Event-triggered consensus of multi-agent systems under jointly connected topology

XIA CHEN, FEI HAO*, AND MINGYUAN SHAO

School of Automation Science and Electrical Engineering, Beihang University, Beijing 100191, P. R. China and The Seventh Research Division, Beihang University, Beijing 100191, P. R. China

*Corresponding author: fhao@buaa.edu.cn  xchen@asee.buaa.edu.cn

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This paper investigates event-triggered leaderless and leader-following consensus problems of multi-agent systems under jointly connected topology. A state-dependent event condition is proposed for each agent, which only relies on its own state and the last sampled data of itself and its neighbours. Based on the Lyapunov technique and algebraic graph theory, the designed event-triggered control strategies are proved to solve the leaderless and leader-following consensus problems when the topology is jointly connected. Moreover, such strategies can exclude Zeno-behaviour. Simulations illustrate the effectiveness of the theoretical results.

Keywords: event-triggered control; event condition; multi-agent systems; leader-following.

1. Introduction

Coordination control of multi-agent systems (MASs) has received great attention in recent years for its broad applications in distributed sensor networks, attitude alignment of clusters of satellites, cooperative control of unmanned air vehicles and so on. For coordinated control, consensus is a critical and fundamental problem, which can be classified as leaderless consensus and leader-following consensus. Leaderless consensus means the agreement of a group of agents on their common states via local interaction, while leader-following consensus means that the agents follow an object set by a virtual leader.

The consensus problem has been intensively investigated; see Vicsek et al. (1995), Jadlbabie et al. (2003), Ren & Beard (2004), Hong et al. (2007), Ni & Cheng (2010) and the references therein. Vicsek et al. presented a simple discrete-time model of autonomous agents and simulated complex dynamics of the model in Vicsek et al. (1995). In Jadlbabie et al. (2003), Jadlbabie et al. proposed an explicit discrete-time consensus algorithm for the consensus behaviour of the Vicsek model. It was proved that if the graph containing the agents and the leader is jointly connected, the leader-following consensus would be achieved. Ren & Beard (2004) showed that consensus under dynamically changing interaction topologies would be achieved if and only if the union of a collection of interaction graphs across some time intervals has a spanning tree frequently enough. In Hong et al. (2007), for a MAS under jointly connected topology, neighbour-based protocols were provided to solve the leader-following consensus with a static leader. Ni & Cheng (2010) considered the leader-following consensus problem for high-order MASs under fixed and jointly connected topologies.

With the development of digital microprocessors, controls tend to be implemented on digital platforms. Traditionally, the control tasks including measurement, communication and control updates are
executed periodically. To guarantee performance for all operation points, the constant sampling period is usually taken to be a conservative value. Thus, it leads to unnecessary usage of computational resources, communication resources and limited bandwidth. In recent years, an alternative approach so-called event-triggered control is applied. Under an event-triggered control scheme, the control task is only executed when ‘needed’. Generally, such a need can be described as a mathematical condition related with the state of the system. Such a condition is named event condition in Velasco et al. (2008). It is shown that event-triggered control could preserve desired performances with fewer samples than time-triggered control; see Lemmon et al. (2007), Henningsson et al. (2008), Wang & Lemmon (2009), Lunze & Lehmann (2010), Mazo Jr et al. (2010) and Chen & Hao (2013).

Recently, event-triggered control has been investigated for consensus problems of MASs. First, event-triggered control was researched under the fixed topology. Event-triggered implementation of the consensus protocol for first-order MASs was developed in Dimarogonas et al. (2012). It has been proved that there is no Zeno-behaviour, which means there is an accumulation point of events on the time axis. However, in Seyboth et al. (2013), it was pointed out that the approach in Dimarogonas et al. (2012) requires continuous communications among neighbouring agents. Thus, the authors proposed static and time-dependent trigger functions to relax this requirement. For high-order MASs, event-based control was examined in Liu & Chen (2010). Besides, event-triggered control was investigated for discrete-time MASs in Chen & Hao (2012).

Besides, consensus problems via event-triggered control were studied when the topology is switching. In Liu & Chen (2011), after assuming that the digraph in each topology switching interval is strongly connected and balanced, an event-triggered control scheme was designed to solve the consensus problem. In Seyboth (2010), under the jointly connected topology, an event-triggered control law with a static trigger function was given which renders the disagreement vector (between the state of each agent and the average consensus point) asymptotically to a bounded region around the average consensus point. In Shi & Johansson (2011), a time-dependent trigger function was applied to ensure the consensus of MASs when the interconnection topology is jointly connected. The aforementioned works on the event-triggered consensus mainly concentrate on the leaderless consensus. For leader-following consensus under fixed interaction topology, an event-triggered tracking control approach was presented for second-order leader-following MASs in Hu et al. (2011).

Motivated by the analysis above, we will further study the event-triggered leaderless and leader-following consensus problems of MASs under jointly connected topology in this paper. The contributions of the paper are as follows. First, the state-dependent event condition is designed. It is noted in Fan et al. (2013) that the state-dependent event condition is better and more natural than the static trigger function in Seyboth (2010) and the time-dependent trigger function in Shi & Johansson (2011). The main reason is that the system cannot achieve asymptotic convergence in Seyboth (2010) and the convergence rate is governed by an external signal in Shi & Johansson (2011). Moreover, the simulations demonstrate that the new event condition can lead to larger average sampling period than those in references Seyboth (2010) and Shi & Johansson (2011). Secondly, for each agent, the event condition relies on its current state, and last sampled data of itself and its neighbours. Therefore, it needs neither global states nor continuous communications among neighbouring agents. Thirdly, to the best of the authors’ knowledge, the problem of event-triggered leader-following consensus has not been addressed when the graph is switching. Thus, in this paper, leader-following consensus under the jointly connected topology is studied via event-triggered control. Fourthly, since the consensus analysis and the design of event condition become difficult when the graph is jointly connected rather than fixed or always connected, we deal with the problem based on components of the graph.
The remainder of the paper is organized as follows. In Section 2, some necessary preliminaries are provided. Event-triggered leaderless consensus for MASs is discussed in Section 3. Moreover, Zeno behaviour is avoided. Then, leader-following consensus for MASs via event-triggered control is addressed in Section 4. Numerical simulations are given in Section 5 to verify the effectiveness of the proposed results. Finally, some conclusions are drawn in Section 6.

2. Notation and preliminaries

Denote by $1_N$ the $N \times 1$ all-one vector. For a vector $x$, $\|x\|$ represents its Euclid norm. If $y$ is a scalar, $|y|$ is its absolute value. For a matrix $A$, $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ denote its maximum and minimum eigenvalues, respectively; $|S|$ is the number of elements of the set $S$.

First, we consider a MAS composed of $N$ agents without a leader. The interconnection topology of $N$ agents can be described by an undirected graph $G = (V, E)$, where $V = \{1, 2, \ldots, N\}$ and $E = \{(i, j) : i, j \in V\} \subset V \times V$ are the sets of vertices and edges of graph $G$, respectively. Recall some concepts in graph theory. If $(j, i) \in E$, $j$ is called a neighbour of $i$. The neighbour set of agent $i$ is defined by $N_i = \{j \in V | (j, i) \in E, j \neq i\}$. The adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ of $G$ is defined element-wise with $a_{ij} = 1$ if $(j, i) \in E$, and $a_{ij} = 0$ otherwise. The degree matrix $D$ of $G$ is given by $D = \text{diag}(d_1, d_2, \ldots, d_N)$ with $d_i = |N_i|$. The Laplacian matrix $L$ is defined as $L = D - A$. For an undirected graph, $L$ is symmetric and $L1_N = 0$. A path is a sequence of connected edges in a graph. Moreover, if there is a path between any two vertices of $G$, $G$ is called connected. A component of graph $G$ is a connected subgraph that is maximal (i.e. no more vertices can be added while preserving its connectedness; see more details in Ni & Cheng (2010)).

For the leader-following case, we consider a graph $\tilde{G}$ associated with the system consisting of $N$ agents and a leader represented by vertex 0. The dynamics of the leader is independent of the followers. The leader adjacency matrix is defined as $B \in \mathbb{R}^{N \times N}$ with diagonal elements $b_i = 1$ if agent $i$ is a neighbour of the leader, and $b_i = 0$ otherwise.

When the interaction topology varies with time, we consider all possible graphs $\{\tilde{G}_p : p \in \mathcal{P}\}$, where $\mathcal{P}$ is an index set for all graphs defined on $N$ agents and the leader. Similarly, $\{G_p : p \in \mathcal{P}\}$ denotes subgraphs defined on $N$ agents. We use $\sigma(t) : [0, \infty) \to \mathcal{P}$ as a switching signal that determines the interaction topology. It is assumed that $\sigma(t)$ switches finite times in any bounded time interval. Denote the underlying graph (subgraph) by $\tilde{G}_{\sigma(t)}(G_{\sigma(t)})$. Consequently, $N_i(\sigma(t))$ represents the neighbour set of agent $i$ and the Laplacian matrix associated with $G_{\sigma(t)}$, respectively; $B_{\sigma(t)}$ denotes the leader adjacency matrix of $\tilde{G}_{\sigma(t)}$.

Consider an infinite sequence of non-empty, bounded and contiguous time intervals $[t_k, t_{k+1})$, $k = 0, 1, \ldots$, with $t_0 = 0, t_{k+1} - t_k \leq T$ for some constant $T > 0$. Assume that, in each interval $[t_k, t_{k+1})$, there exists a sequence of non-overlapping subintervals $[t_k^0, t_k^1] \cdots [t_k^j, t_k^{j+1}] \cdots [t_k^{m_k-1}, t_k^m], t_k = t_k^0, t_{k+1} = t_k^m$, satisfying $t_k^{j+1} - t_k^j \geq \tau$, $0 \leq j \leq m_k - 1$ for some integer $m_k \geq 0$ and a given constant $\tau > 0$ such that, during each of such subintervals, the interaction topology is fixed.

**Definition 2.1** $\tilde{G}_{\sigma(t)}$ is called jointly connected across the time interval $[t, t+S)$, $S > 0$ if the joint graph $\bigcup_{s \in [t,t+S)} \tilde{G}_{\sigma(s)}$ is connected. $G_{\sigma(t)}$ is called jointly connected across the time interval $[t, t+S), S > 0$ if the joint graph $\bigcup_{s \in [t,t+S)} G_{\sigma(s)}$ is connected.

**Lemma 2.1** (Lin et al., 2005) The graph $G$ is connected if and only if the Laplacian matrix $L$ of $G$ has a simple zero eigenvalue.
From Lemma 2.1, the eigenvalues of a connected graph $G$ can be described as $0 = \lambda_1(L) < \lambda_2(L) \leq \cdots \leq \lambda_N(L)$.

**Lemma 2.2** *(Godsil & Royle (2001))*. For a connected graph $G$ that is undirected with the Laplacian matrix $L$ and the vector $x$ satisfying $1^T x = 0$, $\min_{x \neq 0} (x^T L x / x^T x) = \lambda_2(L)$ holds.

3. Event-triggered leaderless consensus under jointly connected topology

The dynamics of each agent under consideration is

$$\dot{x}_i(t) = u_i(t),$$

where $x_i(t) \in \mathbb{R}$ and $u_i(t) \in \mathbb{R}$ are the state vector and the control protocol of agent $i$, respectively. It is said that $u_i$ asymptotically solves the consensus problem of (3.1), if and only if the states of agents satisfy $\lim_{t \to \infty} |x_i(t) - x_j(t)| = 0$, $i, j = 1, \ldots, N$, for any initial condition $x_i(0)$, $i = 1, \ldots, N$. In Olfati-Saber et al. (2007), the protocol is proposed as

$$u_i(t) = \sum_{j \in N_i(\sigma(t))} (x_j(t) - x_i(t)),$$

which has been proved to asymptotically solve the consensus problem of (3.1) when the graph $G$ is connected. In this section, we reformulate the protocol taking into account event-triggered control strategy to solve the consensus problem of (3.1) under a switching topology. The following assumption will be used throughout this section.

**Assumption 3.1** The graph $G_{\sigma(t)}$ is jointly connected across the time interval $[t_k, t_{k+1})$.

In this paper, we assume that there are no communication and actuation update delays. By ‘event’, we mean a communication transmission is triggered once an event condition is satisfied. The event condition can be described by a mathematical inequality. Thus, a communication transmission, which can be represented by a sampling, is triggered when this inequality is violated. Let $r^{t_k}_k$ be the $k$th sampling time instant (the time when the $k$th communication transmission is triggered) of agent $i$. Then $T^t_k = r^{t+1}_k - r^t_k$ represents the $k$th sampling period of agent $i$. Denote by $r^t_k$ the $k$th sampling time instant of the MAS. It is easy to see that the time sequence $\{r^t_k\}$ contains all $r^t_k$ of each agent $i$, $i = 1, 2, \ldots, N$. The protocol is computed based on the last sampled data and held constant or piece-wise constant during each sampling period, i.e.

$$u_i(t) = \sum_{j \in N_i(\sigma(t))} (x_j(t) - x_i(t)), \quad t \in [r^t_k, r^{t+1}_k),$$

where $i = 1, \ldots, N$, $k(t) = \arg \min_{k} \{ t - r^t_k | t \geq r^t_k \}$, and $r^{t}_k$ is the latest sampling time of agent $j$ before $t$.

The communications over the network can be summed as follows. The agent $i$ broadcasts its state $x_i(t)$ during $[r^t_k, r^{t+1}_k)$. Each agent updates the protocol $u_i$ at the sampling times of itself and its neighbours. When the interaction topology switches, the control input of the agent which loses or gains a new edge in the graph is updated.

For agent $i$, we introduce a state measurement error $e_i(t) = x_i(t) - x_i(t), i = 1, \ldots, N, t \in [r^t_k, r^{t+1}_k)$. It follows that $u_i(t) = \sum_{j \in N_i(\sigma(t))} (x_j(t) - x_i(t)) + \sum_{j \in N_i(\sigma(t))} (e_j(t) - e_i(t)), i = 1, \ldots, N$. By introducing
x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^\top and e(t) = [e_1(t), e_2(t), \ldots, e_N(t)]^\top, we have
\[ \dot{x}(t) = -L_{\sigma(t)}x(r_k) = -L_{\sigma(t)}(x(t) + e(t)). \] (3.4)

**Remark 3.1** There is a slight abuse of notation in (3.4). \( x(r_k) \) is not the value of \( x(t) \) at \( t = r_i \) but \((x_1(r_i^1), x_2(r_i^2), \ldots, x_N(r_i^N))^\top\), where \( r_i^j \leq r_k \) (\( i = 1, 2, \ldots, N \)) is the latest sampling time of agent \( i \) before or the same as \( r_k \).

Define \( \tilde{x}(t) = (1/N) \sum_{i=1}^N x_i(t) \). As noted in the preliminaries, for any undirected graph, \( L \) is symmetric and \( L1_N = 0 \). Consequently, \( 1_N^\top L = 0^\top \) holds for any switching undirected topology. Thus, \( \dot{\tilde{x}}(t) = (1/N)1_N^\top \tilde{x}(t) = -(1/N)1_N^\top L_{\sigma(t)}(x(t) + e(t)) = 0 \), which means \( \tilde{x}(t) \) is an invariant quantity. It allows decomposition of \( x(t) \) as \( x(t) = \delta(t) + \tilde{x}(t)1_N \), where \( \delta(t) \) is called the disagreement vector and satisfies \( 1_N^\top \delta(t) = 0 \). Moreover, \( e_i(t) = \delta_i(r_k) - \delta_i(t), i = 1, \ldots, N \), where \( \delta_i(t) \) is the \( i \)th element of \( \delta(t) \). Then \( \delta(t) \) evolves as
\[ \dot{\delta}(t) = -L_{\sigma(t)}\delta(r_k) = -L_{\sigma(t)}(\delta(t) + \tilde{x}(t)1_N) \] (3.5)

Since the isolated vertex in the graph has no control input, we can just concentrate on the vertices having neighbours. Towards this end, we classify the vertices according to the components. For any graph \( G_p \), we define \( S_c = \{ S^j_1, S^j_2, \ldots, S^j_p \} \) as the set of the components containing at least two vertices and \( S_o = \{ S^{q+1}_1, \ldots, S^Q_p \} \) as the set of the components having only one vertex, i.e. the isolated vertex, where \( 1 \leq q \leq Q \leq N \). Let \( V(S^j_p) \) be the vertex set of \( S^j_p \) and \( m^j_p = |V(S^j_p)| \). Without loss of generality, after rearranging the order of all agents, the Laplacian matrix \( L_p \) of \( G_p \) can be written in the form of \( L_p = \text{diag}(L^{j_1}_p, L^{j_2}_p, \ldots, L^{j_q}_p, L^{q+1}_p, \ldots, L^{Q}_p) \), where \( L^{j_q}_p \in \mathbb{R}^{m^j_p \times m^j_p} \) is the Laplacian matrix of the component \( S^{j_q}_p \). It is not difficult to see that \( L^{j_q}_p \) (\( q = 1, 2, \ldots, Q \)) has a simple zero eigenvalue and \( L^{j_q}_p = 0 \) for \( q = 1, 2, \ldots, Q \).

**Remark 3.2** For each vertex in the component \( S^j_p \in S_c \), it has no control input. Thus, we just implement event-triggered control for the vertices in the set \( S_c \).

For the convenience of analysis, we give more notation. Define the composite state vector of \( V(S^j_p) \) by \( x_j(t) = (x_{l_1}, x_{l_2}, \ldots, x_{l_{m^j_p}})^\top \in \mathbb{R}^{m^j_p} \). Accordingly, we have disagreement vector \( \delta_l(t) = (\delta_{l_1}, \delta_{l_2}, \ldots, \delta_{l_{m^j_p}})^\top = x_j(t) - \tilde{x}(t)1_{m^j_p} \in \mathbb{R}^{m^j_p} \). Defining \( \bar{x}_j(t) = (1/m^j_p)1_{m^j_p}^\top x_j(t) \), we have another disagreement vector \( \bar{\delta}_l(t) = x_j(t) - \bar{x}_j(t)1_{m^j_p} \). Obviously, \( \bar{x}_j(t) \) is an invariant quantity and \( 1_{m^j_p}^\top \bar{\delta}_l(t) = 0 \). Moreover, define \( e_l(t) = x_j(r_k) - x_j(t) = (e_{l_1}(t), \ldots, e_{l_{m^j_p}}(t))^\top \). It is easy to see that \( e_l(t) = \delta_l(r_k) - \delta_l(t) = \bar{\delta}_l(r_k) - \bar{\delta}_l(t) \) holds.

**Remark 3.3** Similarly as \( x(r_k) \), \( \delta(r_k) = (\delta_1(r_k^1), \ldots, \delta_N(r_k^N))^\top \) with \( r_i^j \leq r_k \) (\( i = 1, 2, \ldots, N \)) as the latest sampling time of agent \( i \) less than or equal to \( r_k \). The same property holds for \( x_j(r_k) \) and \( \delta_l(r_k) \).

Now, we will design the event condition for each agent. The specific structure of the event condition is given in the following results.

**Theorem 3.1** Consider MAS (3.1) under Assumption 3.1 with \( r_0 = 0 \). For each component \( S^j_p \in S_c \), the event condition is given as
\[ |e_{l_i}(t)| \leq \frac{1}{\|L^j_p\|} \sqrt{\frac{c}{2(1+c)}} \left| \sum_{l_j \in N_i} (x_{l_j}(r_k^j) - x_{l_j}(r_i^j)) \right|, \] (3.6)
where \( c = (2\rho \lambda_2(L_p^e/a\lambda_{\text{max}}(L_p^e)))(1 - 1/2a), \rho \in (0, 1) \) and \( a > \frac{1}{2}, k'(t) = \arg \min_{g} \{ t - r_k^l | t \geq r_k^l \}; \) then the protocol (3.3) asymptotically solves the consensus problem of (3.1).

**Proof.** Consider the closed-loop system (3.5). Choose the common Lyapunov function candidate, \( V(t) = \frac{1}{2} \delta^\top(t) \delta(t). \) It is easy to see that \( V(t) \) is continuous. Moreover, it is continuously differentiable at any time except for the topology switching instants \( t_k \) and the sampling time instants \( r_k. \) Define the derivative of \( V(t) \) at any time instant \( t_k \) or \( r_k \) is its right derivative.

Suppose that the active interconnection topology is \( G_p \) at time \( t. \) As mentioned before, after rearranging the order of all agents, we can rewrite the Laplacian matrix \( L_p \) as \( L_p = \text{diag}\{L_p^1, \ldots, L_p^Q \}, \) where \( L_p^i \in \mathbb{R}^{m_i \times m_i} \) is the Laplacian matrix of the component \( S_p^i. \) Consequently, the time derivative of \( V(t) \) along the trajectory of system (3.5) is

\[
\dot{V}(t) = -\delta^\top(t)L_p\delta(r_k) = -\sum_{l=1}^{Q} \delta^\top_l(t)L_p^l\delta_l(r_k)
\]

\[
= -\sum_{l=1}^{Q} (x_l(t) - \bar{x}(t)1_{m_p}^l - \bar{x}_l(t)1_{m_p}^l + \bar{x}_l(t)1_{m_p}^l)^\top L_p^l(x_l(r_k) - \bar{x}(r_k)1_{m_p}^l)
- \bar{x}_l(t)1_{m_p}^l + \bar{x}_l(t)1_{m_p}^l)
\]

\[
= -\sum_{l=1}^{Q} (\bar{\delta}_l(t) + (\bar{x}_l(t) - \bar{x}(t))1_{m_p}^l)^\top L_p^l(\bar{\delta}_l(r_k) + (\bar{x}_l(t) - \bar{x}(r_k))1_{m_p}^l).
\]

As \( L_p^l1_{m_p}^l = 0 \) and \( e_l(t) = \bar{\delta}_l(r_k) - \bar{\delta}_l(t), \) it implies \( \dot{V}(t) = -\sum_{l=1}^{Q} \delta^\top_l(t)L_p^l\delta_l(r_k) = -\sum_{l=1}^{Q} \delta^\top_l(t)L_p^l\delta_l(r_k) + e_l(t). \) Note that \( L_p^l = 0 \) for \( l = q + 1, \ldots, Q. \) Then, from the well-known inequality \( y^\top z \leq (a/2)y^\top y + (1/2a)z^\top z, \) for any \( y, z \in \mathbb{R}^a, a > 0, \) we have

\[
\dot{V}(t) = \sum_{l=1}^{q} \delta^\top_l(t)(-L_p^l\delta_l(t) - L_p^l e_l(t)) = \sum_{l=1}^{q} (-\delta^\top_l(t)L_p^l\delta_l(t) - \delta^\top_l(t)(L_p^l)^{1/2}(L_p^l)^{1/2}e_l(t))
\]

\[
\leq \sum_{l=1}^{q} \left( -\delta^\top_l(t)L_p^l\delta_l(t) + \frac{1}{2a} \delta^\top_l(t)L_p^l\delta_l(t) + \frac{a}{2} e_l^\top(t)L_p^l e_l(t) \right).
\]

It is easy to see \( \delta^\top_l(t)L_p^l\delta_l(t) = \delta^\top_l(t)(L_p^l)^{1/2}L_p^l(L_p^l)^{1/2}\delta_l(t) \leq \lambda_{\text{max}}(L_p^l)\delta_l^\top(t)L_p^l\delta_l(t). \) Meanwhile, since \((L_p^l)^{1/2}1_{m_p}^l)^\top(L_p^l)^{1/2}1_{m_p}^l = 1_{m_p}^lL_p^l1_{m_p}^l = 0, \) it implies \( 1_{m_p}^l(L_p^l)^{1/2} = 0_{m_p}^l. \) Moreover, \( 1_{m_p}^l(L_p^l)^{1/2}e_l(t) = 0. \) By Lemma 2.2, \( e_l^\top(t)L_p^lL_p^l e_l(t) = e_l^\top(t)(L_p^l)^{1/2}L_p^l(L_p^l)^{1/2}e_l(t) \geq \lambda_2(L_p^l)e_l^\top(t)L_p^l e_l(t) \) holds. Hence, we obtain

\[
\dot{V}(t) \leq \sum_{l=1}^{q} \left( -\frac{(1 - 1/2a)}{\lambda_{\text{max}}(L_p^l)} \delta_l^\top(t)L_p^lL_p^l\delta_l(t) + \frac{a}{2\lambda_2(L_p^l)} e_l^\top(t)L_p^lL_p^l e_l(t) \right).
\]

From the event condition (3.6), we have \( \|L_p^l\|_2\|e_l(t)\| \leq \sqrt{c/2(1 + \delta)}\|L_p^e x_l(r_k)\|. \) Meanwhile, \( \|L_p^l e_l(t)\| \leq \|L_p^l\|_2\|e_l(t)\| \) and \( \|L_p^e x_l(r_k)\| = \|L_p^l\delta_l(r_k)\| \) hold. Thus, we get \( \|L_p^l e_l(t)\|^2 \leq \frac{c}{2(1 + \delta)}\|L_p^e x_l(r_k)\|\|L_p^l\|_2\|e_l(t)\| \).
\begin{equation*}
(c/2(1 + c))\|L_p^T \delta_i(r_k)\|^2. \text{ It leads to}
\end{equation*}

\begin{align*}
e_i^T(t)L_p^T L_p e_i(t) & \leq -c e_i^T(t)L_p^T L_p e_i(t) + \frac{c}{2} \delta_i^T(r_k)L_p^T L_p \delta_i(r_k) \\
& \leq -c e_i^T(t)L_p^T L_p e_i(t) + \frac{c}{2} e_i^T(t)L_p^T L_p e_i(t) + c e_i^T(t)L_p^T L_p \delta_i(t) + \frac{c}{2} \delta_i^T(t)L_p^T L_p \delta_i(t) \\
& \leq -\frac{c}{2} e_i^T(t)L_p^T L_p e_i(t) + \frac{c}{2} \delta_i^T(t)L_p^T L_p \delta_i(t) + \frac{c}{2} \delta_i^T(t)L_p^T L_p \delta_i(t) \\
& = c \delta_i^T(t)L_p^T L_p \delta_i(t),
\end{align*}

i.e. \( e_i^T(t)L_p^T L_p e_i(t) \leq (2\rho \lambda_2(L_p^T)/a \lambda_{\max}(L_p^T))(1 - 1/2a)\|L_p^T \delta_i(t)\|^2, \) which follows that \( \dot{V}(t) \leq \sum_{l=1}^q (\rho - 1/\lambda_{\max}(L_p^T))(1 - 1/2a) \lambda_2(L_p^T) \delta_i^T(t) \delta_i(t). \) Define \( \lambda = \min((\lambda_2(L_p^T)/\lambda_{\max}(L_p^T))(1 - 1/2a), 1), \) \( i \in \mathcal{P}. \) Since \( \mathcal{P} \) is finite, \( \lambda \) actually exists. Consequently, we obtain \( \dot{V}(t) \leq (\rho - 1)\lambda \sum_{l=1}^q \delta_i^T(t) \delta_i(t) \leq 0. \) Therefore, \( \lim_{t \to \infty} V(t) \) exists.

Recalling the time instant \( t_i \) defined before, we get the infinite sequences \( V(t_i), i = 0, 1, \ldots \) Applying Cauchy’s convergence criteria, we know that, for any \( \epsilon > 0, \) there exists a positive integer \( K_\epsilon, \) such that \( |V(t_{k+1}) - V(t_k)| < \epsilon, \) i.e. \( \int_{t_k}^{t_{k+1}} \dot{V}(t) \, dt < \epsilon \) holds for \( k \geq K_\epsilon. \) Consequently, we have \( \int_{t_k}^{t_{k+1}} \dot{V}(t) \, dt < \epsilon. \)

Because \( q \) varies with the switching topology, we denote by \( q_j \) the value of \( q \) during the interval \( [t_k, t_{k+1}]. \) Considering

\begin{align*}
\int_{t_k}^{t_{k+1}} \dot{V}(t) \, dt & \geq (1 - \rho)\lambda \sum_{l=1}^{q_k} \delta_i^T(t) \delta_i(t) \, dt \geq (1 - \rho)\lambda \sum_{l=1}^{q_k} \delta_i^T(t) \delta_i(t) \, dt,
\end{align*}

we conclude

\begin{equation*}
(1 - \rho)\lambda \left( \int_{t_k}^{t_{k+1}} \sum_{l=1}^{q_j} \delta_i^T(t) \delta_i(t) \, dt + \cdots + \int_{t_{k-1}}^{t_k} \sum_{l=1}^{q_j} \delta_i^T(t) \delta_i(t) \, dt \right) < \epsilon.
\end{equation*}

As \( \sigma(t) \) switches finite times in the time interval \([t_k, t_{k+1}), m_k \) is finite for any \( k = 0, 1, \ldots. \) Hence, for \( k \geq K_\epsilon, \) we get \( (1 - \rho)\lambda \sum_{l=1}^{q_j} \delta_i^T(t) \delta_i(t) \, dt < \epsilon, j = 0, 1, \ldots, m_k - 1, \) i.e.

\begin{equation*}
\lim_{k \to \infty} \int_{t_k}^{t_{k+1}} \sum_{l=1}^{q_j} \delta_i^T(s) \delta_i(s) \, ds = 0, j = 0, 1, \ldots, m_k - 1.
\end{equation*}

From \( \dot{V}(t) \leq 0 \) and (3.5), we know that \( \delta_i(t) \) and \( \tilde{\delta}_i(t) \) are bounded. Moreover, it is easy to see that \( \delta_i(t) \) and \( \tilde{\delta}_i(t) \) are bounded. Then, \( \sum_{j=1}^{q_j} \delta_j^T(s) \delta_i(s) \) is uniformly continuous. Employing Barbalat’s Lemma, we conclude \( \lim_{t \to \infty} \sum_{j=1}^{q_j} \delta_j^T(t) \delta_i(t) = 0 \) for \( j = 1, 2, \ldots, m_k - 1. \) It follows that \( \lim_{t \to \infty} \delta_i(t) = 0, l = 1, \ldots, q_j. \) In other words, the states of all agents in each component tend to be consistent. This combining with the joint connectivity of \( G \) in \([t_k, t_{k+1}) \) results in the consensus of MAS (3.1). It can be explained by contradiction. If there exists a state \( x_i \) inconsistent with the other states, it must be isolated all the time in \([t_k, t_{k+1}). \) However, this contradicts the joint connectivity of \( G. \) If there are two states \( x_i \) and \( x_j \) different from the others, they are either isolated or in the same component (containing only the two vertices \( i \) and \( j \)) for any \( t \in [t_k, t_{k+1}). \) It is also against the joint connectivity of \( G. \) Other cases can be proved similarly. This completes the proof. \( \square \)
Theorem

For MAS (3.1), suppose that Assumption 3.1 holds and the state of the leader is defined as

\[ \dot{x}_0(t) = x_0, \quad (4.2) \]

which is a constant reference state.

The leader-following consensus of system (4.1–4.2) is said to be achieved if, for each agent \( i \in \{1, 2, \ldots, N\} \), there is a local state feedback \( u_i(t) \) of \( \{x_j(t) : j \in N_i\} \) such that the closed-loop system
satisfies \( \lim_{t \to \infty} |x_i(t) - x_0| = 0 \) \( (i = 1, \ldots, N) \) for any initial condition \( x_i(0) \), \( i = 1, \ldots, N \). The continuous protocol is described as

\[
u_i(t) = \sum_{j \in N_i(\sigma(t))} (x_j(t) - x_i(t)) + b_i(x_0 - x_i(t)), \tag{4.3}
\]

for \( i = 1, \ldots, N \), which can drive the agents to follow the leader. The following assumption is used throughout this section.

**Assumption 4.1** The graph \( \tilde{G}_{\sigma(t)} \) is jointly connected across the time interval \([t_k, t_{k+1})\).

The event-triggered protocol is given as

\[
u_i(t) = \sum_{j \in N_i(\sigma(t))} (x_j(t) - x_i(t)) + \sum_{j \in N_i(\sigma(t))} (e_j(t) - e_i(t)) + b_i(x_0 - (x_i(t) + e_i(t))), \tag{4.4}
\]

for \( t \in [r^i_k, r^i_{k+1}) \), \( i = 1, \ldots, N \), and \( k'(t) = \arg \min_k \{t - r^i_k \mid t \geq r^i_k\} \), with the same communication strategy presented in the last section.

Define the variables \( e_i(t) = x_i(r^i_k) - x_i(t) \), \( e(t) = [e_1(t), e_2(t), \ldots, e_N(t)]^T \) and \( x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T \). Thus,

\[
\dot{x}(t) = -L_{\sigma(t)}(x(t) + e(t)) - B_{\sigma(t)}(x(t) + e(t)) + B_{\sigma(t)}\bar{x}_0 = -H_{\sigma(t)}(x(t) + e(t)) + B_{\sigma(t)}\bar{x}_0. \tag{4.6}
\]

Since \( L_{\sigma(t)}1_N = 0 \), (4.6) can be rewritten as \( \dot{x}(t) = -H_{\sigma(t)}(x(t) + e(t)) + H_{\sigma(t)}\bar{x}_0 \). Introduce the error vector \( y(t) = x(t) - \bar{x}_0 \) with \( y_i(t) = x_i(t) - x_0 \). It is explicit that \( e_i(t) = y_i(r^i_k) - y_i(t) \) and the dynamics of \( y(t) \) can be described as

\[
\dot{y}(t) = -H_{\sigma(t)}y(r^i_k) = -H_{\sigma(t)}(y(t) + e(t)). \tag{4.7}
\]

For the succeeding analysis, we part the components of \( G_p \) into three sets: \( S_c = \{S^1_p, S^2_p, \ldots, S^q_p\} \), where \( S^p_q \) (\( 1 \leq p \leq q \)) is a component connected to the leader; \( S_n = \{S^{q+1}_p, S^{q+2}_p, \ldots, S^{q+m}_p\} \), where \( S^p_q \) (\( q + 1 \leq p \leq q + m \)) is a component containing no less than two vertices and it is not connected to the leader; \( S_o = \{S^{q+m+1}_p, \ldots, S^{Q}_p\} \) is the set of the isolated vertices.

Without loss of generality, after rearranging the order of all agents, the Laplacian \( L_p \) of \( G_p \) can be written in the form \( L_p = \text{diag}(L^1_p, L^2_p, \ldots, L^Q_p) \), where \( L^i_p \) is Laplacian of \( S^i_p \). Accordingly, the matrix \( B_p \) can be partitioned into \( B_p = \text{diag}(B^1_p, B^2_p, \ldots, B^Q_p) \). Then we obtain \( H_p = \text{diag}(H^1_p, H^2_p, \ldots, H^Q_p) \), where \( H^l_p = L^l_p + B^l_p \), \( l = 1, 2, \ldots, Q \). As mentioned in Ni & Cheng (2010), for \( 1 \leq l \leq q \), \( H^l_p \) is positive definite. For \( q + 1 \leq l \leq q + m \), \( H^l_p = L^l_p \) has a single zero eigenvalue and \( H^l_p = L^l_p = 0 \) for \( q + m + 1 \leq l \leq Q \).
Assume $V(S_p^l)$ to be the vertex set of $S_p^l$ and $m_p^l = |V(S_p^l)|$. Defined by $x_c^l(t) = (x_{l_1}, x_{l_2}, \ldots, x_{l_m^l})^T \in \mathbb{R}^{m_p^l}$, the state vector composed of vertices in $S_p^l \subset S_c$. Similarly, $x_a^l(t)$ and $x_o^l(t)$ are the state vectors of vertices in $S_p^a \subset S_a$ and $S_p^o \subset S_o$, respectively. Subsequently, we define $y_c^l(t) = x_c^l(t) - 1_{m_p^l} x_0$ and $y_a^l(t) = x_a^l(t) - 1_{m_p^l} x_0$. For $S_p^a \subset S_a$, define the disagreement vector $\delta_n^l(t) = x_n^l(t) - \bar{x}_n^l(0)$ with the invariant quantity $\bar{x}_n^l(t) = (1/m_p^l)1_{m_p^l} x_n^l(t)$. We see that $\delta_n^l(t)$ satisfies $1_{m_p^l}^T \delta_n^l(t) \equiv 0$. Define a group of measurement errors $\epsilon_c^l(t) = x_c^l(r_k) - x_c^l(t) = y_c^l(r_k) - y_c^l(t)$ and $\epsilon_a^l(t) = x_a^l(r_k) - x_a^l(t) = y_a^l(r_k) - y_a^l(t)$.

Since the isolated vertices are not controlled, we just design the event conditions for vertices covered in $S_c$ and $S_n$. First, we present preliminary event conditions and then provide the final ones.

**Lemma 4.1** Consider MAS (4.1–4.2) under Assumption 4.1 with $r_0 = 0$. For each component $S_p^l \subset S_c$ and $S_p^l \subset S_n$, the event conditions are given as

\[
\|H_p e_c^l(t)\|^2 \leq s \|H_p y_c^l(t)\|^2, \tag{4.8}
\]

\[
\|L_p e_a^l(t)\|^2 \leq f \|L_p \delta_n^l(t)\|^2, \tag{4.9}
\]

where $s = (2\rho_n \lambda_{\min}(H_p^l)/d \lambda_{\max}(H_p^l))(1 - 1/2d)$, $f = (2\rho_n \lambda_{\max}(L_p^l)/b \lambda_{\max}(L_p^l))(1 - 1/2b)$, $\rho_c \in (0, 1)$, $\rho_n \in (0, 1), d > \frac{1}{2}$ and $b > \frac{1}{2}$, then under the protocol (4.4), the leader-following consensus of (4.1–4.2) is achieved.

**Proof.** We will investigate the dynamics of system (4.7) to examine the evolutions of each agent. The common Lyapunov function candidate is chosen as $V(t) = \frac{1}{2} y^T(t)y(t)$, which is continuous. Assume that $G_p$ is active at time $t$. Define the derivative of $V(t)$ at time instants $t_k$ and $r_k$ is its right derivative. Consider the time derivative of $V(t)$ along the trajectory of closed-loop system (4.7),

\[
\dot{V}(t) = -y^T(t) H_p y(r_k)
\]

\[
= -\sum_{l=1}^{q} y_c^l(t)^T H_p^l y_c^l(r_k) - \sum_{l=q+1}^{q+m} y_n^l(t)^T H_p y_n^l(r_k) - \sum_{l=q+m+1}^{Q} y_o^l(t)^T H_p y_o^l(r_k). \tag{4.10}
\]

Note that $H_p^l = L_p^l$ for $q + 1 \leq l \leq q + m$ and $H_p^l = L_p^l = 0$ for $q + m + 1 \leq l \leq Q$. Thus, (4.10) changes into

\[
\dot{V}(t) = -\sum_{l=1}^{q} y_c^l(t)^T H_p^l y_c^l(r_k) - \sum_{l=q+1}^{q+m} y_n^l(t)^T L_p^l y_n^l(r_k)
\]

\[
= -\sum_{l=1}^{q} y_c^l(t)^T H_p^l y_c^l(r_k) - \sum_{l=q+1}^{q+m} (\delta_n^l(t) + (\bar{x}_n^l(t) - x_0) 1_{m_p^l})^T \times L_p^l (\delta_n^l(r_k) + (\bar{x}_n^l(r_k) - x_0) 1_{m_p^l})
\]

\[
= -\sum_{l=1}^{q} y_c^l(t)^T H_p^l y_c^l(r_k) - \sum_{l=q+1}^{q+m} \delta_n^l(t)^T L_p^l \delta_n^l(r_k)
\]

\[
= -\sum_{l=1}^{q} y_c^l(t)^T H_p^l y_c^l(r_k) - \sum_{l=q+1}^{q+m} \delta_n^l(t)^T L_p^l \delta_n^l(r_k)
\]
\[
V(t) \leq \sum_{l=1}^{q} \frac{1}{\lambda_{\max}(H_p^l)} - \frac{(1 - 1/2d)}{\lambda_{\max}(H_p^l)} y_c^T(t) H_p^l H_p^l y_c(t) + \frac{d}{2\lambda_{\min}(H_p^l)} e_c^T(t) H_p^l e_c(t) \\
+ \sum_{l=q+1}^{q+m} \left( -\frac{1}{\lambda_{\max}(L_p^l)} \delta^{(l)}_n(t) L_p^l L_p^l \delta^{(l)}_n(t) + \frac{b}{2\lambda_{\min}(L_p^l)} e_n^T(t) L_p^l e_n(t) \right).
\]

From the event conditions (4.8–4.9), we have

\[
V(t) \leq \sum_{l=1}^{q} \frac{(\rho_c - 1)}{\lambda_{\max}(H_p^l)} \left( 1 - \frac{1}{2d} \right) y_c^T(t) H_p^l H_p^l y_c(t) \\
+ \sum_{l=q+1}^{q+m} \left( \frac{(\rho_n - 1)}{\lambda_{\max}(L_p^l)} \left( 1 - \frac{1}{2b} \right) \delta^{(l)}_n(t) L_p^l L_p^l \delta^{(l)}_n(t) \right). \tag{4.11}
\]

Define \( \hat{\lambda} = \min\{\lambda_{\min}^2(H_p^l)/\lambda_{\max}(H_p^l)(1 - 1/2d), S_p^l, p \in P\} \), \( \Lambda = \min\{\lambda_{\min}^2(L_p^l)/\lambda_{\max}(L_p^l)(1 - 1/2b), S_p^l, p \in P\} \) and \( \eta = \min\{(1 - \rho_c)\hat{\lambda}, (1 - \rho_n)\Lambda\} > 0 \). Then (4.11) becomes

\[
V(t) \leq (\rho_c - 1 - \hat{\lambda} \sum_{l=1}^{q} y_c^T(t) y_c(t) + (\rho_n - 1) \Lambda \sum_{l=q+1}^{q+m} \delta^{(l)}_n(t) \delta^{(l)}_n(t) \\
\leq -\eta \left( \sum_{l=1}^{q} y_c^T(t) y_c(t) + \sum_{l=q+1}^{q+m} \delta^{(l)}_n(t) \delta^{(l)}_n(t) \right) \\
\leq 0. \tag{4.12}
\]

Therefore, \( \lim_{t \to \infty} V(t) \) exists.
Focus on the infinite sequences $V(t_i), i = 0, 1, \ldots$. In view of Cauchy’s convergence criteria, for any $\epsilon > 0$, there exists a positive integer $K_\epsilon$, such that $|V(t_{k+1}) - V(t_k)| < \epsilon$, i.e. $\int_{t_k}^{t_{k+1}} \dot{V}(t) \, dt < \epsilon$ holds for $k \geq K_\epsilon$. It follows that $\int_{t_k}^{t_{k+1}} [-\dot{V}(t)] \, dt + \cdots + \int_{t_{k-1}}^{t_k} [-\dot{V}(t)] \, dt < \epsilon$.

Denote by $q_j$ and $m_j$ as the values of $q$ and $m$, respectively, during the interval $[t_k, t_{k+1})$. From (4.12), it leads to

$$
\int_{t_k}^{t_{k+1}} [-\dot{V}(t)] \, dt \geq \eta \int_{t_k}^{t_{k+1}} \left( \sum_{l=1}^{q_j} y_c^l(t)y_c^l(t) + \sum_{l=q_j+1}^{q_j+m_j} \delta_n^l(t)\delta_n^l(t) \right) \, dt + \cdots
$$

$$
\eta \int_{t_k}^{t_{k+1}} \left( \sum_{l=1}^{q_j} y_c^l(t)y_c^l(t) + \sum_{l=q_j+1}^{q_j+m_j} \delta_n^l(t)\delta_n^l(t) \right) \, dt + \cdots
$$

$$
+ \int_{t_{k-1}}^{t_k} \left( \sum_{l=1}^{q_j} y_c^l(t)y_c^l(t) + \sum_{l=q_j+1}^{q_j+m_j} \delta_n^l(t)\delta_n^l(t) \right) \, dt < \epsilon.
$$

We conclude

$$
\eta \int_{t_k}^{t_{k+1}} \left( \sum_{l=1}^{q_j} y_c^l(t)y_c^l(t) + \sum_{l=q_j+1}^{q_j+m_j} \delta_n^l(t)\delta_n^l(t) \right) \, dt + \cdots
$$

$$
+ \int_{t_{k-1}}^{t_k} \left( \sum_{l=1}^{q_j} y_c^l(t)y_c^l(t) + \sum_{l=q_j+1}^{q_j+m_j} \delta_n^l(t)\delta_n^l(t) \right) \, dt < \epsilon.
$$

It implies $\eta \int_{t_k}^{t_{k+1}} (\sum_{l=1}^{q_j} y_c^l(t)y_c^l(t) + \sum_{l=q_j+1}^{q_j+m_j} \delta_n^l(t)\delta_n^l(t)) \, ds = 0$, $j = 0, 1, \ldots, m_k - 1$. The equation above is equal to $\lim_{t \to \infty} \int_t^{t+\tau} (\sum_{l=1}^{q_j} y_c^l(s)y_c^l(s) + \sum_{l=q_j+1}^{q_j+m_j} \delta_n^l(s)\delta_n^l(s)) \, ds = 0$, and $\lim_{t \to \infty} \int_t^{t+\tau} (\sum_{l=1}^{q_j} y_c^l(s)y_c^l(s) \, ds = 0$, $j = 0, 1, \ldots, m_k - 1$.

Owing to $\dot{V}(t) \leq 0$ and (4.7), $y_c^l(t)$ and $\dot{y}_c^l(t)$ are bounded. Hence we find that $\sum_{l=1}^{q_j} y_c^l(s)y_c^l(s)$ is uniformly continuous. From Barbalat’s Lemma, $\lim_{t \to \infty} \sum_{l=1}^{q_j} y_c^l(t)y_c^l(t) = 0$ for $j = 1, 2, \ldots, m_k - 1$, i.e. $\lim_{t \to \infty} \sum_{l=1}^{q_j} y_c^l(t) = 0$, $l = 1, \ldots, q_j$. This states that the agents in $S_t$ tend to follow the leader. With the same steps, we obtain $\lim_{t \to \infty} \delta_n^l(t) = 0$, $l = 1, \ldots, q_j$. In other words, the agents in $S_t$ achieve consensus. Under Assumption 4.1, we can conclude that the leader-following consensus is achieved for MAS (4.1–4.2). Otherwise, assume that there exists only one agent that does not follow the leader. Then, it must be isolated all the time, which is contrary to the jointly connectivity. If two agents are not in consensus with the leader, they are either isolated or in the same component of $S_t$ (containing only the two vertices) for $t \in [t_k, t_{k+1})$. It also leads to a contradiction. Similarly, other cases can be excluded. The proof is completed.

**Theorem 4.1** For MAS (4.1–4.2), suppose that Assumption 4.1 holds and $r_0 = 0$. For each component $S_p^l \in S_c$ and $S_p^l \in S_n$, the event conditions are described as

$$
|e_{c_i}(t)| \leq \frac{1}{\|H_p^c\|} \left| \sum_{c_j \in N_{c_i}} \left( x_{c_j}(r_c^j) - x_{c_j}(r_c^j) \right) + b_{c_i}x_0 - b_{c_i}x_c(r_c^i) \right|,
$$

(4.13)
respectively, where \(s\) and \(f\) are defined in Lemma 4.1, \(k'(t) = \arg\min_r \{t - r^c_j \mid t \geq r^c_j\}\) or \(k'(t) = \arg\min_{r_j} \{t - r^c_j \mid t \geq r^c_j\}\); then under the protocol (4.4), the leader-following consensus of (4.1–4.2) is achieved.

**Proof.** From event condition (4.13), we have 
\[
\|H_p^f \| \|e^f(t)\| \leq \sqrt{s/(1 + s)} \|H_p^f y^f(r_k)\|
\]
Considering \(\|H_p^f e^f(t)\| \leq \|H_p^f \|\|e^f(t)\|\), we get \(\|H_p^f e^f(t)\|^2 \leq (s/(1 + s)) \|H_p^f y^f(r_k)\|^2\). It can be written as
\[
e^f_c(t) H_p^f H_p^f e^f_c(t)
\]
\[
\leq -s e^f_c(t) H_p^f H_p^f e^f_c(t) + \frac{s}{2} \gamma^f_c(t) H_p^f H_p^f y^f(r_k)
\]
\[
\leq -s e^f_c(t) H_p^f H_p^f e^f_c(t) + \frac{s}{2} e^f_c(t) H_p^f H_p^f e^f_c(t) + s e^f_c(t) H_p^f H_p^f y^f(r_k)
\]
\[
\leq -s e^f_c(t) H_p^f H_p^f e^f_c(t) + \frac{s}{2} e^f_c(t) H_p^f H_p^f e^f_c(t) + \frac{s}{2} \gamma^f_c(t) H_p^f H_p^f y^f(r_k)
\]
\[
= s \gamma^f_c(t) H_p^f H_p^f y^f(r_k).
\]

In other words, the event condition (4.8) can be derived from (4.13). With a similar derivation, we conclude that event condition (4.9) can also be derived from (4.14). The rest of the proof of the leader-following consensus is the same as that for Lemma 4.1. Thus it is omitted. \(\square\)

**Remark 4.1** In fact, the event conditions in Lemma 4.1 are not practical since they need the global information of the whole component such as \(e^f_c(t), y^f_c(t)\) and so on. Thus, we further investigate the event conditions in Theorem 4.1. Each agent has its own event condition and then it can inspect the next sampling time based on the measurement error of itself, and the last sampling data of itself and its neighbours.

**Theorem 4.2** Consider MAS (4.1–4.2) under Assumption 4.1 with \(r_0 = 0\). Under the protocol (4.4) and the event conditions (4.13–4.14) guaranteeing the leader-following consensus of (4.1–4.2), there is no Zeno-behaviour.

**Proof.** As the first step, we exclude Zeno-behaviour for each agent by contradiction. Without loss of generality, we suppose that \(l_i\) belongs to some component connected to the leader. Thus, the agent \(l_i\) can be noted as \(c_i\) and the event condition is (4.13). The case of \(l_i\) belonging to some component \(S^f_p \in S_n\) can be analysed in the similar approach. Assume that there exists Zeno-behaviour, i.e. infinite events occur in finite time interval.

**Case I:** During the kth sampling period \(T^l_k\) for some agent \(l_i\), neither graph switching nor event triggering of its neighbours occurs. Noting that \(|\dot{e}_c(t)| = |\dot{x}_c(t)| = |\sum_{c_j \in N_{c_i}} (x_{c_j}(r^c_j) - x_{c_i}(r^c_j)) + b_{c_i} x_0 - b_{c_i} x_{c_i}(r^c_j)| = e_{c}(r^c_j) = 0\), we know \(|\dot{e}_c(t)| \leq |\sum_{c_j \in N_{c_i}} (x_{c_j}(r^c_j) - x_{c_i}(r^c_j)) + b_{c_i} x_0 - b_{c_i} x_{c_i}(r^c_j)| \), \(t \geq r^c_j\). Considering event condition (4.13), the kth sampling period \(T^l_k\) is
bounded by the time interval \((t - r^{ci}_k)\) such that 
\[|\sum_{c_j \in N_{ci}} (x_{c_j}(r^{ci}_k) - x_{c_i}(r^{ci}_k)) + b_{c_i} x_0 - b_{c_i} x_{c_i}(r^{ci}_k)|(t - r^{ci}_k) = (1/\|H^p_l\|) \sqrt{s/2(T + s)}|\sum_{c_j \in N_{ci}} (x_{c_j}(r^{ci}_k) - x_{c_i}(r^{ci}_k)) + b_{c_i} x_0 - b_{c_i} x_{c_i}(r^{ci}_k)|\] holds. After defining \(\tau_1 = (1/\|H^p_l\|) \sqrt{s/2(T + s)}\), we get \(T^{li}_k \geq \tau_1\).

Case II: Before the \(k + 1\)th sampling is triggered, the graph switches or some neighbours of agent \(i\) are triggered at time instant \(t^* > r^{li}_k\). In this situation, we have \(T^{li}_k \geq t^* - r^{li}_k \geq |e_i(t^*)|/|\sum_{c_j \in N_{ni}} (x_{c_j}(r^{ni}_k) - x_{c_i}(r^{ni}_k)) + b_{c_i} x_0 - b_{c_i} x_{c_i}(r^{ni}_k)| > 0, i \in S^l_p \in S_c\) and \(T^{li}_k \geq t^* - r^{li}_k \geq |e_i(t^*)|/|\sum_{c_j \in N_{ni}} (x_{c_j}(r^{ni}_k) - x_{c_i}(r^{ni}_k))| > 0, i \in S^l_p \in S_n\).

Obviously, each sampling period is strictly greater than zero. This contradicts with Zeno-behaviour since infinite events with positive period can only occur in infinite time interval. Thus, Zeno-behaviour is excluded.

Finally, we can exclude any accumulation point in the sampling time sequence \(r_k\) because there are finite numbers of agents. In summary, there is no Zeno-behaviour. \(\square\)

5. Simulations

5.1 Event-triggered leaderless consensus under jointly connected topology

Consider a MAS consisting of five agents. Suppose possible interaction graphs are \(\{G_1, G_2, G_3\}\) which are given in Fig. 1. The graphs \(G_1 \cup G_2 \cup G_3\) are connected. Moreover, we assume that the graphs are switched as \(G_1 \to G_2 \to G_3 \to G_1\) and denote the active time of each agent as \(\bar{\tau}\).

We will compare the results in Section 3 with the event-triggered control schemes in Seyboth (2010) and Shi & Johansson (2011). The event-triggered protocols in Seyboth (2010) and Shi & Johansson (2011) are the same as (3.3). In Seyboth (2010), the static trigger function is

\[|e_i(t)| \leq c_0, \quad c_0 > 0. \tag{5.1}\]

In Shi & Johansson (2011), the time-dependent trigger function is

\[|e_i(t)| \leq c_1 e^{-\lambda t}, \tag{5.2}\]

where \(c_1 > 0, 0 < \lambda < -\ln(1 - a^*)/K_s, 0 < a^* < 1, \) and \(K_s = (N - 1)(T + \tau)\).

![Fig. 1. Possible interaction topologies between agents.](https://academic.oup.com/imamci/article-abstract/32/3/537/2357094/535?download=1)
Fig. 2. State trajectories under active time $\bar{\tau} = 0.4s$.

Fig. 3. Sampling periods under event condition (3.6).
Set the initial states as $x_1 = 5, x_2 = 3, x_3 = -6, x_4 = 4, x_5 = 2$ and choose $\bar{\tau} = 0.4$. It is easy to find all the components of the graphs and classify them into the set $S_c$ and $S_o$. For event condition (3.6), we choose $\rho = 0.95$ and $a = 1$. For (5.1–5.2), we set $c_0 = 0.1, c_1 = 0.15, \lambda = 0.09$. These parameters are selected so as to generate similar consensus convergence performances under three different event conditions (3.6), (5.1) and (5.2).

Figure 2 plots the state trajectories of $x_i$ ($i = 1, 2, 3, 4, 5$) under event condition (3.6) (top plot), (5.1) (middle plot) and (5.2) (bottom plot). It implies that three event-triggered control schemes all lead to consensus. The sampling periods of each agent under event condition (3.6) are shown in Fig. 3. The average sampling periods under different event conditions (3.6), (5.1) and (5.2) are 0.9028s, 0.6813s and 0.6326s, respectively. As event condition (3.6) can generate larger average sampling period than (5.1) and (5.2), it implies that the event condition proposed in this paper performs better in resource utilization.
Then, we examine the influence of $\bar{\tau}$ on the average sampling period. Taking parameters $\rho$, $a$, $c_0$, $c_1$ and $\lambda$ of the values as above, we vary $\bar{\tau}$ from $0.3s$ to $0.4s$ to $0.5s$. Figures 2 and 4 show the nearly identical consensus convergence performances under the three different event conditions with $\bar{\tau} = 0.4s$. 

Fig. 5. Possible interaction topologies between leader and agents.

Fig. 6. State trajectories under continuous protocol (4.3) and event-triggered protocol (4.4).
and $\bar{\tau} = 0.3s$, respectively. Table 1 presents the average sampling periods. From Table 1, one can see that the average sampling periods increase with the graph active time. Moreover, Table 1 illustrates that event condition (3.6) always leads to a larger average sampling period with different values of $\bar{\tau}$. This shows the superiority of the event-triggered control scheme proposed in this paper in terms of the resource utilization.

5.2 Event-triggered leader-following consensus under jointly connected topology

Consider a MAS consisting of a leader and five agents. Referring to Ni & Cheng (2010), we suppose that possible interaction graphs are $\{\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, \tilde{G}_4, \tilde{G}_5, \tilde{G}_6\}$, which are given in Fig. 5. Assume the graphs are switched as $\tilde{G}_1 \rightarrow \tilde{G}_2 \rightarrow \tilde{G}_3 \rightarrow \tilde{G}_4 \rightarrow \tilde{G}_5 \rightarrow \tilde{G}_6 \rightarrow \tilde{G}_1 \ldots$ and each graph is active for $1/3s$. Thus, the topology is jointly connected during $1s$. Set the initial states as $x_1 = 5, x_2 = 3, x_3 = -6, x_4 = 4$ and $x_0 = 2$. The state trajectories of $x_i$ $(i = 1, 2, 3, 4)$ under the continuous protocol (4.3) are shown in the top plot of Fig. 6. We can see that the four agents follow the static leader.

It is not difficult to classify all the components into the set $S_c, S_n$ and $S_o$ and calculate the parameters $\|H_p^0\|$ and $\|L_p^0\|$ in event conditions (4.13–4.14). Choose $d = b = 1, \rho_c = \rho_d = 0.8$. Then, under event conditions (4.13–4.14), the simulations are described in the bottom plot of Figs 6 and 7. Figure 6 implies that event-triggered control can lead to a similar convergence rate as the continuous one.

6. Conclusions

In this paper, we have studied the leaderless and leader-following consensus problems of MASs under jointly connected topology by using the event-triggered control method. Based on the analysis of components of the graph, event-triggered protocols and event conditions are designed to ensure leaderless
and leader-following consensus. It is also proved that Zeno-behaviour is avoided under the control strategies proposed. Two examples are given to validate the effectiveness of our results.

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**References**


