special case. This approach is also depending on the initial condition and a similar conclusion can be drawn. In Ref. [10], spline method (SM) is used. Generalization of the classical SM is used to fit with the fractional case. Their absolute error in this technique is 10^{-3} or less. In Ref. [12], they consider only the linear case of Problem (1.1) while in Ref. [13], they use the operational matrix method to solve the case when $\Xi(x, \phi(x))$ is a quadratic polynomial of ϕ.

In order to address the FIVP (1.1), we develop a novel numerical technique in this paper. We take into account the memory effects on the problem at hand and iteratively produce the coefficients of an approximate solution. When compared to other methods like ABM or SM, the computational cost is extremely low. We first prove the existence and uniqueness of Problem (1.1), and then we prove that the series of functions produced by our new strategy uniformly converges to this solution. We also think about the scenario in which $\Xi(x, \phi(x)) = -\alpha\phi(x) + r(x)$. When $r(x)$ is a fractional polynomial, a fractional exponential function, or a fractional trigonometric function, we find the closed form of the solution. Additionally, by using the integral form, we are able to determine the solution for any function of $r(x)$. To demonstrate the effectiveness of the new numerical method, we also offer a number of examples and comparisons with work by other researchers. Additionally, in order to make our conclusions more apparent and applicable, we apply our theoretical results to a number of examples.

Following is how the remainder of the paper is organized. We provide the fundamental definitions and theorems that are required for this paper in Sec. 2. Describe a thorough study for the scenario where $\Xi(x, \phi(x)) = -\alpha\phi(x) + r(x)$ in Sec. 3. We state and prove the results in all instances of $r(x)$. We construct our new numerical technique in Sec. 4. We have shown the theorem's existence, uniqueness, and uniformly convergent properties. In Sec. 5, a number of illustrations and comparisons with other researchers are provided. Finally, we conclude this article with a few closing remarks limitations for the proposed method.

2 Preliminaries

In this section, we define some concepts and we state some results that we use in this paper.

Definition 2.1. *Let $\xi \in (0, 1)$ and $\nu > 0$. Then, the CD of $\beta(\nu)$ is given by [1–3]*

$$D^{\xi}\beta(\nu) = \frac{1}{\Gamma(1-\xi)}\int_0^{\nu}(\nu-\alpha)^{-\xi}\beta'(\alpha)d\alpha$$

and the fractional integral (FI) operator is given by

$$I^{\xi}\beta(\nu) = \frac{1}{\Gamma(\xi)}\int_0^{\nu}(\nu-\alpha)^{\xi-1}\beta(\alpha)d\alpha$$

The following are some properties of the CD and the FI which are given Lemmas 2.1–2.3.

Lemma 2.1. *If $\xi_1, \xi_2, b > 0$, and $r(x) \in L_q(0, b)$, $1 \le q \le \infty$, then*

$$I^{\xi_1}I^{\xi_2}r(x) = I^{\xi_1+\xi_2}r(x)$$

almost everywhere in $[0, b]$. If $\xi_1 + \xi_2 > 1$, then relation is true for all $x \in [0, b]$.

Lemma 2.2. *If $\xi > 0$, then*

$$I^{\xi}x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\xi+1)}x^{\mu+\xi}, \mu \ge 0$$

and

$$D^{\xi}x^{\mu} = \begin{Bmatrix} 0, & \mu < \xi, \mu \in \{0,1,2,\ldots\} \\ \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\xi+1)}x^{\mu-\xi} & \text{otherwise} \end{Bmatrix}$$

Lemma 2.3. *If $\xi > 0$, and $r(x) \in C[0, b]$, then*

$$I^{\xi}D^{\xi}r(x) = r(x) - r(0)$$

and

$$D^{\xi}I^{\xi}r(x) = r(x)$$

In the next three definitions, we define some important functions which we use in this paper.

Definition 2.2. *Let $\xi_1, \xi_2 > 0$, then we define Mittag–Leffler of one and two parameters functions by*

$$E_{\xi_1}(x) = \sum_{l=0}^{\infty}\frac{x^l}{\Gamma(l\xi_1+1)}$$

and

$$E_{\xi_1,\xi_2}(x) = \sum_{l=0}^{\infty}\frac{x^l}{\Gamma(l\xi_1+\xi_2)}$$

respectively.

Definition 2.3. *Let $\xi > 0$ and $n \in \{0, 1, 2, \ldots\}$, then we define the ξ-polynomial of degree n as*

$$P_{\xi,n}(x) = a_0 + a_1x^{\xi} + a_2x^{2\xi} + \cdots + a_nx^{n\xi}$$

In addition, we define the ξ-sine and ξ-cosine functions as

$$\sin_{\xi}(x) = \frac{E_{\xi}(ix) - E_{\xi}(-ix)}{2i}$$

and

$$\cos_{\xi}(x) = \frac{E_{\xi}(ix) + E_{\xi}(-ix)}{2}$$

Definition 2.4. *Let $x_l = l\Delta, l \in \{0, 1, 2, \ldots, L-1\}, \Delta = 1/L$, and $L \in \mathbb{N}$, then the l block pulse function (BPF) is given by [13]*

$$\pi_l(x) = \begin{cases} 1, & x_l \le x < x_{l+1} \\ 0, & \text{otherwise} \end{cases}, 0 \le l < L$$

We end this section by the completeness property.

Lemma 2.4. *If $\beta \in L^2[0, 1)$, then [14]*

$$\beta(x) = \sum_{j=0}^{\infty}\beta_j\pi_j(x) \tag{2.1}$$

where

$$\beta_j = \frac{1}{\Delta}\int_{j\Delta}^{(j+1)\Delta}\beta(x)dx \tag{2.2}$$

3 Exact Solutions for a Class of Fractional Initial Value Problems

In this section, we study the solution of the following problem:

$$D^{\xi}\phi(x) + \alpha\phi(x) = r(x), x > 0, 0 < \xi < 1 \tag{3.1}$$

with

$$\phi(0) = \gamma \tag{3.2}$$

First, we study the case when $r(x) = 0$. The result is given in the following theorem.

THEOREM 3.1. *For $0 < \xi < 1$ and $\alpha \in \mathbb{R}$, the solution of the following IVP:*

$$D^{\xi}\phi(x) + \alpha\phi(x) = 0, \phi(0) = \gamma, x > 0 \tag{3.3}$$

is

$$\phi(x) = \gamma E_{\xi}(-\alpha x^{\xi})$$

Proof. By taking the fractional integral for both sides and using Lemma 2.3, we get

$$\phi(x) - \phi(0) + \alpha I^{\xi}\phi(x) = 0$$

which can be written as

$$\phi(x) = \gamma - \frac{\alpha}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\phi(s)ds$$

Using the method of successive approximation with

$$\phi_{k+1}(x) = \gamma - \frac{\alpha}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\phi_k(s)ds, \phi_0(x) = \gamma, k = 0, 1, 2, \ldots$$

we get

$$\phi_1(x) = \gamma - \frac{\alpha}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\phi_0(s)ds = \gamma - \frac{\alpha\gamma x^{\xi}}{\xi\Gamma(\xi)} = \gamma - \frac{\alpha\gamma x^{\xi}}{\Gamma(\xi+1)}$$

since $\Gamma(\xi+1) = \xi\Gamma(\xi)$. Similarly,

$$\begin{aligned}\phi_2(x) &= \gamma - \frac{\alpha}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\phi_1(s)ds\\ &= \gamma - \frac{\alpha}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\left(\gamma - \frac{\alpha\gamma s^{\xi}}{\Gamma(\xi+1)}\right)ds\\ &= \gamma - \frac{\alpha\gamma x^{\xi}}{\Gamma(\xi+1)} + \frac{\alpha^2\gamma x^{2\xi}}{\Gamma(2\xi+1)} = \gamma\sum_{k=0}^{2}\frac{(-\alpha)^k x^{k\xi}}{\Gamma(k\xi+1)}\end{aligned}$$

If we continue in this process, we find the nth term

$$\phi_n(x) = \gamma\sum_{k=0}^{n}\frac{(-\alpha)^k x^{k\xi}}{\Gamma(k\xi+1)}$$

If we take the limit as n approaches to infinity, we find the solution of Problem (3.3), which is given as

$$\phi(x) = \lim_{n\to\infty}\phi_n(x) = \gamma\sum_{k=0}^{\infty}\frac{(-\alpha)^k x^{k\xi}}{\Gamma(k\xi+1)} = \gamma E_{\xi}(-\alpha x^{\xi})$$

■

Now, we will use the same technique to prove the general case when $r \in L(0, b)$ where $b > 0$.

THEOREM 3.2. *For $0 < \xi < 1, \alpha \in \mathbb{R}$, and $r \in L(0, b), b > 0$, then the solution of the following IVP:*

$$D^{\xi}\phi(x) + \alpha\phi(x) = r(x), \phi(0) = \gamma, x > 0 \tag{3.4}$$

is

$$\phi(x) = \gamma E_{\xi}(-\alpha x^{\xi}) + \int_0^x (x-s)^{\xi-1}E_{\xi,\xi}(-\alpha(x-s)^{\xi})r(s)ds$$

Proof. By taking the fractional integral for both sides and using Lemma 2.3, we get

$$\phi(x) - \phi(0) + \alpha I^{\xi}\phi(x) = I^{\xi}r(x)$$

which can be written as

$$\phi(x) = \gamma + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}r(s)ds - \frac{\alpha}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\phi(s)ds$$

Using the method of successive approximation, we have

$$\begin{aligned}\phi_{k+1}(x) &= \gamma + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}r(s)ds\\ &\quad - \frac{\alpha}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\phi_k(s)ds\\ &= \gamma + I^{\xi}r(x) - \alpha I^{\xi}\phi_k(x), k = 0, 1, 2, \ldots\end{aligned}$$

where

$$\phi_0(x) = \gamma + I^{\xi}r(x)$$

Using Lemmas 2.1 and 2.2

$$\begin{aligned}\phi_1(x) &= \gamma + I^{\xi}r(x) - \alpha I^{\xi}\phi_0(x) = \gamma + I^{\xi}r(x) - \alpha I^{\xi}\left(\gamma + I^{\xi}r(x)\right)\\ &= \gamma - \alpha I^{\xi}(\gamma) + I^{\xi}r(x) - \alpha I^{\xi}\left(I^{\xi}r(x)\right)\\ &= \gamma - \frac{\gamma\alpha x^{\xi}}{\Gamma(\xi+1)} + I^{\xi}r(x) - \alpha I^{2\xi}r(x)\end{aligned}$$

Similarly,

$$\begin{aligned}\phi_2(x) &= \gamma + I^{\xi}r(x) - \alpha I^{\xi}\left(\gamma - \frac{\gamma\alpha x^{\xi}}{\Gamma(\xi+1)} + I^{\xi}r(x) - \alpha I^{2\xi}r(x)\right)\\ &= \gamma - \alpha I^{\xi}(\gamma) + \alpha I^{\xi}\left(\frac{\gamma\alpha x^{\xi}}{\Gamma(\xi+1)}\right) + I^{\xi}r(x) - \alpha I^{2\xi}r(x) + \alpha^2 I^{3\xi}r(x)\\ &= \gamma - \frac{\gamma\alpha x^{\xi}}{\Gamma(\xi+1)} + \frac{\gamma\alpha^2 x^{2\xi}}{\Gamma(2\xi+1)} + I^{\xi}r(x) - \alpha I^{2\xi}r(x) + \alpha^2 I^{3\xi}r(x)\\ &= \gamma\sum_{k=0}^{2}\frac{(-\alpha)^k x^{k\xi}}{\Gamma(k\xi+1)} + \sum_{k=0}^{2}(-\alpha)^k I^{(k+1)\xi}r(x)\\ &= \gamma\sum_{k=0}^{2}\frac{(-\alpha)^k x^{k\xi}}{\Gamma(k\xi+1)} + \int_0^x\sum_{k=0}^{2}\frac{(-\alpha)^k}{\Gamma((k+1)\xi)}(x-s)^{(k+1)\xi-1}g(s)ds\\ &= \gamma\sum_{k=0}^{2}\frac{(-\alpha)^k x^{k\xi}}{\Gamma(k\xi+1)} + \int_0^x (x-s)^{\xi-1}\left(\sum_{k=0}^{2}\frac{(-\alpha)^k(x-s)^{k\xi}}{\Gamma(k\xi+\xi)}\right)g(s)ds\end{aligned}$$

If we continue in this process, we find the nth term is

$$\phi_n(x) = \gamma\sum_{k=0}^{n}\frac{(-\alpha)^k x^{k\xi}}{\Gamma(k\xi+1)} + \int_0^x (x-s)^{\xi-1}\left(\sum_{k=0}^{n}\frac{(-\alpha)^k(x-s)^{k\xi}}{\Gamma(k\xi+\xi)}\right)g(s)ds$$

If we take the limit as n approaches to infinity, we find the solution of Problem (3.4) which is given by

$$\phi(x) = \lim_{n\to\infty}\phi_n(x)$$
$$= \gamma E_\xi(-\alpha x^\xi) + \int_0^x (x-s)^{\xi-1}E_{\xi,\xi}(-\alpha(x-s)^\xi)r(s)ds$$

■

THEOREM 3.3. *Let* $0 < \xi < 1$, $\alpha \in \mathbb{R}$, *and* $P_{\xi,n} = \sum_{k=0}^{n} a_k x^{k\xi}$ *be a* ξ*-polynomial of degree* $n \geq 0$, *then the solution of the following IVP:*

$$D^\xi\phi(x) + \alpha\phi(x) = P_{\xi,n}(x), \phi(0) = \gamma, x > 0 \qquad (3.5)$$

is

$$\phi(x) = \gamma E_\xi(-\alpha x^\xi) + \sum_{k=0}^{n} a_k\Gamma(k\xi+1)x^{(k+1)\xi}E_{\xi,\xi(k+1)+1}(-\alpha x^\xi)$$

Proof. For any $k \in \{0, 1, 2, \ldots, n\}$, let $s = tx$ to have

$$\int_0^x (x-s)^{\xi-1}(x-s)^{m\xi}s^{k\xi}ds = \int_0^1 (x-tx)^{\xi-1}(x-tx)^{m\xi}t^{k\xi}x^{k\xi}xdt$$
$$= x^{(m+k+1)\xi}\int_0^1 t^{k\xi}(1-t)^{(m+1)\xi-1}dt$$
$$= x^{(m+k+1)\xi}B(k\xi+1,(m+1)\xi) \qquad (3.6)$$

where B is beta function. Theorem 3.2 implies that

$$\phi(x) = \gamma E_\xi(-\alpha x^\xi) + \int_0^x (x-s)^{\xi-1}E_{\xi,\xi}\left(-\alpha(x-s)^\xi\right)P_{\xi,n}(s)ds$$
$$= \gamma E_\xi(-\alpha x^\xi) + \sum_{k=0}^{n} a_k \sum_{m=0}^{\infty}\frac{(-\alpha)^m}{\Gamma(m\xi+\xi)}$$
$$\int_0^x (x-s)^{\xi-1}(x-s)^{m\xi}s^{k\xi}ds$$
$$= \gamma E_\xi(-\alpha x^\xi) + \sum_{k=0}^{n} a_k \sum_{m=0}^{\infty}\frac{(-\alpha)^m}{\Gamma(m\xi+\xi)}x^{(m+k+1)\xi}$$
$$B(k\xi+1,(m+1)\xi)$$
$$= \gamma E_\xi(-\alpha x^\xi) + \sum_{k=0}^{n} a_k\Gamma(k\xi+1)x^{(k+1)\xi}$$
$$\sum_{m=0}^{\infty}\frac{(-\alpha)^m x^{m\xi}}{\Gamma(m\xi+(k+1)\xi+1)}$$
$$= \gamma E_\xi(-\alpha x^\xi) + \sum_{k=0}^{n} a_k\Gamma(k\xi+1)x^{(k+1)\xi}E_{\xi,(k+1)\xi+1}(-\alpha x^\xi)$$

■

THEOREM 3.4. *Let* $0 < \xi < 1$, α, β *be constants such that* $\beta \neq 0$ *and* $\alpha + \beta \neq 0$. *Then the solution of the following IVP:*

$$D^\xi\phi(x) + \alpha\phi(x) = E_\xi(\beta x^\xi), \phi(0) = \gamma, x > 0 \qquad (3.7)$$

is

$$\phi(x) = \gamma E_\xi(-\alpha x^\xi) + \frac{\beta}{\alpha+\beta}x^\xi E_{\xi,\xi+1}(\beta x^\xi) + \frac{\alpha}{\alpha+\beta}x^\xi E_{\xi,\xi+1}(-\alpha x^\xi)$$

Proof. The Cauchy product of two infinite series yields

$$E_{\xi,\xi}\left(-\alpha(x-s)^\xi\right)E_\xi(\beta s^\xi) = \left(\sum_{i=0}^{\infty}\frac{(-\alpha)^i(x-s)^{i\xi}}{\Gamma(i\xi+\xi)}\right)\left(\sum_{j=0}^{\infty}\frac{\beta^j s^{j\xi}}{\Gamma(j\xi+1)}\right)$$
$$= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\frac{(-\alpha)^l\beta^{(k-l)}(x-s)^{l\xi}s^{(k-l)\xi}}{\Gamma(l\xi+\xi)\Gamma((k-l)\xi+1)}\right)$$

Using formula (3.6), the particular solution is

$$\phi_p(x) = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\frac{(-\alpha)^l\beta^{k-l}\int_0^x (x-s)^{\xi-1}(x-s)^{l\xi}s^{(k-l)\xi}ds}{\Gamma(l\xi+\xi)\Gamma((k-l)\xi+1)}\right)$$
$$= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\frac{(-\alpha)^l\beta^{k-l}x^{k\xi+\xi}B((k-l)\xi+1, l\xi+\xi)}{\Gamma(l\xi+\xi)\Gamma((k-l)\xi+1)}\right)$$
$$= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\frac{(-\alpha)^l\beta^{k-l}x^{k\xi+\xi}}{\Gamma((k+1)\xi+1)}\right)$$
$$= \sum_{k=0}^{\infty}\frac{\beta^k x^{k\xi+\xi}}{\Gamma((k+1)\xi+1)}\sum_{l=0}^{k}\left(\frac{-\alpha}{\beta}\right)^l$$
$$= \sum_{k=0}^{\infty}\frac{\beta^k x^{k\xi+\xi}}{\Gamma((k+1)\xi+1)}\frac{1-\left(\frac{-\alpha}{\beta}\right)^{k+1}}{1+\frac{\alpha}{\beta}}$$
$$= \frac{\beta}{\alpha+\beta}x^\xi\left(\sum_{k=0}^{\infty}\frac{\beta^k x^{k\xi}}{\Gamma(1+k\xi+\xi)} + \frac{\alpha}{\beta}\sum_{k=0}^{\infty}\frac{(-\alpha)^k x^{k\xi}}{\Gamma(1+k\xi+\xi)}\right)$$
$$= \frac{\beta}{\alpha+\beta}x^\xi E_{\xi,\xi+1}(\beta x^\xi) + \frac{\alpha}{\alpha+\beta}x^\xi E_{\xi,\xi+1}(-\alpha x^\xi)$$

Therefore, the solution of problem (3.7) is

$$\phi(x) = \gamma E_\xi(-\alpha x^\xi) + \frac{\beta}{\alpha+\beta}x^\xi E_{\xi,\xi+1}(\beta x^\xi) + \frac{\alpha}{\alpha+\beta}x^\xi E_{\xi,\xi+1}(-\alpha x^\xi)$$

■

THEOREM 3.5. *Let* $0 < \xi < 1$, α, β *be constants such that* $\beta \neq 0$ *and* $\alpha + \beta \neq 0$. *Then the solution of the following IVP:*

$$D^\xi\phi(x) + \alpha\phi(x) = \sin_\xi(\beta z^\xi), \phi(0) = \gamma, x > 0 \qquad (3.8)$$

is

$$\phi(x) = \gamma E_\xi(-\alpha x^\xi) + \frac{1}{\alpha^2+\beta^2}$$
$$\left(\alpha\sin_\xi(\beta x^\xi) + \beta\left(1-\cos_\xi(\beta x^\xi)\right) - \alpha\beta x^\xi E_{\xi,\xi+1}(-\alpha x^\xi)\right)$$

Proof. Since

$$\sin_\xi(\beta x^\xi) = \frac{E_\xi(i\beta x^\xi) - E_\xi(-i\beta\xi)}{2i}$$

using Theorem 3.4 and superposition principle, the particular solution is

$$\phi_p(x) = \frac{x^\xi}{2i}\left(\frac{i\beta}{\alpha+i\beta}E_{\xi,\xi+1}(i\beta x^\xi) + \frac{\alpha}{\alpha+i\beta}E_{\xi,\xi+1}(-\alpha x^\xi)\right)$$
$$-\frac{x^\xi}{2i}\left(\frac{-i\beta}{\alpha-i\beta}E_{\xi,\xi+1}(-i\beta x^\xi) + \frac{\alpha}{\alpha-i\beta}E_{\xi,\xi+1}(-\alpha x^\xi)\right)$$
$$= \frac{i\beta x^\xi}{2i(\alpha^2+\beta^2)}\left((\alpha-i\beta)E_{\xi,\xi+1}(i\beta x^\xi)\right.$$
$$\left.+(\alpha+i\beta)E_{\xi,\xi+1}(-i\beta x^\xi)\right)$$
$$+\frac{\alpha x^\xi E_{\xi,\xi+1}(-\alpha x^\xi)}{2i(\alpha^2+\beta^2)}((\alpha-i\beta)-(\alpha+i\beta))$$
$$= \frac{\alpha\beta x^\xi}{2(\alpha^2+\beta^2)}\left(E_{\xi,\xi+1}(i\beta x^\xi) + E_{\xi,\xi+1}(-i\beta x^\xi)\right)$$
$$-\frac{i\beta^2 x^\xi}{2(\alpha^2+\beta^2)}\left(E_{\xi,\xi+1}(i\beta x^\xi) - E_{\xi,\xi+1}(-i\beta x^\xi)\right)$$
$$-\frac{\alpha\beta}{\alpha^2+\beta^2}x^\xi E_{\xi,\xi+1}(-\alpha x^\xi)$$

When $x > 0$, then

$$E_{\xi,\xi+1}(\lambda x^{\xi}) = \sum_{k=0}^{\infty} \frac{\lambda^k x^{k\xi}}{\Gamma(\xi k+\xi+1)} = \sum_{l=1}^{\infty} \frac{\lambda^{(l-1)} x^{(l-1)\xi}}{\Gamma(l\xi+1)} = \frac{1}{\lambda x^{\xi}}\left(E_{\xi}(\lambda x^{\xi})-1\right)$$

Also

$$\lim_{x\to 0} \frac{1}{\lambda x^{\xi}}\left(E_{\xi}(\lambda x^{\xi})-1\right) = \frac{1}{\Gamma(\xi+1)}$$

which is the value of $E_{\xi,\xi+1}(\beta x^{\xi})$ at $x = 0$. Then

$$\phi_p(x) = \frac{\alpha}{2i(\alpha^2+\beta^2)}\left(E_{\xi}(i\beta x^{\xi}) - E_{\xi}(-i\beta x^{\xi})\right) - \frac{\beta}{2(\alpha^2+\beta^2)}\left(E_{\xi}(i\beta x^{\xi}) + E_{\xi}(-i\beta x^{\xi}) - 2\right) - \frac{\alpha\beta}{\alpha^2+\beta^2} x^{\xi} E_{\xi,\xi+1}(-\alpha x^{\xi})$$

Using Definition 2.3, we have

$$\phi_p(x) = \frac{1}{\alpha^2+\beta^2}\Big(\alpha \sin_{\xi}(\beta x^{\xi}) + \beta\left(1-\cos_{\xi}(\beta x^{\xi})\right) - \alpha\beta x^{\xi} E_{\xi,\xi+1}(-\alpha x^{\xi})\Big)$$

and the solution to Problem (3.9) is

$$\phi(x) = \gamma E_{\xi}(-\alpha x^{\xi}) + \frac{1}{\alpha^2+\beta^2}\Big(\alpha \sin_{\xi}(\beta x^{\xi}) + \beta\left(1-\cos_{\xi}(\beta x^{\xi})\right) - \alpha\beta x^{\xi} E_{\xi,\xi+1}(-\alpha x^{\xi})\Big)$$

■

THEOREM 3.6. *Let $0 < \xi < 1$, α, β be constants such that $\beta \neq 0$ and $\alpha + \beta \neq 0$. Then the solution of the following IVP:*

$$D^{\xi}\phi(x) + \alpha\phi(x) = \cos_{\xi}(\beta z^{\xi}), \phi(0) = \gamma, x > 0 \quad (3.9)$$

is

$$\phi(x) = \gamma E_{\xi}(-\alpha x^{\xi}) + \frac{1}{\alpha^2+\beta^2}\Big(\beta \sin_{\xi}(\beta x^{\xi}) + \alpha\left(\cos_{\xi}(\beta x^{\xi})-1\right) + \alpha^2 x^{\xi} E_{\xi,\xi+1}(-\alpha x^{\xi})\Big)$$

Proof. The proof is similar to the proof of Theorem 3.5.

■

We end this section by the following remark.

Remark 3.1. We should note that if $r(t)$ is a linear combination of $t^{k\xi}, E_{\xi}(\beta x^{\xi})$, $\sin_{\xi}(\beta x^{\xi})$, or $\cos_{\xi}(\beta x^{\xi})$, then the solution of Problem (3.1) is

$$\phi(x) = \gamma E_{\xi}(-\alpha x^{\xi}) + \phi_p(x)$$

where $\phi_p(x)$ can be computed using the superposition principle and Theorems 3.1–3.6.

4 A Numerical Scheme for the Nonlinear Fractional Initial Value Problems

In the following, we study the nonlinear problem:

$$D^{\xi}\phi(x) = \Xi(x, \phi(x)), \phi(0) = \gamma, 0 < x \leq 1, 0 < \xi \leq 1 \quad (4.1)$$

Let us define the space $H = C((0,1), \mathbb{R})$ with the norm

$$||q(x)||_{(0,1)} = \sup_{x\in(0,1)} |g(x)|$$

Then, H is a Banach space. For simplicity, we denote this norm by $||.||$. First, we prove the existence and uniqueness of the solution of Problem (4.1).

THEOREM 4.1. *Let $\Xi(x, \phi(x)) \in H$ be a function that satisfy the Lipschitz condition with respect to ϕ with Lipschitz constant θ, then the FIVP (4.1) has a unique solution if*

$$\frac{\theta}{\Gamma(\xi+1)} < 1 \quad (4.2)$$

Proof. Using Lemma 2.2, we can rewrite Eq. (4.1) as

$$\phi(x) = \gamma + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\Xi(s, \phi(s))ds \quad (4.3)$$

Define the operator Υ as

$$\Upsilon(\phi) = \gamma + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\Xi(s, \phi(s))ds$$

Then

$$\begin{aligned} &|\Upsilon(\phi_1) - \Upsilon(\phi_2)| \\ &= \frac{1}{\Gamma(\xi)}\left|\int_0^x (x-s)^{\xi-1}(\Xi(s,\phi_1(s)) - \Xi(s,\phi_2(s)))ds\right| \\ &\leq \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}|\Xi(s,\phi_1(s)) - \Xi(s,\phi_2(s)))|ds \\ &\leq \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}ds||\Xi(x,\phi_1(x)) - \Xi(x,\phi_2(x)))|| \\ &\leq \frac{\theta x^{\xi}}{\xi\Gamma(\xi)}||\phi_1(x) - \phi_2(x)|| \leq \frac{\theta}{\Gamma(\xi+1)}||\phi_1(x) - \phi_2(x)|| \end{aligned}$$

Since $\frac{\theta}{\Gamma(\xi+1)} < 1$, then Υ is contraction on H. Hence, Banach fixed point theorem will guarantee the existence and uniqueness of Problem (4.1).

■

To find the unique solution to Problem (4.1), Eq. (4.3) proposed the following sequence of functions:

$$\phi_0(x) = \gamma \quad (4.4)$$

$$\phi_k(x) = \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\Xi(s, \phi_{k-1}(s))ds, k = 1, 2, \ldots \quad (4.5)$$

which is based on the method of successive approximation. However, modification for it is necessary to reduce the computational cost. A numerical method will be derived to solve

$$D^{\xi}\phi(x) = \Xi(x, \phi(x)), \phi(0) = \gamma, x > 0 \quad (4.6)$$

where $\Xi \in H$. By taking the fractional integral, Lemma 2.3 gives

$$\begin{aligned} \phi(x) &= \phi(0) + I^{\xi}(\Xi(x, \phi(x))) \\ &= \gamma + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\Xi(s, \phi(s))ds \end{aligned} \quad (4.7)$$

Approximate ϕ in terms of the BPFs as

$$\phi_L(x) = \sum_{l=0}^{L-1} \phi_l \pi_l(x) \quad (4.8)$$

Substitute Eq. (4.8) in Eq. (4.7) to get

$$\sum_{l=0}^{L-1}\phi_l\pi_l(x) = \gamma + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\Xi\left(s, \sum_{l=0}^{L-1}\phi_l\pi_l(s)\right)ds \quad (4.9)$$

Collocate Eq. (4.9) at $x_j, 1 \le j < L$, to get

$$\begin{aligned}\phi_j &= \gamma + \frac{1}{\Gamma(\xi)}\int_0^{x_j}(x_j-s)^{\xi-1}\Xi\left(s, \sum_{l=0}^{L-1}\phi_l\pi_l(s)\right)ds \\ &= \gamma + \frac{1}{\Gamma(\xi)}\sum_{k=1}^{j}\int_{x_{k-1}}^{x_k}(x_j-s)^{\xi-1}\Xi\left(s, \sum_{l=0}^{L-1}\phi_l\pi_l(s)\right)ds\end{aligned} \quad (4.10)$$

since

$$\pi_l(x_k) = \begin{cases}1, & l = k\\ 0, & l \ne k\end{cases}$$

Using Definition 2.4, Eq. (4.10) yields

$$\phi_j = \gamma + \frac{1}{\Gamma(\xi)}\sum_{k=1}^{j}\int_{x_{k-1}}^{x_k}(x_j-s)^{\xi-1}\Xi(s, \phi_{k-1})ds \quad (4.11)$$

Note that $\gamma = \phi(0) = \phi_0$. Then, Eq. (4.11) becomes

$$\phi_j = \gamma + \frac{1}{\Gamma(\xi)}\sum_{k=1}^{j}T_{j,k} \quad (4.12)$$

where

$$T_{j,k} = \int_{x_{k-1}}^{x_k}(x_j-s)^{\xi-1}\Xi(s, \phi_{k-1})ds \quad (4.13)$$

Since ϕ_j depends on the previous values of $\phi_k, k = 0, 1, \ldots, j-1$, then we can find the solution in direct way without solving nonlinear algebraic system. The solution will have cheap computational cost. We can summarize our numerical approach in the following Algorithm.

Algorithm 4.1.

Input: ξ, γ.
Output: $\phi(x)$: approximate solution
Step 1 Let $\phi(x) = 0$.
Step 2: For $j = 0 : L-1$, do steps 3–7.
Step 3: Let $\phi_j = \gamma$ and $w_j = 0$.
Step 4: For $k = 1 : j$, do step 5.
Step 5: $w_j = w_j + T_{j,k}$ where $T_{j,k}$ is given in Eq. (4.13).
Step 6: Let $\phi_j = \phi_j + \frac{w_j}{\Gamma(\xi)}$.
Step 7: Let $\phi(x) = \phi(x) + \phi_j\pi_j(x)$.
Step 8: Stop.

The next theorem, we prove the convergence of the proposed solution in Eqs. (4.8) and (4.11).

Theorem 4.2. *Let $\Xi(x, \phi(x)) \in H$ be a function that satisfy the Lipschitz condition with respect to ϕ with Lipschitz constant θ, then the sequence $\{\phi_L(x)\}_{L=0}^{\infty}$ converges uniformly to the unique solution of the FIVP (4.1).*

Proof. Assume that $\phi(x)$ is the exact solution of Problem (4.1). Then, it satisfies the integral Eq. (4.3)

$$\Phi(x) = \gamma + \frac{1}{\Gamma(\xi)}\int_0^x (x-s)^{\xi-1}\Xi(s, \phi(s))ds \quad (4.14)$$

Also, using Lemma 2.4, we have

$$\Phi(x) = \sum_{k=0}^{\infty}\lambda_k\pi_k(x), \lambda_0 = \gamma \quad (4.15)$$

For any $0 \le j < L-1$, then

$$\Phi(x_j) = \sum_{k=0}^{\infty}\lambda_k\pi_k(x_j) = \lambda_j$$

Then

$$\lambda_j = \Phi(x_j) = \gamma + \frac{1}{\Gamma(\xi)}\int_0^{x_j}(x_j-s)^{\xi-1}\Xi(s, \phi(s))ds \quad (4.16)$$

Subtract Eq. (4.10) from Eq. (4.16) to have

$$\begin{aligned}|\lambda_j - \phi_j| &= \frac{1}{\Gamma(\xi)}\int_0^{x_j}(x_j-s)^{\xi-1}|\Xi(s, \phi(s)) - \Xi(s, \phi_L(s))|ds \\ &\le \frac{\theta||\phi-\phi_L||}{\Gamma(\xi)}\int_0^{x_j}(x_j-s)^{\xi-1}ds \\ &\le \frac{x_j\theta||\phi-\phi_L||}{\xi\Gamma(\xi)} \le \frac{\theta||\phi-\phi_L||}{\Gamma(\xi+1)}\end{aligned} \quad (4.17)$$

Then

$$\begin{aligned}|\Phi(x) - \phi_L(x)| &= \left|\sum_{k=0}^{L-1}(\lambda_j-\phi_j)\pi(x) + \sum_{k=L}^{\infty}\lambda_k\pi_k(x)\right| \\ &\le \sum_{k=0}^{L-1}|\lambda_j-\phi_j| + \left|\sum_{k=L}^{\infty}\lambda_k\pi_k(x)\right|\end{aligned}$$

From Eq. (4.17), we get

$$\begin{aligned}|\Phi(x) - \phi_L(x)| &\le \sum_{k=0}^{L-1}\frac{\theta||\phi-\phi_L||}{\Gamma(\xi+1)} + \left|\sum_{k=L}^{\infty}\lambda_k\pi_k(x)\right| \\ &\le \frac{L\theta||\phi-\phi_L||}{\Gamma(\xi+1)} + \left|\sum_{k=L}^{\infty}\lambda_k\pi_k(x)\right|\end{aligned} \quad (4.18)$$

Since the series in Eq. (4.15) converges uniformly, then for $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that

$$\left|\sum_{k=L}^{\infty}\lambda_k\pi_k(x)\right| \le 1, L \ge N \quad (4.19)$$

Hence, Inequalities (4.18) and (4.19) imply that

$$|\Phi(x) - \phi_L(x)| \le \frac{L\theta||\phi-\phi_L||}{\Gamma(\xi+1)} + 1, L \ge N \quad (4.20)$$

Take the supreme on $[0, 1]$ for both sides of inequality (4.20) to get

$$||\Phi - \phi_L|| \le \frac{L\theta||\phi-\phi_L||}{\Gamma(\xi+1)} + 1, L \ge N$$

which gives

$$\lim_{L\to\infty}||\Phi-\phi_L|| \le \lim_{L\to\infty}\frac{\Gamma(\xi+1)}{\Gamma(\xi+1) - L\theta} = 0$$

Therefore, the sequence $\{\phi_L(x)\}_{L=0}^{\infty}$ converges uniformly to the unique solution of the FIVP (4.1). ■

5 Illustrative Examples

In the following, we solve several examples to show the efficiency of the proposed method. Also, compassion with other methods will be presented.

Example 5.1. Consider the following nonlinear Riccati equation [11]:

$$D^{\xi}\phi(x) = 1 - \phi^2(x), \phi(0) = 0$$

Let us approximate $\phi(x)$ as in Eq. (4.8)

$$\phi_L(x) = \sum_{j=0}^{L-1} \phi_j \pi_j(x)$$

where ϕ_l is given by formula (4.11) as

$$\phi_j = \gamma + \frac{1}{\Gamma(\xi)} \sum_{k=1}^{j} \int_{x_{k-1}}^{x_k} (x_j - s)^{\xi-1} \Xi(s, \phi_{k-1}) ds$$

Since $\Xi(x, \phi(x)) = 1 - \phi^2(x)$ and $\gamma = 0$, then

$$\phi_j = \frac{1}{\Gamma(\xi)} \sum_{k=1}^{j} \int_{x_{k-1}}^{x_k} (x_j - s)^{\xi-1} \left(1 - \phi_{k-1}^2\right) ds$$
$$= \frac{\Delta^{\xi}}{\Gamma(\xi+1)} \sum_{k=1}^{j} \left(1 - \phi_{k-1}^2\right)\left((j-k+1)^{\xi} - (j-k)^{\xi}\right)$$

As a special case, if we take $\xi = 1$, then

$$\phi_j = \Delta \sum_{k=1}^{j} (1 - \phi_{k-1}^2)$$

Direct calculations imply that

$$\phi_0 = 0, \phi_1 = \Delta, \phi_2 = 2\Delta - \Delta^3, \phi_3 = 3\Delta - 5\Delta^3 + 4\Delta^5 - \Delta^7, \ldots.$$

The exact solution when $\xi = 1$ is $\phi_e(x) = \frac{e^{2x}-1}{e^{2x}+1}$. In Fig. 1, we plot the graph of the exact and approximate solutions when $\xi = 1$. Now, we compare between the absolute error in our results with Adams Bashforth method of order 2 (AM2) [9], variational iteration method (VIM) [11], and modified homotopy perturbation method (MHPM) [7] in Table 1. In Fig. 2, we plot the graph of approximate solutions when $\xi = 0.75, 0.85, 0.90, 0.95, 1$, and the exact solution when $\xi = 1$.

Example 5.2. Consider the following FIVP [10]:

$$D^{\xi}\phi(x) = (1 - \phi(x))^4, \phi(0) = 0$$

Let us approximate $\phi(x)$ as in Eq. (4.8)

$$\phi_L(x) = \sum_{j=0}^{L-1} \phi_j \pi_j(x)$$

where ϕ_l is given by formula (4.11) as

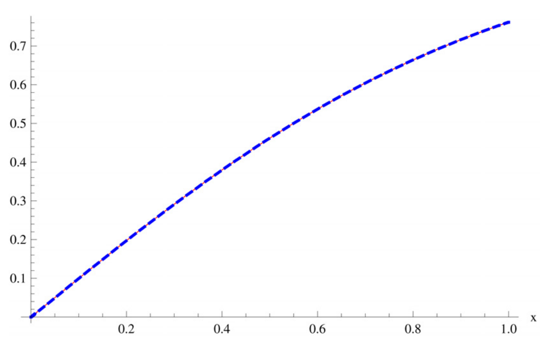

Fig. 1 The exact and approximate solutions for $\xi = 1$

Table 1 Comparison between absolute errors for $\xi = 1$ for Example 5.1

x	Our results	AM2	VIM	MHPM
0.1	0.0000×10^{0}	0.0000×10^{0}	0.0000×10^{0}	0.0000×10^{0}
0.2	$4.940, 52 \times 10^{-7}$	9.5000×10^{-5}	0.0000×10^{0}	0.0000×10^{0}
0.3	$1.909, 14 \times 10^{-7}$	1.8121×10^{-4}	8.0000×10^{-6}	0.0000×10^{0}
0.4	$4.057, 48 \times 10^{-7}$	2.5921×10^{-4}	5.7000×10^{-5}	4.0000×10^{-6}
0.5	$6.669, 752 \times 10^{-7}$	3.2978×10^{-4}	2.5800×10^{-4}	3.9000×10^{-5}
0.6	$4.446, 26 \times 10^{-7}$	3.9363×10^{-4}	8.7400×10^{-4}	1.9200×10^{-4}
0.7	$1.121, 06 \times 10^{-7}$	4.5139×10^{-4}	2.4010×10^{-3}	7.3600×10^{-4}
0.8	$1.442, 60 \times 10^{-7}$	5.0366×10^{-4}	5.6590×10^{-3}	2.3300×10^{-3}
0.9	$1.752, 13 \times 10^{-7}$	5.5094×10^{-4}	1.1842×10^{-2}	6.3780×10^{-3}
1	$1.761, 59 \times 10^{-7}$	5.9372×10^{-4}	2.2532×10^{-2}	1.5562×10^{-2}

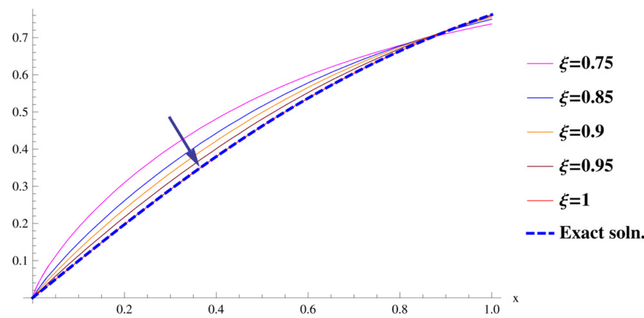

Fig. 2 The approximate solutions for different values of ξ

$$\phi_j = \frac{1}{\Gamma(\xi)} \sum_{k=1}^{j} \int_{x_{k-1}}^{x_k} (x_j - s)^{\xi-1} (1 - \phi_{k-1})^4 ds$$
$$= \frac{\Delta^{\xi}}{\Gamma(\xi+1)} \sum_{k=1}^{j} (1 - \phi_{k-1})^4 \left((j-k+1)^{\xi} - (j-k)^{\xi}\right)$$

As a special case, if we take $\xi = 1$, then

$$\phi_j = \Delta \sum_{k=1}^{j} (1 - \phi_{k-1})^4$$
$$= \phi_{j-1} + \Delta(1 - \phi_{j-1})^4$$

Direct calculations imply that

$$\phi_0 = 0, \phi_1 = \Delta, \phi_2 = 2\Delta - 4\Delta^2 + 6\Delta^3 - 4\Delta^4 + \Delta^5, \ldots.$$

The exact solution when $\xi = 1$ is $\phi_e(x) = 1 - \frac{1}{\sqrt[3]{1+3x}}$. In Fig. 3, we plot the graph of the exact and approximate solutions when $\xi = 1$. Now, we compare between the absolute error in our results with Adams Bashforth method of order 2 (AM2) [9], spline method (SM)

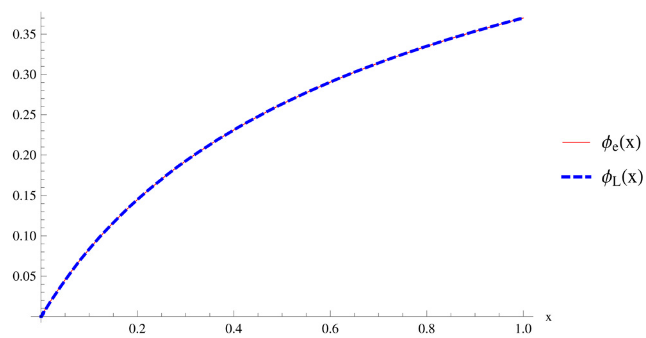

Fig. 3 The exact and approximate solutions for $\xi = 1$

[10] in Table 2. In Fig. 4, we plot the graph of approximate solutions when $\xi = 0.75, 0.85, 0.90, 0.95, 1,$ and the exact solution when $\xi = 1$.

In the next example, we will implement our numerical technique to system of two equations to show the possibility of generalize it to system of fractional equations.

Example 5.3. Consider the following system of equation [9,12]:

$$D^{\xi}\phi(x) = \phi(x) + \psi(x), \phi(0) = 0$$
$$D^{\xi}\psi(x) = -\phi(x) + \psi(x), \psi(0) = 1$$

Let us approximate $\phi(x)$ and $\psi(x)$ as in Eq. (4.8)

$$\phi_L(x) = \sum_{j=0}^{L-1}\phi_j\pi_j(x), \psi_L(x) = \sum_{j=0}^{L-1}\psi_j\pi_j(x)$$

where ϕ_j and ψ_j are given by formula (4.11) as

$$\phi_j = \frac{1}{\Gamma(\xi)}\sum_{k=1}^{j}(\phi_{k-1} + \psi_{k-1})\int_{x_{k-1}}^{x_k}(x_j - s)^{\xi-1}ds$$
$$\psi_j = 1 + \frac{1}{\Gamma(\xi)}\sum_{k=1}^{j}(\psi_{k-1} - \phi_{k-1})\int_{x_{k-1}}^{x_k}(x_j - s)^{\xi-1}ds$$

Perform the integration in the last two equations to get

$$\phi_j = \frac{\Delta^{\xi}}{\Gamma(\xi+1)}\sum_{k=1}^{j}(\phi_{k-1} + \psi_{k-1})\left((j-k+1)^{\xi} - (j-k)^{\xi}\right)$$
$$\psi_j = 1 + \frac{\Delta^{\xi}}{\Gamma(\xi+1)}\sum_{k=1}^{j}(\psi_{k-1} - \phi_{k-1})\left((j-k+1)^{\xi} - (j-k)^{\xi}\right)$$

As a special case, if we take $\xi = 1$, then

Table 2 Comparison between absolute errors for $\xi = 1$ for Example 5.2

x	Our results	AM2	SM
0.1	0.0000×10^{0}	8.6456×10^{-5}	9.3660×10^{-5}
0.2	1.2356×10^{-7}	8.0214×10^{-5}	1.1397×10^{-4}
0.3	1.6773×10^{-7}	7.7033×10^{-5}	1.1162×10^{-4}
0.4	1.8210×10^{-7}	7.5360×10^{-5}	1.0297×10^{-4}
0.5	1.8393×10^{-7}	6.4478×10^{-5}	9.3223×10^{-5}
0.6	1.8022×10^{-7}	6.4037×10^{-5}	8.4033×10^{-5}
0.7	1.1741×10^{-7}	5.3851×10^{-5}	7.5850×10^{-5}
0.8	1.6701×10^{-7}	4.3814×10^{-5}	6.8710×10^{-5}
0.9	1.1524×10^{-7}	3.3865×10^{-5}	6.2520×10^{-5}
1	1.1738×10^{-7}	3.1967×10^{-5}	5.7152×10^{-5}

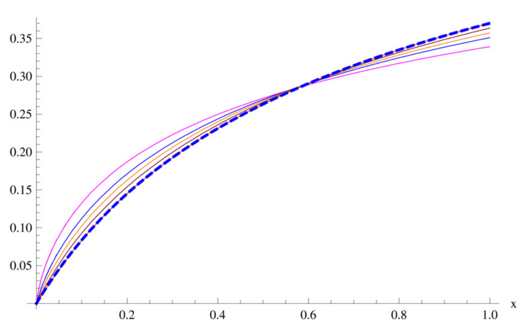

Fig. 4 The approximate solutions for different values of ξ

$$\phi_j = \Delta\sum_{k=1}^{j}(\phi_{k-1} + \psi_{k-1}) = \phi_{j-1} + \Delta(\phi_{j-1} + \psi_{j-1})$$
$$\psi_j = 1 + \Delta\sum_{k=1}^{j}(\psi_{k-1} - \phi_{k-1}) = \psi_{j-1} + \Delta(\psi_{j-1} - \phi_{j-1})$$

Direct calculations implies that

$$\phi_0 = 0, \quad \phi_1 = \Delta, \quad \phi_2 = 2\Delta + 2\Delta^2, \quad \phi_3 = 3\Delta + 6\Delta^2 + 4\Delta^3, \ldots$$
$$\psi_0 = 1, \quad \psi_1 = 1 + \Delta, \quad \psi_2 = 1 + 2\Delta + 2\Delta^2,$$
$$\phi_3 = 1 + 3\Delta + 6\Delta^2 + 4\Delta^3, \ldots.$$

The exact solutions when $\xi = 1$ are $\phi_e(x) = e^x\sin(x)$ and $\psi_e(x) = e^x\cos(x)$. In Fig. 5, we plot the graph of the exact and approximate solutions when $\xi = 1$. In Figs. 6 and 7, we plot the graph of approximate solutions of ϕ and ψ, respectively, when $\xi = 0.75, 0.85, 0.90, 0.95, 1,$ and the exact solution when $\xi = 1$. Let us define the maximum error as follows:

$$error = \max\left\{\left\|\begin{pmatrix} D^{\xi}\phi(x_i) - \phi(x_i) - \psi(x_j) \\ D^{\xi}\psi(x_i) + \phi(x_i) - \psi(x_j)\end{pmatrix}\right\| : x_j = \frac{j-1}{m}, j = 1, 2, \ldots, 11\right\}$$

Then, we compare the error in our approach with Refs. [9] and [12] and results are reported in Table 3.

Example 5.4. Consider the following FIVP [10]:

$$D^{\frac{1}{2}}\phi(x) + \phi(x) = r(x), \phi(0) = 0, x > 0 \tag{5.1}$$

where

$$r(x) = x^4 - \frac{x^3}{2} - \frac{3x^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} + \frac{24x^{\frac{7}{2}}}{\Gamma(\frac{9}{2})}$$

is $\frac{1}{2}$-polynomial of degree eight, $\alpha = 1$, and $\gamma = 0$. Theorem 3.3 yields to

$$a_5 = \frac{-3}{\Gamma(\frac{7}{2})}, a_6 = \frac{1}{2}, a_7 = \frac{24}{\Gamma(\frac{9}{2})}, a_8 = 1, a_k = 0, k = 0, 1, \ldots, 4$$

and

$$\begin{aligned}\phi(x) &= \frac{-3}{\Gamma(\frac{7}{2})}\Gamma(\frac{7}{2})x^3E_{\frac{1}{2},4}(-x^{\frac{1}{2}}) + \frac{1}{2}\Gamma(4)x^{\frac{7}{2}}E_{\frac{1}{2},\frac{9}{2}}(-x^{\frac{1}{2}}) \\ &\quad + \frac{24}{\Gamma(\frac{9}{2})}\Gamma(\frac{9}{2})x^4E_{\frac{1}{2},5}(-x^{\frac{1}{2}}) + \Gamma(5)x^{\frac{9}{2}}E_{\frac{1}{2},\frac{11}{2}}(-x^{\frac{1}{2}}) \\ &= -3x^3E_{\frac{1}{2},4}(-x^{\frac{1}{2}}) + 3x^{\frac{7}{2}}E_{\frac{1}{2},\frac{9}{2}}(-x^{\frac{1}{2}}) \\ &\quad + 24x^4E_{\frac{1}{2},5}(-x^{\frac{1}{2}}) + 24x^{\frac{9}{2}}E_{\frac{1}{2},\frac{11}{2}}(-x^{\frac{1}{2}}) \\ &= -3x^3\left(\sum_{k=0}^{\infty}\frac{(-1)^kx^{\frac{k}{2}}}{\Gamma(\frac{1}{2}k+4)} - \sum_{k=0}^{\infty}\frac{(-1)^kx^{\frac{k+1}{2}}}{\Gamma(\frac{1}{2}(k+1)+4)}\right) \\ &\quad + 24x^4\left(\sum_{k=0}^{\infty}\frac{(-1)^kx^{\frac{k}{2}}}{\Gamma(\frac{1}{2}k+5)} - \sum_{k=0}^{\infty}\frac{(-1)^kx^{\frac{k+1}{2}}}{\Gamma(\frac{1}{2}(k+1)+5)}\right) \\ &= -3x^3\left(\sum_{k=0}^{\infty}\frac{(-1)^kx^{\frac{k}{2}}}{\Gamma(\frac{1}{2}k+4)} - \sum_{l=1}^{\infty}\frac{(-1)^{l-1}x^{\frac{l}{2}}}{\Gamma(\frac{1}{2}l+4)}\right) \\ &\quad + 24x^4\left(\sum_{k=0}^{\infty}\frac{(-1)^kx^{\frac{k}{2}}}{\Gamma(\frac{1}{2}k+5)} - \sum_{l=1}^{\infty}\frac{(-1)^{l-1}x^{\frac{l}{2}}}{\Gamma(\frac{1}{2}(l)+5)}\right)\end{aligned}$$

Cancel the similar terms to get

$$\phi(x) = \frac{-3x^3}{\Gamma(4)} + \frac{24x^4}{\Gamma(5)} = \frac{-x^3}{2} + x^4$$

It is easy to verify that $\phi(x) = \frac{-x^3}{2} + x^4$ is a solution to the IVP (5.1). In Ref. [10], the authors solved the same Problem and their absolute error is given in Table 4.

Example 5.5. Consider the following FIVP:

$$D^{\xi}\phi(x) + 3\phi(x) = r(x), \phi(0) = 1, x > 0 \qquad (5.2)$$

In the following, we study FIVP (5.2) for different choices of $r(x)$.

Case 1

If $r(x) = E_{\xi}(2x^{\xi})$, then using Theorem 3.4, we have

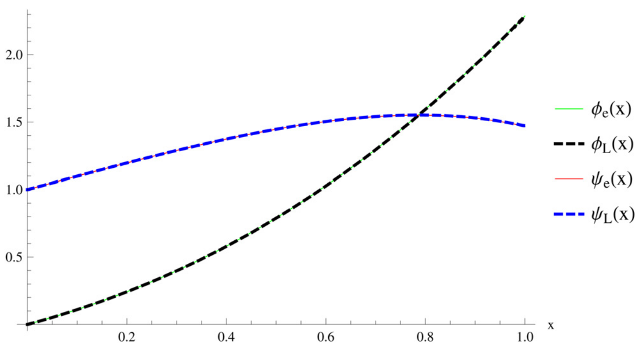

Fig. 5 The exact and approximate solutions for $\xi = 1$

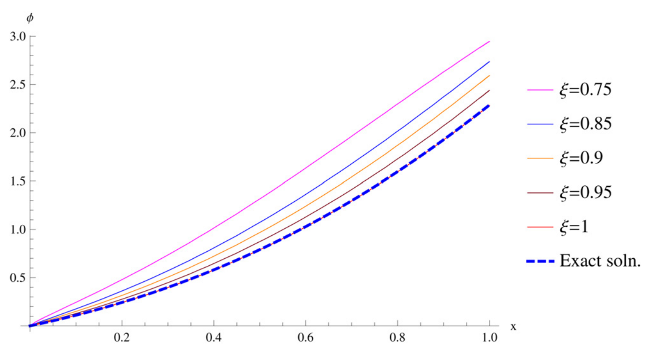

Fig. 6 The approximate solutions of ϕ for different values of ξ

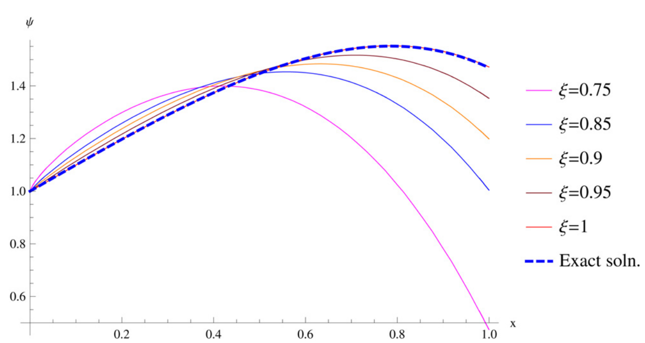

Fig. 7 The approximate solutions of ψ for different values of ξ

Table 3 Comparing the error with Refs. [10] and [12] for ξ=0.4, 0.5, 0.7 for Example 5.3

ξ	Error in Ref. [10]	Error in Ref. [12]
0.3	2.3848×10^{-4}	8.407×10^{-2}
0.5	2.2925×10^{-5}	1.668×10^{-14}
0.7	2.1809×10^{-6}	7.124×10^{-3}

$$\phi(x) = E_{\xi}(-3x^{\xi}) + \frac{2}{5}x^{\xi}E_{\xi,\xi+1}(2x^{\xi}) + \frac{3}{5}x^{\xi}E_{\xi,\xi+1}(-3x^{\xi})$$

Then

$$\begin{aligned}\lim_{\xi\to 1}\phi(x) &= \lim_{\xi\to 1}\left(E_{\xi}(-3x^{\xi}) + \frac{2}{5}x^{\xi}E_{\xi,\xi+1}(2x^{\xi}) + \frac{3}{5}x^{\xi}E_{\xi,\xi+1}(-3x^{\xi})\right)\\ &= e^{-3x} + \frac{2}{5}\left(\frac{e^{2x}-1}{2}\right) - \frac{3}{5}\left(\frac{e^{-3x}-1}{3}\right) = \frac{e^{2x}+4e^{-3x}}{5}\end{aligned}$$

since

$$\begin{aligned}\lim_{\xi\to 1}x^{\xi}E_{\xi,\xi+1}(\mu z) &= xE_{1,2}(\mu z)\\ &= \sum_{k=0}^{\infty}\frac{\mu^k z^{k+1}}{\Gamma(k+2)} = \sum_{n=1}^{\infty}\frac{\mu^{n-1}z^n}{n!} = \frac{e^{\mu z}-1}{\mu}\end{aligned}$$

One can easily verify that $\lim_{\xi\to 1}\phi(x)$ is a solution to the IVP (5.2) when ξ approaches to one which is

$$\phi'(x) + 3\phi(x) = e^{2x}, \phi(0) = 1, x > 0$$

Case 2

If $r(x) = \sin_{\xi}(2x^{\xi})$, then using Theorem 3.5, we have

$$\begin{aligned}\phi(x) &= E_{\xi}(-3x^{\xi})\\ &\quad + \frac{1}{13}\left(3\sin_{\xi}(2x^{\xi}) + 2\left(1-\cos_{\xi}(2x^{\xi})\right) - 6x^{\xi}E_{\xi,\xi+1}(-3x^{\xi})\right)\end{aligned}$$

Then

$$\begin{aligned}\lim_{\xi\to 1}\phi(x) &= \lim_{\xi\to 1}\Bigg(E_{\xi}(-3x^{\xi}) + \frac{1}{13}\Big(3\sin_{\xi}(2x^{\xi}) + 2\left(1-\cos_{\xi}(2x^{\xi})\right)\\ &\quad -6x^{\xi}E_{\xi,\xi+1}(-3x^{\xi})\Big)\Bigg)\\ &= e^{-3x} + \frac{1}{13}\left(3\sin(2x) + 2(1-\cos(2x)) + 6\frac{e^{-3x}-1}{3}\right)\\ &= \frac{15}{13}e^{-3x} + \frac{3}{13}\sin(2x) - \frac{2}{13}\cos(2x)\end{aligned}$$

since

$$\lim_{\xi\to 1}\sin_{\xi}(\mu x^{\xi}) = \sin(\mu x), \lim_{\xi\to 1}\cos_{\xi}(\mu x^{\xi}) = \cos(\mu x)$$

One can easily verify that $\lim_{\xi\to 1}\phi(x)$ is a solution to the IVP (5.2) when ξ approaches to one which is

$$\phi'(x) + 3\phi(x) = \sin(2x), \phi(0) = 1, x > 0$$

6 Conclusions and Limitations

In this paper, we present a novel numerical approach for solving nonlinear problems with a singular kernel. We prove the existence

Table 4 Absolute errors in Ref. [10] for $\xi = \frac{1}{2}$ for Example 5.4

x	Absolute error in Ref. [10]
0.1	4.0000×10^{-4}
0.2	1.1668×10^{-3}
0.3	2.9299×10^{-4}
0.4	4.5080×10^{-4}
0.5	2.2930×10^{-3}
0.6	6.5464×10^{-3}
0.7	1.5311×10^{-2}
0.8	2.7885×10^{-2}
0.9	4.5954×10^{-2}
1	6.832×10^{-2}

and the uniqueness of the solution for these models as well as the uniform convergence of the function sequence produced by our novel approach to the unique solution. Additionally, we offer a closed form and prove these results for a specific class of these problems where the free term is a fractional polynomial, a fractional exponential, or a fractional trigonometric function. These findings are new to the best of our knowledge. To demonstrate the effectiveness of our numerical method and how to apply our theoretical findings, we solved a number of examples. Comparisons with various researchers are reported. We end this article by the following remarks:

(1) We test our new numerical technique by three examples. Since the exact solution is unknown for any $0 < \xi < 1$, we plot the exact and approximate solutions for $\xi = 1$ in Figs. 1 and 3. We notice that they are coincide for $\xi = 1$. This gives us a numerical evidence that the new approach is working efficiently.
(2) We compare our results with the results in Refs. [7] and [9–11] using AM2, VIM, MHPM, and SM methods. These results are reported in Tables 1 and 2 and they show that our method has more accuracy and less computational cost.
(3) We sketch the graph of the approximate solutions for $\xi = 0.75, 0.85.0.95, 1$, and the exact solution at $\xi = 1$. We notice that the approximate solutions converges to the exact solution at $\xi = 1$ when ξ approaches to one. This fact is clear from Figs. 2 and 4.
(4) We generalize the idea of the numerical method into a system of two equations. This is just to show that the method can be generalized and will be suitable for future work and investigation. We compare our error with the errors in Refs. [10] and [12] as in Table 3. Results show that our method has more accuracy and less computational cost. In addition, we sketch the graphs of ϕ and ψ for several values of ξ. We notice that the approximate solution converges to the exact solution at $\xi = 1$ when ξ approaches to one. Also, the approximate and exact solutions are coincide when $\xi = 1$. This fact is clear from Figs. 5–7.
(5) In Example 5.4, we compare between our results with Refs. [10] and [12]. We use Theorem 3.3 to find the exact solution while the max error in Ref. [10] is of order 10^{-6} and in Ref. [12] is of order 10^{-3} when $\xi = 0.3, 0.7$ and of order 10^{-14} when $\xi = 0.5$. These results are reported in Table 4.
(6) We implement Theorems 3.4 and 3.5 in the last example. We find the exact solution for two problems mentioned in this example.
(7) As we see from the previous discussion that we can find the exact solution for a class of FIVP using Theorems 3.1–3.6. Also, our numerical technique is accurate and can be generalize to other problems such as fractional partial equations and system of FIVP.

6.1 Limitations of the Proposed Method. The proposed method, particularly in the context of solving differential and integral equations, is a powerful mathematical tool. However, like any method, it has its limitations, and there are situations where it may not be the most effective or suitable approach. Here are some limitations and situations where the operational matrix method might face challenges or fail:

1. The method may become less effective when dealing with highly complex or nonlinear problems, as the operational matrix method is often more suitable for problems that are not highly nonlinear systems.
2. Certain boundary conditions may not be easily incorporated into the operational matrix framework, leading to difficulties in solving problems with nonstandard or complex boundary conditions. We attempted to use the linear shooting method, but sometimes it is not possible.
3. The method might struggle when the solution contains discontinuities or singularities, as it relies on approximating functions using basis functions that may not capture such behavior accurately.
4. Additionally, the proposed method may also struggle when the nonlinear term is not Lipschitz continuous or when the minimum value of the Lipschitz constant is very large. This will affect either the speed of convergence or the existence of the numerical solution.

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