Robust stochastic stabilization and $H_\infty$ control of uncertain stochastic interval time-delay systems

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This paper establishes a new robust delay-dependent stabilization and $H_\infty$ control methods for a class of uncertain stochastic time-delay systems. The delays in state and control input are time-varying continuous functions varying in an interval and uncertainties are norm-bounded. An appropriate Lyapunov–Krasovskii functional is constructed and a new technique to estimate the upper bound of the stochastic differential of Lyapunov–Krasovskii functional is obtained. Based on the idea of delay decomposition and free-weighting matrices, new and less conservative solutions to the stabilization and $H_\infty$ control problem of uncertain stochastic time-delay systems are provided in terms of linear matrix inequalities. A robust state feedback controller is proposed to guarantee mean-square asymptotic stability as well as the prescribed $H_\infty$ performance for the closed-loop systems. The advantages of the results proposed in this paper, which differ greatly from most of the existing results, lie in their reduced conservatism by thinning the delay partitioning. Numerical examples are provided to show the advantages of the proposed technique.

Keywords: stochastic systems; $H_\infty$ control; interval time delay; linear matrix inequality.

1. Introduction

During the past decades, stochastic systems have been an active research topic. Considerable attention has been paid to it and a great number of results related to this subject have been reported in Xu & Chen (2002), Xia et al. (2008), Yang et al. (2008), Qiu et al. (2011) and Huang & Feng (2007). In many practical systems, such as electronic systems and aircraft systems, there always exist some unavoidable perturbations that should be taken into consideration in system design and performance analysis. Therefore, stochastic systems play an important role in many aspects of science and engineering applications. The phenomena of time delays are often encountered in various systems, such as aircraft stabilization, chemical engineering systems, population dynamics and so on. They are often the source of instability, oscillations and poor performance (Gao et al., 2006, 2008; Lam et al., 2007, Park & Ko, 2007; Peng & Tian, 2008; Li et al., 2009). Thus, problems of stability and control of time-delay systems have been of great importance and interest. Recently, much attention has been focused on delay-dependent conditions for the analysis and control of time-delay systems (Mahmoud & Ismail, 2006; Mahmoud & Nounou, 2006; Mahmoud & Almutairi, 2010; Mahmoud, 2010; Gao & Chen, 2007; Li et al., 2012, 2014, 2010),
because delay-dependent conditions are generally less conservative than delay-independent ones. Many effective methods have been proposed such as the model transformation methods (Gao & Wang, 2003), the free-weighting-matrix method (Xu & Lam, 2005; He et al., 2007) and so on.

As is well known, the problem of $H_\infty$ control for systems has been regarded as an important issue in control community and $H_\infty$ performance is an important index for evaluating the disturbance rejection attenuation property. The study of stability analysis and controller synthesis for stochastic systems with time delay has been investigated in past years, and great deals of results related to such systems have been reported in the literature (Xie & Xie, 2000; Xu & Chen, 2003; Chen et al., 2004; Li et al., 2009). When the parameter uncertainties were considered, the robust stochastic stabilization problem was solved via linear matrix inequality approach in Xie & Xie (2000). The problem of robust $H_\infty$ control and $H_\infty$ filtering were investigated in Xu & Chen (2002, 2003), respectively. It is noted that all these results are delay independent, which is usually conservative, especially when delays are small. This motivates the development of delay-dependent conditions for stochastic systems. To improve the performance of delay-dependent criteria, much effort has been devoted in recent works. For instance, the Moon’s inequalities approach was applied to investigate the problem of delay-dependent robust stochastic stabilization and $H_\infty$ control for stochastic systems with norm-bounded uncertainties and state delay in Chen et al. (2004), but it does not take the effect of the control input delays into account when the controllers were designed for stochastic systems. Although the problem of the $H_\infty$ control for uncertain stochastic systems with state and input delays was studied in Li et al. (2009) the time delay does not vary in a range. To the best of the authors’ knowledge, the delay-range-dependent robust $H_\infty$ control for uncertain stochastic systems with interval time-varying delay in state and control input has not been fully investigated to date, which has motivated the present study.

Recently, some up-to-date techniques, the so-called delay partitioning or delay decomposition, have been developed to investigate different systems with time delay (Zhao et al., 2009; Gao & Chen, 2007; Li et al., 2012, 2014, 2010). Although, the appealing idea has been verified to be more effective, it is the first attempt to utilize this idea to study the $H_\infty$ control problem for uncertain stochastic systems with interval time-varying delay in state and control input. In this paper, we consider the problems of robust stochastic stabilization and robust $H_\infty$ control for uncertain stochastic systems with interval time-varying delay in state and control input. For the robust $H_\infty$ control problem, the purpose is to design a state-feedback controller such that the resulting closed-loop system is mean-square asymptotically stable while a prescribed $H_\infty$ performance level is satisfied. By using the stochastic analysis technique, a novel Lyapunov–Krasovskii functional is introduced. In order to reduce conservatism in the derivation of criteria, this work resorts to the idea of partitioning the known bounds of the delay interval in multiple regions or subintervals. This allows to reduce the conservatism due to the fact that less restrictive bounds for specific terms in the Lyapunov–Krasovskii functional are derived separately in each subinterval. Less conservative conditions for the solvability of $H_\infty$ control problem are proposed. In addition to delay dependence, the obtained results are also dependent on the decomposition size. Numerical examples are given to illustrate the effectiveness of the results presented. Through the given examples, it can be seen that increasing the number of subintervals may result in the reduction of conservativeness of the derived criteria.

1.1 Notation

Through this paper, the superscript $^T$ stands for matrix transposition and the symbol $^*$ is used to denote the transposed elements in the symmetric positions of a matrix. $R^n$ denotes the $n$-dimensional Euclidean space and $R^{n\times n}$ is the set of all $n \times n$ real matrices. The notation $X \geq Y$ (respectively, $X > Y$) means...
that the matrix $X - Y$ is positive semi-definite (respectively, positive definite), $I$ is the identity matrix with appropriate dimension. $\| \cdot \|$ stands for the Euclidean norm. $L_2[0, \infty)$ is the space of square-integrable vector functions over $[0, \infty)$. $(\Omega, F, \{F_t\}_{t \geq 0}, \mathbb{P})$ is a complete probability space with a filtration $\{F_t\}_{t \geq 0}$ satisfying the conditions that it is right continuous and $F_0$ contains all $\mathbb{P}$-null sets. $h > 0$ and $C([-h, 0]; \mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued functions $\varphi$ on $[-h, 0]$ with the norm $\| \varphi \| = \sup_{-h \leq \theta \leq 0} |\varphi(\theta)|$. $L_2^0([-h, 0]; \mathbb{R}^n)$ denotes the family of all $F_0$-measurable $C([-h, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h \leq \theta \leq 0} E|\xi(\theta)|^2 < \infty$. $E\{\cdot\}$ denotes the expectation operator. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Problem formulation

Consider the following uncertain stochastic system with interval time-varying delays in state and control input:

\[
\begin{align*}
\dot{x}(t) &= [A(t)x(t) + A_d(t)x(t - h(t))] + B(t)u(t) + B_d(t)u(t - h(t)) + Ev(t)] \, dt \\
&\quad + [F(t)x(t) + F_d(t)x(t - h(t))] + E_dv(t)] \, d\omega(t) \\
y(t) &= Cx(t) + C_d x(t - h(t)) + Du(t) + D_d u(t - h(t)) \\
x(t) &= \phi(t), \quad t \in [-h_M, 0]
\end{align*}
\]  

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^q$ is the controlled output, $v(t) \in \mathbb{R}^p$ is the disturbance input which belongs to $L_2[0, \infty)$. $\omega(t)$ is a zero-mean Wiener process defined on a complete probability space $(\Omega, F, \{F_t\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{F_t\}_{t \geq 0}$ satisfying $E[d\omega(t)] = 0$ and $E[d\omega^2(t)] = 0$. $\varphi(t)$ is a real-valued continuous initial function on $[-h_M, 0]$. The bounded function $h(t)$ represents interval time-varying delay satisfying $h_m \leq h(t) \leq h_M$, $\dot{h}(t) \leq d$, where $h_m, h_M, d$ are the real known constant scalars. $E, E_d, C, C_d, D, D_d$, are known as the real constant matrices, and

\[
\begin{align*}
A(t) &= A + \Delta A(t), & A_d(t) &= A_d + \Delta A_d(t), & B(t) &= B + \Delta B(t), & B_d(t) &= B_d + \Delta B_d(t) \\
F(t) &= F + \Delta F(t), & F_d(t) &= F_d + \Delta F_d(t)
\end{align*}
\]

where the matrices $A, A_d, B, B_d, F, F_d$ are known as the real constant matrices with appropriate dimensions, and $\Delta A(t), \Delta A_d(t), \Delta B(t), \Delta B_d(t), \Delta F(t), \Delta F_d(t)$ represent the parameter uncertainties of the system, which are assumed to be of the form

\[
\begin{bmatrix}
\Delta A(t) & \Delta A_d(t) & \Delta B(t) & \Delta B_d(t) & \Delta F(t) & \Delta F_d(t)
\end{bmatrix} = MF(t)[N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6]
\]  

(2)

In the above equation, $M, N_1, N_2, N_3, N_4, N_5, N_6$ are known as the real constant matrices with appropriate dimensions and $F(t)$ is the time-varying uncertain matrix satisfying $F^T(t)F(t) \leq I$.

Throughout the paper the following concepts and lemmas will be needed.

DEFINITION 1 For the system (1) with $u(t) = 0$ and $v(t) = 0$, the equilibrium point is said to be robustly stable mean square if for any $\varepsilon > 0$, there is a scalar $\delta(\varepsilon) > 0$ such that

\[
E[\|x(t)\|^2] < \varepsilon, \quad t > 0
\]
for all admissible uncertainties, when \( \sup_{-h_y \leq t \leq 0} E[\|\varphi(s)\|^2] < \delta(\varepsilon) \). If, in addition, \( \lim_{t \to \infty} E[\|x(t)\|^2] = 0 \), for any initial conditions and all admissible uncertainties, the system (1) with \( u(t) = 0 \) and \( v(t) = 0 \) is said to be robustly asymptotically stable in the mean square sense.

**Definition 2** Given a scalar \( \gamma > 0 \), the uncertain stochastic system (1) with \( u(t) = 0 \) is said to be robustly stochastically stable with disturbance attenuation \( \gamma \), if it is robustly stochastically stable and under zero initial conditions, \( \|y(t)\|_2 < \gamma \|v(t)\|_2 \) holds for all nonzero \( v(t) \in L_2[0, \infty) \) and all admissible uncertainties, where

\[
\|y(t)\|_2 = \left( E\left\{ \int_0^\infty \|y(t)\|_2^2 dt \right\} \right)^{1/2}.
\]

**Lemma 1** (Wang et al., 1992) For any vector \( x, y \in \mathbb{R}^n \), matrices \( P > 0, D, E, F \) with appropriate dimensions, and any scalars \( \varepsilon > 0 \), if \( F^T F \leq I \), then

\[
2x^T y \leq x^T P^{-1} x + y^T P y, \tag{3}
\]

\[
DFE + E^T FD^T < \varepsilon DD^T + \varepsilon^{-1} EE^T. \tag{4}
\]

**Lemma 2** (Huang & Feng, 2007) For any positive-definite matrix \( M \in \mathbb{R}^{n \times n} \), scalar \( \gamma > 0 \) and vector function \( f(t) : [0, \gamma] \to \mathbb{R}^n \), such that the integrations concerned are well-defined, the following inequality holds

\[
\left( \int_0^\gamma f(s) \, ds \right)^T M \left( \int_0^\gamma f(s) \, ds \right) \leq \gamma \int_0^\gamma f^T(s) M f(s) \, ds.
\]

**Lemma 3** (Ramakrishnan & Ray, 2011) Suppose \( h_1 \leq h(t) \leq h_2 \), where \( h(t) : \mathbb{R}_+ \to \mathbb{R}_+ \). Then, for any constant matrices \( \mathcal{S}_1, \mathcal{S}_2, \) and \( \Omega \) with proper dimensions, the following matrix inequality

\[
\Omega + (h(t) - h_1) \mathcal{S}_1 + (h_2 - h(t)) \mathcal{S}_2 < 0
\]

holds, if and only if

\[
\Omega + (h_2 - h_1) \mathcal{S}_1 < 0,
\]

\[
\Omega + (h_2 - h_1) \mathcal{S}_2 < 0.
\]

In this paper, our aim is to develop criteria of robust stochastic stabilization and robust \( H_\infty \) control for the uncertain stochastic interval time-delay system (1). More specifically, we are concerned with the following two problems:

1. Robust stochastic stabilization problem: design a memoryless state feedback controller \( u(t) = Kx(t) \) for the system (1) with \( v(t) = 0 \) such that the resulting closed-loop system

\[
\begin{align*}
dx(t) &= [(A(t) + B(t)K)x(t) + (A_d(t) + B_d(t)K)x(t - h(t))]dt \\
&\quad + [F(t)x(t) + F_d(t)x(t - h(t))]d\omega(t) \\
y(t) &= (C + DK)x(t) + (C_d + D_d K)x(t - h(t)) \\
x(t) &= \phi(t), t \in [-h_M, 0]
\end{align*}
\]

(5)
is robustly asymptotically stable for all admissible uncertainties in the mean square sense. In this case, the system (1) with \( v(t) = 0 \) is said to be robustly stochastically stabilizable.

(2) Robust \( H_\infty \) control problem: given a constant scalar \( \gamma > 0 \), design a state feedback controller \( u(t) = Kx(t) \) for the system (1) such that, for all admissible uncertainties, the resulting closed-loop system

\[
\begin{align*}
\dot{x}(t) &= [A(t) + B(t)K)x(t) + (A_d(t) + B_d(t)K)x(t - h(t)) + Ev(t)] \, dt \\
&\quad + [F(t)x(t) + F_d(t)x(t - h(t)) + E_dv(t)] \, d\omega(t) \\
y(t) &= (C + DK)x(t) + (C_d + D_dK)x(t - h(t)) \\
x(t) &= \phi(t), \quad t \in [-h_M, 0]
\end{align*}
\]

is robustly asymptotically stable in the mean square sense and for any nonzero \( v(t) \in L_2[0, \infty) \), \( \|y(t)\|_2 < \gamma \|v(t)\|_2 \) is satisfied under the zero-initial condition. In this case, the system (1) is said to be robustly stochastically stabilizable with disturbance attenuation level \( \gamma \).

3. Main results

This section investigates the problem of robust stochastic stabilization and \( H_\infty \) control for uncertain stochastic interval time-delay system. An appropriate Lyapunov–Krasovskii functional is proposed based on the delay partitioning technique. The results obtained here are delay-dependent and are expressed as the feasibility of linear matrix inequalities. They are considerably less conservative than similar expressions appearing in the literature (Fridman & Shaked, 2002; Lee et al., 2004; Xu et al., 2006; Yang et al., 2010).

3.1 Robust stochastic stabilization

This subsection is concerned with the robust stochastic stabilization problem. In order to obtain less conservative sufficient conditions, we decompose the variation interval of delay into multiple subintervals of the same size,

\[
[h_m, h_M] = [h_1, h_2] \cup \cdots \cup [h_i, h_{i+1}] \cup \cdots \cup [h_N, h_{N+1}]
\]

where \( h_1 = h_m \), \( h_{N+1} = h_M \), \( h_{i+1} - h_i = h = (h_M - h_m)/N \), and \( N \) is a given integer.

As will be shown, in the following three subsections, stability criteria of the system (5) for \( N = 2 \), \( N = 3 \), \( N \) will be derived, respectively, based on the Lyapunov–Krasovskii stability theorem and delay decomposing method.

3.1.1 Stability criteria for \( N = 2 \) For \( N = 2 \), define \( h_1 = h_m \), \( h_3 = h_M \), \( h = (h_3 - h_1)/2 \), then \( h_2 = h_1 + h = (h_3 + h_1)/2 \). We have the following result.

THEOREM 1 Consider the uncertain stochastic time-delay system (5). If for prescribed scalars \( 0 < h_m < h_M \), \( d \), there exist symmetric positive-definite matrices \( \bar{X}, Q_1, Q_2, Q_3, R_1, R_2, R_3 \), matrices \( Y, L_i, G_i \),
$i = 1, 2, 3$, and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the following LMIs hold:

\[
\begin{bmatrix}
\Gamma_i + \bar{\Gamma}_i + \bar{\Gamma}_i^T & \Gamma_{12} & h\tilde{U}_i \\
* & \Gamma_{22} & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2 \tag{7}
\]

\[
\begin{bmatrix}
\Gamma_i + \bar{\Gamma}_i + \bar{\Gamma}_i^T & \Gamma_{12} & h\bar{V}_i \\
* & \Gamma_{22} & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2 \tag{8}
\]

where

\[
\Gamma_1 = \begin{bmatrix}
\Gamma_{11} & A_d\tilde{X} + B_dY & R_1 & 0 & 0 & 0 & \tilde{X}A^T + Y^TB^T + \varepsilon_1MM^T \\
* & -(1 - d)Q_1 & 0 & 0 & 0 & 0 & \tilde{X}A_d^T + Y^TB_d^T \\
* & * & -Q_2 - R_1 & 0 & 0 & 0 & 0 \\
* & * & * & -Q_3 - \delta R_3 & \delta R_3 & 0 & 0 \\
* & * & * & * & -Q_4 - \delta R_3 & 0 & 0 \\
* & * & * & * & * & \Gamma_{66}
\end{bmatrix},
\]

\[
\Gamma_2 = \begin{bmatrix}
\Gamma_{11} & A_d\tilde{X} + B_dY & R_1 & 0 & 0 & 0 & \tilde{X}A^T + Y^TB^T + \varepsilon_1MM^T \\
* & -(1 - d)Q_1 & 0 & 0 & 0 & 0 & \tilde{X}A_d^T + Y^TB_d^T \\
* & * & -Q_2 - R_1 - \delta R_2 & \delta R_2 & 0 & 0 & 0 \\
* & * & * & -Q_3 - \delta R_2 & 0 & 0 & 0 \\
* & * & * & * & -Q_4 & 0 & 0 \\
* & * & * & * & * & \Gamma_{66}
\end{bmatrix}, \quad \delta = h^{-1}
\]

\[
\Gamma_{12} = \begin{bmatrix} H^T & 0_{3 \times 4} \end{bmatrix}^T, \quad \Gamma_{22} = \text{diag}\{-\tilde{X} + \varepsilon_2MM^T - \varepsilon_1I - \varepsilon_2I\},
\]

\[
\bar{\Gamma}_1 = \begin{bmatrix} 0 & \bar{V}_1 - \bar{U}_1 & \bar{V}_1 & 0 & 0 \end{bmatrix}, \quad \bar{\Gamma}_2 = \begin{bmatrix} 0 & \bar{V}_2 - \bar{U}_2 & 0 & \bar{U}_2 & -\bar{V}_2 & 0 \end{bmatrix}
\]

with

\[
H = \begin{bmatrix}
\tilde{X}F^T & \tilde{X}N_1^T + YN_3^T & XN_5^T \\
\tilde{X}F_d^T & \tilde{X}N_2^T + YN_4^T & XN_6^T
\end{bmatrix}, \quad \Gamma_{66} = h_1^2R_1 + hR_2 + hR_3 - 2\tilde{X} + \varepsilon_1MM^T,
\]

\[
\Gamma_{11} = AX + \tilde{X}A^T + BY + Y^TB^T + Q_1 + Q_2 + Q_3 + Q_4 - R_1 + \varepsilon_1MM^T,
\]

\[
\bar{V}_1 = \begin{bmatrix} 0 & G_2^T & G_1^T & G_3^T & 0 & 0 \end{bmatrix}^T, \quad \bar{U}_1 = \begin{bmatrix} 0 & L_2^T & L_1^T & L_3^T & 0 & 0 \end{bmatrix}^T,
\]

\[
\bar{V}_2 = \begin{bmatrix} 0 & G_2^T & 0 & G_1^T & G_3^T & 0 \end{bmatrix}^T, \quad \bar{U}_2 = \begin{bmatrix} 0 & L_2^T & 0 & L_1^T & L_3^T & 0 \end{bmatrix}^T.
\]
then the system (5) is robustly stochastically stabilizable. Moreover, the state feedback controller is presented by $u(t) = Y\dot{X}^{-1}x(t)$.

**Proof.** Construct the following Lyapunov–Krasovskii functional as

$$V(x(t), t) = x^T(t)Px(t) + \int_{t-h(t)}^t x^T(s)P_x(s) ds$$

$$+ \int_{t-h_1}^t x^T(s)P_2x(s) ds + \int_{t-h_2}^t x^T(s)P_3x(s) ds$$

$$+ \int_{t-h_3}^t x^T(s)P_4x(s) ds + \int_{-h_1}^t \int_{t+h_1}^t z^T(s)S_1 z(s) ds d\theta$$

$$+ \int_{-h_2}^t \int_{t+h_2}^t z^T(s)S_2 z(s) ds d\theta + \int_{-h_3}^t \int_{t+h_3}^t z^T(s)S_3 z(s) ds d\theta$$

where $P, P_1, P_2, P_3, P_4, S_1, S_2$ and $S_3$ are all symmetric positive-definite matrices with appropriate dimensions to be determined.

Define $\xi(t) = [x^T(t) x^T(t-h(t)) x^T(t-h_1) x^T(t-h_2) x^T(t-h_3) z^T(t)]^T$. Then the proof can be completed by the same technique as in Theorem 2, and is omitted for brevity. \qed

### 3.1.2 Stability criteria for $N=3$

For $N=3$, define $h_1 = h_m$, $h_4 = h_M$, $h = (h_4 - h_1)/3$, then $h_2 = h_1 + h$, $h_3 = h_1 + 2h$.

**Theorem 2** Consider the uncertain stochastic time-delay system (5). If for prescribed scalars $0 < h_m < h_M, d$, there exist symmetric positive-definite matrices $\bar{X}, Q_1, Q_2, Q_3, Q_4, Q_5, R_1, R_2, R_3, R_4$, matrices $Y, L_i, G_i, (i = 1, 2, 3)$, and scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that the following LMIs hold:

$$\begin{bmatrix}
\Sigma_1 + \bar{\Sigma}_1 + \bar{\Sigma}_1^T & \Sigma_{12} & hU_i \\
* & \Sigma_{22} & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2, 3 \quad (9)$$

$$\begin{bmatrix}
\Sigma_1 + \bar{\Sigma}_1 + \bar{\Sigma}_1^T & \Sigma_{12} & hV_i \\
* & \Sigma_{22} & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2, 3 \quad (10)$$

where

$$\Sigma_1 = \begin{bmatrix}
\Sigma_{11} & A_d \bar{X} + B_d Y & R_1 & 0 & 0 & 0 & 0 & \bar{X} \Lambda^T + Y^T B_d^T + \varepsilon_1 M M^T \\
* & -(1-d)Q_1 & 0 & 0 & 0 & 0 & 0 & \bar{X} A_d^T + Y^T B_d^T \\
* & * & -Q_2 - R_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -Q_3 - \delta R_3 & \delta R_3 & 0 & 0 & 0 \\
* & * & * & * & -Q_4 - \delta R_3 - \delta R_4 & \delta R_4 & 0 & 0 \\
* & * & * & * & * & -Q_5 - \delta R_4 & 0 & 0 \\
* & * & * & * & * & * & \Sigma_{77}
\end{bmatrix}.$$
\[
\Sigma_1 = \begin{bmatrix}
A_d \dot{X} + B_d Y & R_1 & 0 & 0 & 0 & \dot{X} A^T + Y^T B^T + \varepsilon_1 M M^T
\end{bmatrix}
\]
\[
\Sigma_2 = \begin{bmatrix}
* & -(1-d)Q_1 & 0 & 0 & 0 & 0 & \dot{X} A_d^T + Y^T B_d^T
* & * & -Q_2 - R_1 - \delta R_2 & \delta R_2 & 0 & 0 & 0
* & * & * & -Q_3 - \delta R_2 & 0 & 0 & 0
* & * & * & * & -Q_4 - \delta R_4 & \delta R_4 & 0
* & * & * & * & * & -Q_5 - \delta R_4 & 0
* & * & * & * & * & * & \Sigma_{77}
\end{bmatrix}
\]
\[
\Sigma_3 = \begin{bmatrix}
\Sigma_1 & A_d \dot{X} + B_d Y & R_1 & 0 & 0 & 0 & \dot{X} A^T + Y^T B^T + \varepsilon_1 M M^T
\end{bmatrix}
\]
\[
\Sigma_{12} = \begin{bmatrix} H^T & 0_{3 \times 5} \end{bmatrix}^T, \quad \Sigma_{22} = \text{diag}\{-\dot{X} + \varepsilon_2 M M^T - \varepsilon_3 I - \varepsilon_3 I\},
\]
\[
\bar{S}_1 = \begin{bmatrix} 0 & V_1 - U_1 & U_1 - V_1 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{S}_2 = \begin{bmatrix} 0 & V_2 - U_2 & U_2 - V_2 & 0 & 0 & 0 \end{bmatrix}, \quad \bar{S}_3 = \begin{bmatrix} 0 & V_3 - U_3 & U_3 - V_3 & 0 & 0 & 0 \end{bmatrix}
\]

with \( \Sigma_1 = A \dot{X} + \dot{X} A^T + B Y + Y^T B^T + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 - R_1 + \varepsilon_1 M M^T \), \( h = (h_M - h_m) / 3 \), \( \Sigma_{77} = h_1^2 R_1 + h R_2 + h R_3 + h R_4 - 2 \dot{X} + \varepsilon_1 M M^T \), \( U_1 = [0 \ L_2^T \ L_1^T \ L_3^T \ 0 \ 0 \ 0]^T \), \( V_1 = [0 \ G_2^T \ G_1^T \ 0 \ 0 \ 0]^T \), \( U_2 = [0 \ L_2^T \ L_1^T \ L_3^T \ 0 \ 0 \ 0]^T \), \( V_2 = [0 \ G_2^T \ 0 \ G_1^T \ G_3^T \ 0 \ 0]^T \), \( U_3 = [0 \ L_2^T \ 0 \ 0 \ L_1^T \ L_3^T \ 0 \ 0]^T \), \( V_3 = [0 \ G_2^T \ 0 \ 0 \ 0 \ G_1^T \ G_3^T \ 0 \ 0]^T \), and \( H \) is defined in Theorem 1, then the system (5) is robustly stochastically stabilizable. Moreover, the state feedback controller is presented by \( u(t) = Y \dot{X}^{-1} x(t) \).

**Proof.** Set a new vector \( z(t) \in \mathbb{R}^n \), such that
\[
z(t) \, dt = dx(t)
\]
Integrating equation (11) on both sides from \( t - h(t) \) to \( t \) gives
\[
\int_{t-h(t)}^{t} z(s) \, ds = x(t) - x(t - h(t))
\]
Taking the Lyapunov–Krasovskii functional for the system (5) as
\[
V(x(t), t) = x^T(t) P x(t) + \int_{t-h(t)}^{t} x^T(s) P_1 x(s) \, ds
+ \int_{t-h_1}^{t} x^T(s) P_2 x(s) \, ds + \int_{t-h_2}^{t} x^T(s) P_3 x(s) \, ds
+ \int_{t-h_3}^{t} x^T(s) P_4 x(s) \, ds + \int_{t-h_4}^{t} x^T(s) P_5 x(s) \, ds
\]
\[ + h_1 \int_{-h_1}^{0} \int_{t+h}^{t} z^T(s)S_1z(s) \, ds \, d\theta + \int_{-h_1}^{0} \int_{t-h}^{t} z^T(s)S_2z(s) \, ds \, d\theta + \int_{-h_2}^{0} \int_{t+h}^{t} z^T(s)S_3z(s) \, ds \, d\theta + \int_{-h_2}^{0} \int_{t-h}^{t} z^T(s)S_4z(s) \, ds \, d\theta \] 

where \( P, P_1, P_2, P_3, P_4, P_5, S_1, S_2, S_3 \) and \( S_4 \) are all the symmetric positive-definite matrices with appropriate dimensions to be determined. Using Itô’s differential formula, we obtain the stochastic differential

\[ dV(x(t), t) = LV(x(t), t) \, dt + 2x(t)P[F(t)x(t) + F_d(t)x(t - h(t))] \, d\omega(t) \] 

where

\[ LV(x(t), t) = 2x^T(t)P[(A(t) + B(t)K)x(t) + (A_d(t) + B_d(t)K)x(t - h(t))] \]

\[ + x^T(t)(P_1 + P_2 + P_3 + P_4 + P_5)x(t) - (1 - h(t))x^T(t - h(t))P_1x(t - h(t)) + x^T(t - h_1)P_2x(t - h_1) - x^T(t - h_2)P_3x(t - h_2) - x^T(t - h_3)P_4x(t - h_3) - x^T(t - h_4)P_5x(t - h_4) + y^T(t)(h_1^2S_1 + hS_2 + hS_3 + hS_4)y(t) \]

\[ - h_1 \int_{t-h_1}^{t} y^T(s)S_1y(s) \, ds - \int_{t-h_1}^{t-h_2} y^T(s)S_2y(s) \, ds \]

\[ - \int_{t-h_2}^{t-h_3} y^T(s)S_3y(s) \, ds - \int_{t-h_3}^{t-h_4} y^T(s)S_4y(s) \, ds \]

\[ + [F(t)x(t) + F_d(t)x(t - h(t))]^T P[F(t)x(t) + F_d(t)x(t - h(t))] \]

It is noted that, for any \( t > 0, h(t) \in [h_1, h_2] \) or \( h(t) \in [h_2, h_3] \) or \( h(t) \in [h_3, h_4] \). We will discuss \( LV(x(t), t) \) in the following three cases.

**Case 1.** For \( h(t) \in [h_1, h_2] \).

According to Lemma 2, it can be easily verified that

\[ - h_1 \int_{t-h_1}^{t} z^T(s)S_1z(s) \, ds \leq \int_{t-h_1}^{t} z^T(s) \, ds \cdot S_1 \cdot \int_{t-h_1}^{t} z(s) \, ds \]

\[ = \begin{bmatrix} x(t) \\ x(t - h_1) \end{bmatrix}^T \begin{bmatrix} -S_1 & S_1 \\ S_1 & -S_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t - h_1) \end{bmatrix} \] 

\[ \leq \begin{bmatrix} x(t - h_2) \\ x(t - h_3) \end{bmatrix}^T \begin{bmatrix} -\delta S_3 & \delta S_3 \\ \delta S_3 & -\delta S_3 \end{bmatrix} \begin{bmatrix} x(t - h_2) \\ x(t - h_3) \end{bmatrix} \] 

\[ \leq \begin{bmatrix} x(t - h_2) \\ x(t - h_3) \end{bmatrix}^T \begin{bmatrix} -\delta S_4 & \delta S_4 \\ \delta S_4 & -\delta S_4 \end{bmatrix} \begin{bmatrix} x(t - h_2) \\ x(t - h_3) \end{bmatrix} \]

Define \( \xi(t) = [x^T(t - h_1) x^T(t - h(t)) x^T(t - h_2)]^T, X = [X_1^T X_2^T X_3^T]^T, Y = [Y_1^T Y_2^T Y_3^T]^T, \)

\[ x = X^T \xi(t), \quad z = \int_{t-h_2}^{t-h(t)} z(s) \, ds, \quad \text{and} \quad P = S_2/(h_2 - h(t)). \]
From (3), we have
\[-2\xi^T(t)X \int_{t-h(t)}^{t-h(t)} z(s) \, ds \leq (h_2 - h(t))\xi^T(t)XS_2^{-1}X^T\xi(t)
+ \frac{1}{(h_2 - h(t))} \left( \int_{t-h(t)}^{t-h(t)} z(s) \, ds \right)^T S_2 \left( \int_{t-h(t)}^{t-h(t)} z(s) \, ds \right).
\]  \hspace{1cm} (18)

Using Lemma 2, we can replace the right-hand side of (18) as
\[-2\xi^T(t)X[x(t - h(t)) - x(t - h_2)] \leq (h_2 - h(t))\xi^T(t)XS_2^{-1}X^T\xi(t) + \int_{t-h(t)}^{t-h(t)} y^T(s)S_2y(s) \, ds.
\]

which, in other words, can be expressed as
\[-\int_{t-h(t)}^{t-h(t)} z^T(s)S_2z(s) \, ds \leq (h_2 - h(t))\xi^T(t)XS_2^{-1}X^T\xi(t) + 2\xi^T(t)X[x(t - h(t)) - x(t - h_2)].
\]

Similarly, we can deduce the following inequality as well:
\[-\int_{t-h(t)}^{t-h(t)} z^T(s)S_2z(s) \, ds \leq (h(t) - h_1)\xi^T(t)YS_2^{-1}Y^T\xi(t) + 2\xi^T(t)Y[x(t - h_1) - x(t - h(t))].
\]

Summation of the last two equations yields
\[-\int_{t-h(t)}^{t-h(t)} z^T(s)S_2z(s) \, ds = -\int_{t-h(t)}^{t-h_1} z^T(s)S_2z(s) \, ds - \int_{t-h(t)}^{t-h(t)} z^T(s)S_2z(s) \, ds
\leq \xi^T(t)\{(h_2 - h(t))XS_2^{-1}X^T + (h(t) - h_1)YS_2^{-1}Y^T\}\xi(t)
+ 2\xi^T(t)X[x(t - h(t)) - x(t - h_2)]
+ 2\xi^T(t)Y[x(t - h_1) - x(t - h(t))].
\]  \hspace{1cm} (19)

From (11), we obtain
\[2z^T(t)P[[\{A(t) + B(t)K\}x(t) + (A_d(t) + B_d(t)K)x(t - h(t)) - z(t)] \, dt
+ [F(t)x(t) + F_d(t)x(t - h(t))] \, d\omega(t)] = 0
\]  \hspace{1cm} (20)

Adding the left-hand side of (20) to (13) results in
\[d\tilde{V}(x(t), t) = L\tilde{V}(x(t), t) \, dt + 2(x(t)P + z^T(t)P)[F(t)x(t) + F_d(t)x(t - h(t))] \, d\omega(t)
\]  \hspace{1cm} (21)

where
\[L\tilde{V}(x(t), t) = LV(x(t), t) + 2z^T(t)P[(A(t) + B(t)K)x(t) + (A_d(t) + B_d(t)K)x(t - h(t)) - z(t)].
\]  \hspace{1cm} (22)
Taking the mathematical expectation of both sides of (21), and combining (14)–(17), (19) together, we can easily get

\[ E[L\hat{V}(x(t), t)] \leq E[\zeta(t)(\Omega + (h_2 - h(t))X_aS_2^{-1}X_a^T + (h(t) - h_1)Y_aS_2^{-1}Y_a^T)\zeta(t)] \]

where

\[ \zeta(t) = [x^T(t)(t - h(t))x^T(t - h_1)x^T(t - h_2)x^T(t - h_2)(t - h_4)z^T(t)]^T, \quad \Omega = (\Omega_{ij})_{7 \times 7}, \quad \Omega_{ij} = \Omega_{ji}, \]

with

\[
\begin{align*}
\Omega_{11} & = P_1 + P_2 + P_3 + P_4 + P_5 - S_1 + P(A(t) + B(t)K) + (A(t) + B(t)K)^TP + F^T(t)PF(t), \\
\Omega_{12} & = P(A_d(t) + B_d(t)K) + F^T(t)PF_d(t), \quad \Omega_{14} = \Omega_{15} = \Omega_{16} = 0, \quad \Omega_{17} = (A(t) + B(t)K)^TP, \\
\Omega_{13} & = S_1, \quad \Omega_{22} = -(1 - d)P_1 + T_2 + T_2^T - Y_2 - Y_2^T + F^T(t)PF_d(t), \quad \Omega_{23} = -Y_1^T + Y_2 + T_1^T, \\
\Omega_{13} & = -T_2 + T_3^T - Y_3^T, \quad \Omega_{25} = \Omega_{26} = 0, \quad \Omega_{27} = (A_d(t) + B_d(t)K)^TP, \quad \Omega_{33} = -P_2 - S_1 + Y_1 + Y_1^T, \\
\Omega_{34} & = -T_1 + Y_1^T, \quad \Omega_{35} = \Omega_{36} = \Omega_{37} = 0, \quad \Omega_{44} = -P_3 - \delta S_3 - T_3 - T_3^T, \\
\Omega_{45} & = \delta S_3, \quad \Omega_{46} = \Omega_{47} = 0, \\
\Omega_{55} & = -P_4 - \delta S_3 - \delta S_4, \quad \Omega_{56} = \delta S_4, \quad \Omega_{57} = \Omega_{67} = 0, \\
\Omega_{66} & = -P_5 - \delta S_4, \\
\Omega_{77} & = -2P + h_1^2S_1 + hS_2 + hS_3 + hS_4, \quad \delta = h^{-1}, \\
Y_a & = [0 \quad Y_2^T \quad Y_1^T \quad Y_3^T \quad 0 \quad 0 \quad 0]^T, \quad X_a = [0 \quad X_2^T \quad X_1^T \quad X_3^T \quad 0 \quad 0 \quad 0]^T
\end{align*}
\]

If

\[ \Omega + (h_2 - h(t))X_aS_2^{-1}X_a^T + (h(t) - h_1)Y_aS_2^{-1}Y_a^T < 0 \] (23)

then \( \hat{L}\hat{V}(x(t), t) < 0 \), which means that the system (5) with \( \nu(t) = 0 \) is robustly stochastically stable.

By using Lemma 4, the convex LMI condition of (23) can be solved nonconservatively at boundary conditions as follows:

\[ \Omega + (h_2 - h_1)Y_aS_2^{-1}Y_a^T < 0 \] (24)
\[ \Omega + (h_2 - h_1)X_aS_2^{-1}X_a^T < 0 \] (25)

Then, pre- and post-multiply (24) and (25) by diag\{P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}\}. Let \( \tilde{X} = P^{-1}, \quad \tilde{X}P\tilde{X} = Q, \quad S_k^{-1} = R_k, \quad \tilde{X}Y_j\tilde{X} = L_i, \quad \tilde{X}X_j\tilde{X} = G_j, \quad i = 1, \cdots, 5, \quad k = 1, \cdots, 4 \text{ and } j = 1, 2, 3. \) By Schur complement and (4) of Lemma 1, it is easy to check that (9) and (10) with \( i = 1 \) are equivalent to (24) and (25), respectively.

Case 2. For \( h(t) \in [h_2, h_3] \).
By using Lemma 2, we get

\[-h_1 \int_{t-h_1}^t \dot{z}(s) S_1 z(s) \, ds \leq \begin{bmatrix} x(t) \\ x(t-h_1) \end{bmatrix}^T \begin{bmatrix} -S_1 & S_1 \\ S_1 & -S_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h_1) \end{bmatrix} \] (26)

\[- \int_{t-h_1}^{t-h_2} \dot{z}(s) S_2 z(s) \, ds \leq \begin{bmatrix} x(t-h_1) \\ x(t-h_2) \end{bmatrix}^T \begin{bmatrix} -\delta S_2 & \delta S_2 \\ \delta S_2 & -\delta S_2 \end{bmatrix} \begin{bmatrix} x(t-h_1) \\ x(t-h_2) \end{bmatrix} \] (27)

\[- \int_{t-h_3}^{t-h_4} \dot{z}(s) S_4 z(s) \, ds \leq \begin{bmatrix} x(t-h_3) \\ x(t-h_4) \end{bmatrix}^T \begin{bmatrix} -\delta S_4 & \delta S_4 \\ \delta S_4 & -\delta S_4 \end{bmatrix} \begin{bmatrix} x(t-h_3) \\ x(t-h_4) \end{bmatrix} . \] (28)

Note that

\[- \int_{t-h_2}^{t-h_3} \dot{z}(s) S_3 z(s) \, ds = - \int_{t-h(t)}^{t-h_2} \dot{z}(s) S_3 z(s) \, ds - \int_{t-h_3}^{t-h(t)} \dot{z}(s) S_3 z(s) \, ds \leq \dot{\xi}^T(t) \{(h_3 - h(t)) X S_3^{-1} X^T + (h(t) - h_2) Y S_3^{-1} Y^T\} \dot{\xi}(t) + 2\dot{\xi}^T(t) X [x(t-h(t)) - x(t-h_3)] + 2\dot{\xi}^T(t) Y [x(t-h_2) - x(t-h(t))]] \] (29)

where

\[\dot{\xi}(t) = [x^T(t-h_2) \quad x^T(t-h(t)) \quad x^T(t-h_3)]^T, \quad X = [X_1^T \quad X_2^T \quad X_3^T]^T \text{ and } Y = [Y_1^T \quad Y_2^T \quad Y_3^T]^T.\]

From (22) and (26)–(29), it is easy to show that

\[E[L \tilde{V}(x(t), t)] \leq E[\dot{\xi}^T(t) (\tilde{\Omega} + (h_3 - h(t)) X_b S_3^{-1} X_b^T + (h(t) - h_2) Y_b S_3^{-1} Y_b^T) \dot{\xi}(t)]\]

where

\[Y_b = [0 \quad Y_2^T \quad 0 \quad Y_1^T \quad Y_3^T \quad 0 \quad 0]^T, \quad X_b = [0 \quad X_2^T \quad 0 \quad X_1^T \quad X_3^T \quad 0 \quad 0]^T,\]

\[
\tilde{\Omega} = \begin{bmatrix}
\Omega_{11} & \Omega_{12} & S_1 & 0 & 0 & 0 & \Omega_{17} \\
\Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} & 0 & \Omega_{27} \\
\Omega_{31} & \Omega_{32} & \Omega_{33} & \delta S_2 & 0 & 0 & 0 \\
\Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} & \Omega_{34} & 0 & 0 \\
\Omega_{51} & \Omega_{52} & \Omega_{53} & \Omega_{54} & \Omega_{55} & \delta S_4 & 0 \\
\Omega_{61} & \Omega_{62} & \Omega_{63} & \Omega_{64} & \Omega_{65} & \Omega_{66} & -P_5 - \delta S_4 \\
\Omega_{71} & \Omega_{72} & \Omega_{73} & \Omega_{74} & \Omega_{75} & \Omega_{76} & \Omega_{77} \\
\end{bmatrix},
\]

with

\[\tilde{\Omega}_{33} = -P_2 - S_1 - \delta S_2, \quad \tilde{\Omega}_{44} = -P_3 - \delta S_2 + Y_1 + Y_1^T, \quad \tilde{\Omega}_{55} = -P_4 - \delta S_4 - T_3 - T_3^T.\]
Then, using a similar method in Case 1, it can be concluded that (9) and (10) with \( i = 2 \) are equivalent to
\[
\hat{\Omega} + (h_3 - h(t))X_bS_3^{-1}X_b^T + (h(t) - h_2)Y_bS_4^{-1}Y_b^T < 0,
\]
which means \( L\hat{V}(x(t), t) < 0 \).

Case 3. For \( h(t) \in [h_3, h_4] \). Similarly, we also have
\[
- \int_{t-h_4}^{t-h_3} z^T(s)S_4z(s) \, ds = - \int_{t-h(t)}^{t-h_4} z^T(s)S_4z(s) \, ds - \int_{t-h_4}^{t-h_3} z^T(s)S_4z(s) \, ds
\]
\[
\leq \xi^T(t)((h_4 - h(t))X_4^{-1}X^T + (h(t) - h_3)Y_4^{-1}Y^T)\xi(t)
\]
\[
+ 2\xi^T(t)X[x(t - h(t)) - x(t - h_4)]
\]
\[
+ 2\xi^T(t)Y[x(t - h_3) - x(t - h_4)]
\]
(30)
where
\[
\xi(t) = [x^T(t - h_3) x^T(t - h(t)) x^T(t - h_4)]^T, \quad X = [X_1^T \ X_2^T \ X_3^T]^T \text{ and } Y = [Y_1^T \ Y_2^T \ Y_3^T]^T.
\]
Substituting (15), (16), (27) and (30) into (22), it can be shown that
\[
E[L\hat{V}(x(t), t)] \leq E[\xi^T(t)(\hat{\Omega} + (h_4 - h(t))X_4^{-1}X^T + (h(t) - h_3)Y_4^{-1}Y^T)\xi(t)]
\]
where
\[
\hat{\Omega} =
\begin{bmatrix}
\Omega_{11} & \Omega_{12} & S_1 & 0 & 0 & 0 & \Omega_{17} \\
* & \Omega_{22} & 0 & \Omega_{23} & \Omega_{24} & \Omega_{27} \\
* & * & \hat{\Omega}_{33} & \delta S_2 & 0 & 0 \\
* & * & * & \hat{\Omega}_{44} & \delta S_3 & 0 \\
* & * & * & * & \hat{\Omega}_{55} & \Omega_{34} \\
* & * & * & * & * & \hat{\Omega}_{66} \\
* & * & * & * & * & * & \Omega_{77}
\end{bmatrix},
\]
\[
\hat{\Omega}_{33} = -P_2 - S_1 - \delta S_2, \quad \hat{\Omega}_{44} = -P_3 - \delta S_2 - \delta S_3,
\]
\[
\hat{\Omega}_{55} = -P_4 - \delta S_3 + Y_1 + Y_1^T, \quad \hat{\Omega}_{66} = -P_5 - T_3 - T_3^T,
\]
\[
Y_c = [0 \ Y_2^T \ 0 \ 0 \ Y_1^T \ Y_3^T \ 0]^T,
\]
\[
T_c = [0 \ T_2^T \ 0 \ 0 \ T_1^T \ T_3^T \ 0]^T.
\]

Similar to the above analysis methods, it can be seen that (9) and (10) with \( i = 3 \) can guarantee
\( L\hat{V}(x(t), t) < 0 \).

From the above discussion, we can see that (9) and (10) with \( i = 1, 2, 3 \) can guarantee the robust
stochastic stability of the system (5).

\[\square\]

3.1.3 Stability criteria for \( N \) subintervals

Define \( h_m = h_1, \ h_M = h_{N+1}, \ h = (h_M - h_m)/N \), then \( h_i = h_1 + (i-1)(h_M - h_m)/N, \ i = 1, 2, \ldots, N \).
We have the following general theorem.

**Theorem 3** Consider the uncertain stochastic time-delay system (5). If for prescribed scalars \( 0 < h_m < h_M, \ d, \) there exist symmetric positive-definite matrices \( \tilde{X}, \ Q_i, \ i = 1, 2, \ldots, N + 2, \ \tilde{R}_i, \ i = 1, 2, \ldots, N + 1, \) matrices \( Y, \ G_i, \ L_i, \ (i = 1, 2, 3) \) and scalars \( \varepsilon_1 > 0, \ \varepsilon_2 > 0 \) such
that the following LMIs hold:

\[
\begin{bmatrix}
\Pi_i + \tilde{\Pi}_i + \tilde{\Pi}_i^T & \Pi_{12} & hX_i \\
* & \Pi_{22} & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0,
\begin{bmatrix}
\Pi_i + \tilde{\Pi}_i + \tilde{\Pi}_i^T & \Pi_{12} & hY_i \\
* & \Pi_{22} & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2, \ldots, N + 1
\]

where

\[
\Pi_{12} = \begin{bmatrix} H^T & 0_{3 \times (N+2)} \end{bmatrix}^T, \quad \Pi_{22} = \text{diag}(-\bar{X} + \varepsilon_2 MM^T - \varepsilon_1 I - \varepsilon_2 I),
\]

\[
\tilde{\Pi}_i = \begin{bmatrix} 0 & Y_1 & -X_1 & -Y_1 & 0 \cdots 0 \end{bmatrix}^T, \quad \tilde{\Pi}_2 = \begin{bmatrix} 0 & Y_2 & -X_2 & 0 & X_2 & -Y_2 \cdots 0 \end{bmatrix}^T, \ldots,
\]

\[
\tilde{\Pi}_N = \begin{bmatrix} 0 & Y_N - X_N & 0 \cdots 0 & X_N - Y_N & 0 \end{bmatrix}^T,
\]

\[
\begin{bmatrix}
\Pi_{11} & A_d \bar{X} + B_d Y & -R_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
* & -(1 - d)Q_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \varphi_1 \\
* & * & -Q_2 - R_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
* & * & * & \Pi_{1,4} & \delta R_3 & 0 & \cdots & 0 & 0 & 0 \\
* & * & * & * & \Pi_{1,5} & \delta R_4 & \cdots & 0 & 0 & 0 \\
* & * & * & * & * & \Pi_{1,6} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & * & * & * & * & * & * & \Pi_{1,N+2} & \delta R_{N+1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & * & * & * & * & * & * & * & \Pi_{1,N+3} & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Pi_{11} & A_d \bar{X} + B_d Y & -R_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
* & -(1 - d)Q_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \varphi_1 \\
* & * & \Pi_{2,3} & \delta R_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
* & * & * & \Pi_{2,4} & 0 & 0 & \cdots & 0 & 0 & 0 \\
* & * & * & \Pi_{2,5} & \delta R_4 & \cdots & 0 & 0 & 0 & 0 \\
* & * & * & * & \Pi_{1,6} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & * & * & * & * & * & * & \Pi_{1,N+2} & \delta R_{N+1} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
* & * & * & * & * & * & * & * & * & * & \Pi_{1,N+3} & 0
\end{bmatrix}
\]
with \( \Pi_1 = A\tilde{X} + \tilde{X}A^T + BY + Y^TB^T + Q_1 + \cdots + Q_{N+2} - R_1 + \varepsilon_1 MM^T, \) \( \varphi_1 = \tilde{X}A^T + Y^TB^T + \varepsilon_1 MM^T, \) \( \varphi_2 = \tilde{X}A^T + Y^TB^T, \) \( \varphi_{N+4} = h_1^2 R_1 + h R_2 + h R_3 + \cdots + h R_{N+1} - 2\tilde{X} + \varepsilon_1 MM^T, \) \( \Pi_{N,N+2} = -Q_{N+1} - \delta R_N, \) \( \Pi_{1,4} = -Q_3 - \delta R_3, \) \( \Pi_{1,5} = -Q_4 - \delta R_3 - \delta R_4, \) \( \Pi_{1,6} = -Q_5 - \delta R_4 - \delta R_5, \) \( \Pi_{1,1+j} = -Q_{j+1} - \delta R_{j+1} - \delta R_{j+2}, \) \( \Pi_{1,1+N} = -Q_{N+1} - \delta R_N - \delta R_{N+1}, \) \( \Pi_{1,1+N+3} = -Q_{N+2} - \delta R_{N+1}, \) \( \Pi_{2,3} = -Q_2 - R_1 - \delta R_2, \) \( \Pi_{2,4} = -Q_3 - \delta R_2, \) \( \Pi_{2,5} = -Q_4 - \delta R_4, \) \( \Pi_{N,N+4} = -Q_3 - \delta R_2 - \delta R_3, \) \( X_1 = \begin{bmatrix} L_2 & L_1 & L_3 & 0 & \cdots & 0 & \cdots & 0 \\ \end{bmatrix}^T, \) \( X_2 = \begin{bmatrix} 0 & L_2 & 0 & L_1 & L_3 & 0 & \cdots & 0 \\ \end{bmatrix}^T, \) \( \cdots, X_N = \begin{bmatrix} 0 & L_2 & 0 & \cdots & 0 & L_1 & L_3 & 0 & \cdots & 0 \\ \end{bmatrix}^T, \) \( Y_1 = \begin{bmatrix} 0 & G_2^T & G_1^T & G_3^T & 0 & \cdots & 0 & \cdots & 0 \\ \end{bmatrix}^T, \) \( Y_2 = \begin{bmatrix} 0 & G_2^T & G_1^T & G_3^T & 0 & \cdots & 0 & \cdots & 0 \\ \end{bmatrix}^T, \) \( h = (h_M - h_m) / N, \) \( Y_2 = \begin{bmatrix} 0 & G_2^T & G_1^T & G_3^T & 0 & \cdots & 0 & \cdots & 0 \\ \end{bmatrix}^T, \) \( \delta = h^{-1} \) and \( H \) is defined in Theorem 1, then the system (5) is robustly stochastically stabilizable. Moreover, the state feedback controller is presented by \( u(t) = Y\tilde{X}^{-1}x(t). \)

**Proof.** Construct the Lyapunov–Krasovskii functional as

\[
V(x(t), t) = x^T(t)P_1x(t) + \int_{t-h(t)}^{t} x^T(s)P_1x(s)\, ds + \int_{t-h_i}^{t} x^T(s)P_{i+1}x(s)\, ds + h_1 \int_{-h_i}^{0} \int_{t+\theta}^{t} z^T(s)S_1z(s)\, ds\, d\theta + \int_{-h_{i-1}}^{t-h_i} \int_{t+\theta}^{t} z^T(s)S_1z(s)\, ds\, d\theta \]

where \( P, P_i, i = 1, 2, \cdots, N + 2 \) and \( S_i, i = 1, 2, \cdots, N + 1 \) are all symmetric positive-definite matrices with appropriate dimensions. Choose

\[
\eta(t) = [x^T(t)x^T(t-h(t))x^T(t-h_1) \cdots x^T(t-h_i)x^T(t-h_{i+1}) \cdots x^T(t-h_N)x^T(t-h_{N+1})z^T(t)]^T.
\]
In the following, we will discuss the stochastic differential of Lyapunov–Krasovskii functional (31) for $N$ cases, that is, $h(t) \in [h_1, h_2]$ or $h(t) \in [h_2, h_3]$ or . . . or $h(t) \in [h_N, h_{N+1}]$. Then, the proof can be completed by following similar line as in the proof of Theorem 2, we omit it here.

By definition, it has been supposed that $h_m > 0$. In the case that the lower bound on the time delay was unknown or exactly zero, it would be necessary to consider the case $h_m = 0$. In such circumstances, LMIs in Theorems 1–3 cannot be solved. So, we can draw the following corollaries.

Corollary 1 Consider the uncertain stochastic time-delay system (5). If for prescribed scalars $h_M > 0$, $d$, there exist symmetric positive-definite matrices $X, Q_1, Q_2, Q_3, R_2, R_3$, matrices $Y, L_i, G_i (i = 1, 2, 3)$ and scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that the following LMIs hold:

$$
\begin{bmatrix}
\tilde{\Gamma}_1 + \tilde{\Gamma}_1^T & \tilde{\Gamma}_{12} & h\tilde{U}_i \\
* & \tilde{\Gamma}_{22} & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0,
$$

$$
\begin{bmatrix}
\tilde{\Gamma}_i + \tilde{\Gamma}_i^T & \tilde{\Gamma}_{12} & h\tilde{V}_i \\
* & \tilde{\Gamma}_{22} & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2
$$

where

$$
\tilde{\Gamma}_1 = \begin{bmatrix}
A_d\tilde{X} + B_dY & 0 & 0 & \tilde{\Gamma}_{15} \\
* & -(1-d)Q_1 & 0 & 0 & \tilde{\Gamma}_{25} \\
* & * & -Q_3 - \delta R_3 & \delta R_3 & 0 \\
* & * & * & -Q_3 - \delta R_3 & 0 \\
* & * & * & * & \tilde{\Gamma}_{55}
\end{bmatrix},
$$

$$
\tilde{\Gamma}_2 = \begin{bmatrix}
\tilde{\Gamma}_{11} - \delta R_2 & A_d\tilde{X} + B_dY & \delta R_2 & 0 & \tilde{\Gamma}_{15} \\
* & -(1-d)Q_1 & 0 & 0 & \tilde{\Gamma}_{25} \\
* & * & -Q_3 - \delta R_2 & 0 & 0 \\
* & * & * & -Q_3 - \delta R_2 & 0 \\
* & * & * & * & \tilde{\Gamma}_{55}
\end{bmatrix},
$$

$$
\tilde{\Gamma}_{12} = \begin{bmatrix}
H^T & 0_{3 \times 3}
\end{bmatrix}^T,
$$

$$
\tilde{\Gamma}_1 = \begin{bmatrix}
\tilde{U}_1 & \tilde{V}_1 - \tilde{U}_1 & -\tilde{V}_1 & 0 & 0
\end{bmatrix},
$$

$$
\tilde{\Gamma}_2 = \begin{bmatrix}
0 & \tilde{V}_2 - \tilde{U}_2 & \tilde{U}_2 & \tilde{V}_2 & 0
\end{bmatrix},
$$

with

$$
\tilde{\Gamma}_{11} = A\tilde{X} + \tilde{X}A^T + BY + Y^TB^T + Q_1 + Q_3 + Q_4 + \varepsilon_1MM^T,
$$

$$
\tilde{\Gamma}_{15} = \tilde{X}A^T + Y^TB^T + \varepsilon_1MM^T,
$$

$$
\tilde{\Gamma}_{25} = \tilde{X}A_d^T + Y^TB_d^T,
$$

$$
\tilde{\Gamma}_{55} = hR_2 + hR_3 - 2\tilde{X} + \varepsilon_1MM^T,
$$

$$
\tilde{\Gamma}_i = [L_1^T \quad L_2^T \quad L_3^T \quad 0 \quad 0]^T, \quad h = h_M/2,
$$

$$
\tilde{\Gamma}_i = [G_1^T \quad G_2^T \quad G_3^T \quad 0 \quad 0]^T, \quad \tilde{U}_2 = [0 \quad L_2^T \quad L_1^T \quad L_3^T \quad 0]^T,
$$

$$
\tilde{\Gamma}_{ii} = [G_1^T \quad G_2^T \quad G_3^T \quad 0 \quad 0]^T.
$$

$\delta = h^{-1}$ and $H, \Gamma_{22}$ are defined in Theorem 1. Then the system (5) is robustly stochastically stabilizable. Moreover, the state feedback controller is presented by $u(t) = YY^{-1}x(t)$.

Proof. It is easy to prove it by the similar idea used for Theorem 1. □

Corollary 2 Consider the uncertain stochastic time-delay system (5). If for prescribed scalars $h_M > 0$, $d$, there exist symmetric positive-definite matrices $X, Q_1, Q_2, Q_3, R_2, R_3, R_4$, matrices $Y, L_i, G_i$
\((i = 1, 2, 3)\) and scalars \(\varepsilon_1 > 0, \varepsilon_2 > 0\) such that the following LMI holds:

\[
\begin{bmatrix}
\tilde{\Sigma}_i + \tilde{\Sigma}_i + \tilde{\Sigma}_i^T & \Sigma_{12} & h\tilde{U}_i \\
\ast & \Sigma_{22} & 0 \\
\ast & \ast & -hR_{i+1}
\end{bmatrix} < 0, \quad \begin{bmatrix}
\tilde{\Sigma}_i + \tilde{\Sigma}_i + \tilde{\Sigma}_i^T & \Sigma_{12} & h\tilde{V}_i \\
\ast & \Sigma_{22} & 0 \\
\ast & \ast & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2, 3
\]

where

\[
\tilde{\Sigma}_1 = \begin{bmatrix}
\tilde{\Sigma}_{11} & A_d\tilde{X} + B_dY & 0 & 0 & 0 & \tilde{\Sigma}_{16} \\
\ast & -(1-d)Q_1 & 0 & 0 & 0 & \tilde{\Sigma}_{26} \\
\ast & \ast & -Q_3 - \delta R_3 & \delta R_3 & 0 & 0 \\
\ast & \ast & \ast & -Q_4 - \delta R_3 - \delta R_4 & \delta R_4 & 0 \\
\ast & \ast & \ast & \ast & -Q_5 - \delta R_4 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\]

\[
\tilde{\Sigma}_2 = \begin{bmatrix}
\tilde{\Sigma}_{11} - \delta R_2 & A_d\tilde{X} + B_dY & \delta R_2 & 0 & 0 & \tilde{\Sigma}_{16} \\
\ast & -(1-d)Q_1 & 0 & 0 & 0 & \tilde{\Sigma}_{26} \\
\ast & \ast & -Q_3 - \delta R_2 & \delta R_3 & 0 & 0 \\
\ast & \ast & \ast & -Q_4 - \delta R_3 & \delta R_4 & 0 \\
\ast & \ast & \ast & \ast & -Q_5 - \delta R_4 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\]

\[
\tilde{\Sigma}_3 = \begin{bmatrix}
\tilde{\Sigma}_{11} - \delta R_2 & A_d\tilde{X} + B_dY & \delta R_2 & 0 & 0 & \tilde{\Sigma}_{16} \\
\ast & -(1-d)Q_1 & 0 & 0 & 0 & \tilde{\Sigma}_{26} \\
\ast & \ast & -Q_3 - \delta R_2 - \delta R_3 & \delta R_3 & 0 & 0 \\
\ast & \ast & \ast & -Q_4 - \delta R_3 & \delta R_4 & 0 \\
\ast & \ast & \ast & \ast & -Q_5 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast
\end{bmatrix}
\]

\[
\tilde{\Sigma}_1 = [\tilde{U}_1 \quad \tilde{V}_1 - \tilde{U}_1 \quad -\tilde{V}_1 \quad 0 \quad 0 \quad 0], \quad \tilde{\Sigma}_2 = [0 \quad \tilde{V}_2 - \tilde{U}_2 \quad \tilde{U}_2 - \tilde{V}_2 \quad 0 \quad 0], \quad h = h_M/3
\]

\[
\tilde{\Sigma}_3 = [0 \quad \tilde{V}_3 - \tilde{U}_3 \quad 0 \quad \tilde{U}_3 - \tilde{V}_3 \quad 0], \quad \tilde{\Sigma}_{12} = [H^T \quad 0_{3 \times 4}]^T, \quad \delta = h^{-1}
\]

with

\[
\tilde{\Sigma}_{11} = \tilde{A}\tilde{X} + \tilde{X}A^T + BY + Y^TB^T + Q_1 + Q_3 + Q_4 + Q_5 + \varepsilon_1 MM^T, \quad \tilde{\Sigma}_{16} = \tilde{X}A^T + Y^TB^T + \varepsilon_1 MM^T, \quad \tilde{\Sigma}_{26} = \tilde{X}A_dY + Y^TB_d^T, \quad \tilde{\Sigma}_{66} = hR_2 + hR_3 + hR_4 - 2\tilde{X} + \varepsilon_1 MM^T, \quad \tilde{\Sigma}_1 = [L_{11}^T \quad L_{12}^T \quad L_{13}^T \quad 0 \quad 0]^T, \quad \tilde{\Sigma}_2 = [0 \quad G_1^T \quad G_1^T \quad 0 \quad 0]^T, \quad \tilde{\Sigma}_3 = [0 \quad L_2^T \quad L_2^T \quad L_3^T \quad 0 \quad 0]^T, \quad \tilde{\Sigma}_4 = [0 \quad G_2^T \quad G_2^T \quad G_3^T \quad 0 \quad 0]^T, \quad \text{and } H, \Sigma_{22}, \Sigma_{26} \text{ are defined in Theorem 2.}
\]

**Proof.** In the case of \(h_m = h_1 = 0\), the integral \(\int_{t-h_1}^{t} x^T(s)P_2x(s) \, ds, \int_{t-h_1}^{t} \int_{t-h_1}^{t} z^T(s)S_1z(s) \, ds \, d\theta \) disappear from (12). Obviously, the state vectors \(x(t - h_1)\) and \(\int_{t-h_1}^{t} z^T(s)S_1z(s) \, ds\) have to be suppressed from the augmented state vector, taking into account that in the first subinterval \(x(t - h_1) = x(t - 0) = 0\).

Then, the proof is similar to that of Theorem 2, and is therefore omitted here. \(\square\)
Remark 1 Note that Theorems 1–3 present new delay-dependent stabilization criteria for uncertain stochastic time-delay systems. On the one hand, some weighting matrices are introduced to reduce the conservatism of the results; on the other hand, a new technique, which estimates the upper bound of the stochastic differential of Lyapunov–Krasovskii functional without neglecting any useful terms, is established to deduce the possible conservatism. It is also worth mentioning that the reduced conservatism of Theorems 1–3 lies in the construction of the appropriate Lyapunov–Krasovskii functional and the idea of delay partitioning technique. The conservatism reduction becomes more obvious as the delay partition number increases, which will be illustrated via several examples later.

Remark 2 In the case that \(h_m = 0\), when the interval time-varying delay is decomposed into two equidistant subintervals, Theorem 1 reduces to equidistant subintervals, Theorem 2 reduces to Corollary 2. Therefore, when the interval time-varying delay is decomposed into \(N\) equidistant subintervals, proceeding in a similar manner as above, we also have new results about Theorem 3 in \(h_m = 0\). For brevity, we omit it here.

3.2 Robust \(H_\infty\) control

In this subsection, we focus on the design of \(H_\infty\) controller for uncertain stochastic time-delay systems. Sufficient conditions for the solvability of the robust \(H_\infty\) problem are proposed and an LMI approach for designing the desired state feedback controller is developed.

3.2.1 Controller design for \(N = 2\) For \(N = 2\), define \(h_1 = h_m, h_3 = h_M, h = (h_3 - h_1)/2\).

Theorem 4 If for prescribed scalars \(\gamma > 0, 0 < h_m < h_M, d\), there exist symmetric positive-definite matrices \(\bar{X}, Q_1, Q_2, Q_3, Q_4, R_1, R_2, R_3\), matrices \(Y, L_i, G_i, (i = 1, 2, 3)\), and scalars \(\varepsilon_1 > 0, \varepsilon_2 > 0\) such that the following LMIs hold:

\[
\begin{bmatrix}
\Gamma_i + \bar{\Gamma}_i + \bar{\Gamma}_i^T & \Lambda & h\bar{U}_i \\
* & \Lambda & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \\
\begin{bmatrix}
\Gamma_i + \bar{\Gamma}_i + \bar{\Gamma}_i^T & \Lambda & h\bar{V}_i \\
* & \Lambda & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2
\]

where

\[
\Lambda_{11} = \begin{bmatrix}
E & \bar{X}F^T & Y^T & X^T & Y^TN_3^T + XN_1^T & XN_5^T \\
0 & \bar{X}F_d^T & Y^TD_d^T + X^T & Y^TN_4^T + XN_2^T & XN_6^T
\end{bmatrix},
\Lambda = \begin{bmatrix}
A_1 & 0 \\
* & A_2
\end{bmatrix},
\]

\[
A_1 = \begin{bmatrix}
-\gamma^2 I & E_d^T \\
* & -\bar{X} + \varepsilon_2 LL^T
\end{bmatrix},
\]

\[
B = [E \ 0 \ 0 \ 0 \ 0], \Lambda_1 = [A_1^T \ 0_{5 \times 3}], \Lambda_2 = \text{diag}(-I - \varepsilon_1 I - \varepsilon_2 I), \text{ and } \Gamma_i, \bar{\Gamma}_i, \bar{U}_i, \bar{V}_i \text{ are defined in Theorem 1. Then the system (6) is robustly stochastically stable with disturbance attenuation } \gamma. \text{ In this case, a suitable stabilizing state feedback controller can be chosen by } K = Y\bar{X}^{-1}.
\]

Proof. It is easy to see that the proof can be completed by using the similar way in Theorem 5, and it is omitted for brevity. \(\square\)
3.2.2 Controller design for $N = 3$

For $N = 3$, define $h_1 = h_m$, $h_4 = h_M$, $h = (h_4 - h_1)/3$.

**Theorem 5** If for prescribed scalars $\gamma > 0$, $0 < h_m < h_M$, $d$, there exist symmetric positive-definite matrices $\bar{X}, Q_1, Q_2, Q_3, Q_4, Q_5, R_1, R_2, R_3, R_4$, matrices $Y, L_i, G_i (i = 1, 2, 3)$ and scalars $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that the following LMIs hold:

$$
\begin{bmatrix}
\Sigma_i + \bar{\Sigma}_i + \bar{\Sigma}_i^T & \Lambda_2 & hU_i \\
* & \Lambda & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2, 3 \quad (32)
$$

$$
\begin{bmatrix}
\Sigma_i + \bar{\Sigma}_i + \bar{\Sigma}_i^T & \Lambda_2 & hV_i \\
* & \Lambda & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2, 3 \quad (33)
$$

where $\Lambda_2 = [I_{11}^T \ 0_{3 \times 4} \ B^T]$ and all the other involved matrices are defined as in Theorems 2 and 4. Then the system (6) is robustly stochastically stable with disturbance attenuation $\gamma$. In this case, a suitable stabilizing state feedback controller can be chosen by $K = Y \bar{X}^{-1}$.

**Proof.** For $t > 0$, we set

$$J(t) = E \left\{ \int_0^t \left[ y^T(s) [y(s) - v^T(s) v(s)] \right] ds \right\}
$$

Then, it is easy to get

$$J(t) = E \left\{ \int_0^t \left[ y^T(s) y(s) - v^T(s) v(s) + LV(x(s), s) \right] ds \right\} - E \{ LV(x(t), t) \}
$$

$$\leq E \left\{ \int_0^t \left[ y^T(s) y(s) - v^T(s) v(s) + LV(x(s), s) \right] ds \right\}, \quad h(t) \in [h_1, h_2]
$$

If $v(t) = 0$, then by Theorem 2, we conclude that the system (6) is robustly stochastically stable. Next, we proceed to prove that the system (6) verifies noise attenuation $\gamma$. Applying Lyapunov–Krasovskii functional (12) to the system (6) and using the same technique as in the proof of Theorem 2, we can easily deduce that

$$J(t) \leq E \left\{ \int_0^t \eta^T(t) \left[ \Omega^* + (h_2 - h(t)) T_a^* S_2^{-1} T_a^* + (h(t) - h_1) Y_a^* S_2^{-1} Y_a^* \right] \eta(t) ds \right\}
$$

where

$$\eta(t) = [\xi^T(t), v^T(t)]^T, \quad \Omega^* = \begin{bmatrix} \Omega_1^* & \Omega_1^T \\ * & \Omega_2^* \end{bmatrix}
$$

$$\Omega_1^* = [E^T P + E_d^T PF(t) \ E_d^T P F_d(t) \ 0 \ 0 \ 0 \ 0 \ E^T P] \Omega_2^* = -\gamma^2 I + E_d^T P F_d,
$$

$$Y_a^* = [0 \ Y_2^T \ Y_1^T \ Y_3^T \ 0 \ 0 \ 0 \ 0]^T, \quad T_a^* = [0 \ T_2^T \ T_1^T \ T_3^T \ 0 \ 0 \ 0 \ 0]^T.
$$

Using the same technique as in the proof of Theorem 2, we can show that (2) and (3) with $i = 1$ imply $\Omega^* + (h_2 - h(t)) T_a^* R_2^{-1} T_a^* + (h(t) - h_1) Y_a^* R_2^{-1} Y_a^* < 0$. It follows that $J(t) < 0$. When $h(t) \in [h_2, h_3]$ or $h(t) \in [h_3, h_4]$, along the similar line as above, we know that (2) and (3) with $i = 2, 3$ can also
guarantee $J(t) < 0$, which implies that for any nonzero $v(t) \in L^2[0, \infty)$, $\|y(t)\|_2 < \gamma \|v(t)\|_2$. Therefore, the system (6) is robustly stochastically stable with disturbance attenuation $\gamma$. This completes the proof. \hfill \square

3.2.3 Controller design for $N$ subintervals Define $h_m = h_1$, $h_M = h_{N+1}$, $h = (h_M - h_m)/N$, then $h_i = h_1 + (i - 1)(h_M - h_m)/N$, $i = 1, 2, \ldots, N$.

**Theorem 6** If for prescribed scalars $\gamma > 0$, $0 < h_m < h_M$, $d$, there exist symmetric positive-definite matrices $\bar{X}, Q_i$, $i = 1, 2, \ldots, N+2$, $R_i$, $i = 1, 2, \ldots, N+1$, matrices $Y$, $L_i$, $G_i$ ($i = 1, 2, 3$) and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the following LMIs hold:

$$
\begin{bmatrix}
\Omega_i + \mathcal{E}_i + \mathcal{E}_i^T & \Lambda N & hX_i \\
* & \Lambda & 0 \\
* & * & -hS_{i+1}
\end{bmatrix} < 0,
\begin{bmatrix}
\Omega_i + \mathcal{E}_i + \mathcal{E}_i^T & \Lambda N & hT_i \\
* & \Lambda & 0 \\
* & * & -hS_{i+1}
\end{bmatrix} < 0,
$$

where $\Lambda_N = [\Lambda_{11}^T \quad 0_{5 \times (N+1)} \quad B^T]$ and all the other involved matrices are defined as in Theorems 3 and 4. Then the system (6) is robustly stochastically stable with disturbance attenuation $\gamma$. In this case, a suitable stabilizing state feedback controller can be chosen by $K = Y\bar{X}^{-1}$.

Similarly, the results of the above theorems can also be reduced to the case of $h_m = 0$ and yield the delay-dependent criteria as follows.

**Corollary 3** If for prescribed scalars $\gamma > 0$, $h_M > 0$, $d$, there exist symmetric positive-definite matrices $\bar{X}, Q_1, Q_2, Q_4$, $R_2, R_3$, matrices $Y$, $L_i$, $G_i$ ($i = 1, 2, 3$), and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the following LMIs hold:

$$
\begin{bmatrix}
\tilde{\Gamma}_i + \tilde{\Gamma}_i + \tilde{\Gamma}_i^T & \Lambda_3 & h\tilde{U}_i \\
* & \Lambda & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0,
\begin{bmatrix}
\tilde{\Gamma}_i + \tilde{\Gamma}_i + \tilde{\Gamma}_i^T & \Lambda_3 & h\tilde{V}_i \\
* & \Lambda & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2,
$$

where $\Lambda_3 = [\Lambda_{11}^T \quad 0_{5 \times 2} \quad B^T]$ and all the other involved matrices are defined as in Corollary 1 and Theorem 4. Then the system (6) is robustly stochastically stable with disturbance attenuation $\gamma$. In this case, a suitable stabilizing state feedback controller can be chosen by $K = Y\bar{X}^{-1}$.

**Corollary 4** If for prescribed scalars $\gamma > 0$, $0 < h_m < h_M$, $d$, there exist symmetric positive-definite matrices $\bar{X}, Q_1, Q_3, Q_4$, $Q_5$, $R_2, R_3, R_4$, matrices $Y$, $L_i$, $G_i$ ($i = 1, 2, 3$), and scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that the following LMIs hold:

$$
\begin{bmatrix}
\bar{\Sigma}_i + \bar{\Sigma}_i + \bar{\Sigma}_i^T & \Lambda_4 & h\bar{U}_i \\
* & \Lambda & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0,
\begin{bmatrix}
\bar{\Sigma}_i + \bar{\Sigma}_i + \bar{\Sigma}_i^T & \Lambda_4 & h\bar{V}_i \\
* & \Lambda & 0 \\
* & * & -hR_{i+1}
\end{bmatrix} < 0, \quad i = 1, 2, 3,
$$

where $\Lambda_4 = [\Lambda_{11}^T \quad 0_{5 \times 3} \quad B^T]$ and all the other involved matrices are defined as in Corollary 2 and Theorem 4. Then the system (6) is robustly stochastically stable with disturbance attenuation $\gamma$. In this case, a suitable stabilizing state feedback controller can be chosen by $K = Y\bar{X}^{-1}$. 
Remark 3 When the time-delay is known but it is not differentiable, namely the delay $h(t)$ satisfies $0 < h(t) < h$. All results still hold by suppressing integral $\int_{t-h(t)}^{t} x^T(s)P_1x(s)\,ds$ from the Lyapunov–Krasovskii functional and by setting $P_1 = 0$ in LMIs. We also develop $H_\infty$ controller for the system (6) with delay satisfying $0 < h(t) < h$. In the case that $h_m = 0$, when $N = 2$, Theorem 4 reduces to Corollary 3; when $N = 3$, Theorem 5 reduces to Corollary 4.

Remark 4 Theorems 4–6 establish the new delay-dependent conditions for the existence of an $H_\infty$ controller. How to develop the less-conservative condition is still a challenging research topic. It is worth pointing out that our criteria extend the constant case as shown in Li et al. (2010), 2014 to the time-varying case, and also extend time-varying case with the restriction $\dot{h}(t) \leq d < 1$ in Gao & Wang (2003) to the case that the derivative for time-delay may be more than 1. The restriction as $d < 1$ limits the application scope of their results. Therefore, our results are more general than the existing ones in the literature. By numerical examples given in the following section, it is shown that, for the same example, the criteria for $N = 3$ can lead to much less conservative results than those by the criteria for $N = 2$. Therefore, larger $N$ would yield better results. However, comparing Theorems 3–5, it can be seen that the criteria become more complicated when $N$ increases. In the following examples, we just provide the criteria for $N = 2$ and 3, which are sufficient enough to show the effectiveness and less conservatism of our method.

4. Numerical examples

In this section, the following numerical examples are given to demonstrate the effectiveness and less conservativeness of our results.

Example 1 Consider the time-delay system with the following parameters borrowed from Xu et al. (2006):

$$
A = \begin{bmatrix}
-0.6238 & -1.0132 \\
2.0116 & -0.2106
\end{bmatrix}, \quad
A_d = \begin{bmatrix}
-0.5011 & -0.7871 \\
-0.3002 & 0.5231
\end{bmatrix}, \quad
B = \begin{bmatrix}
-0.4326 & 0.1253 \\
-1.6656 & 0.2877
\end{bmatrix}, \\
C = \begin{bmatrix}
0.2134 & -0.0191 \\
0.1119 & -0.1665
\end{bmatrix}, \quad
C_d = \begin{bmatrix}
0.0816 & 0.1290 \\
0.0712 & 0.0669
\end{bmatrix}.
$$

Compare our results of $H_\infty$ control with those in Fridman & Shaked (2002), Lee et al. (2004), Xu et al. (2006) and Yang et al. (2010) to show less conservativeness of our results. Our purpose is to find the upper bound of delay and the minimum guaranteed $H_\infty$ performance $\gamma$. Some existing results can be applied to this system. By using Corollaries 1 and 2, we can compute the corresponding maximum allowed delay for different $\gamma$ and the corresponding minimum allowed $\gamma$ for different $h$. The computed results are shown in Tables 1 and 2. Tables 1 and 2 give the comparison results on the maximum allowed delay $h$ for a given $\gamma > 0$ and the minimum allowed $\gamma$ for a given $h > 0$, respectively, via the methods in Fridman & Shaked (2002), Lee et al. (2004) and Xu et al. (2006) and Corollaries 1 and 2.

For a given $\gamma = 2.5$, the maximum allowed delay was found to be 0.581 in Yang et al. (2010), while the maximum allowed delays, based on the Corollaries 1 and 2 in our paper, are 0.609 and 0.647, respectively.

From Tables 1 and 2, it can be seen that our method can lead to less conservative results than those obtained by other existing methods. Furthermore, it is also shown by Tables 1 and 2 that, when $N$
Table 1  Comparison of maximum allowed delay for different given $\gamma$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
<th>4.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fridman &amp; Shaked (2002)</td>
<td>0.406</td>
<td>0.466</td>
<td>0.505</td>
<td>0.532</td>
<td>0.552</td>
<td>0.567</td>
</tr>
<tr>
<td>Lee et al. (2004)</td>
<td>0.406</td>
<td>0.466</td>
<td>0.505</td>
<td>0.532</td>
<td>0.552</td>
<td>0.567</td>
</tr>
<tr>
<td>Xu et al. (2006)</td>
<td>0.4206</td>
<td>0.478</td>
<td>0.515</td>
<td>0.541</td>
<td>0.559</td>
<td>0.573</td>
</tr>
<tr>
<td>Corollary 3</td>
<td>0.717</td>
<td>0.730</td>
<td>0.738</td>
<td>0.745</td>
<td>0.750</td>
<td>0.754</td>
</tr>
<tr>
<td>Corollary 4</td>
<td>0.767</td>
<td>0.775</td>
<td>0.781</td>
<td>0.785</td>
<td>0.789</td>
<td>0.791</td>
</tr>
</tbody>
</table>

Table 2  Comparison of minimum allowed $\gamma$ on fixed time delay $h$

<table>
<thead>
<tr>
<th>$h$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fridman &amp; Shaked (2002)</td>
<td>1.071</td>
<td>1.243</td>
<td>1.507</td>
<td>1.963</td>
<td>2.298</td>
</tr>
<tr>
<td>Lee et al. (2004)</td>
<td>1.071</td>
<td>1.243</td>
<td>1.507</td>
<td>1.963</td>
<td>2.298</td>
</tr>
<tr>
<td>Xu et al. (2006)</td>
<td>1.058</td>
<td>1.211</td>
<td>1.452</td>
<td>1.873</td>
<td>2.776</td>
</tr>
<tr>
<td>Corollary 3</td>
<td>0.028</td>
<td>0.065</td>
<td>0.116</td>
<td>0.195</td>
<td>0.329</td>
</tr>
<tr>
<td>Corollary 4</td>
<td>$2.589 \times 10^{-3}$</td>
<td>$1.181 \times 10^{-2}$</td>
<td>$3.226 \times 10^{-2}$</td>
<td>$7.303 \times 10^{-2}$</td>
<td>$1.521 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Increases, the derived criteria can lead to an improvement on the results by the criteria for the case when $N$ is smaller.

Example 2 Consider the uncertain time-delay stochastic systems with the following parameters:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.2 \\ -0.1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.01 & 0 \\ 0 & -0.01 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.01 & 0.01 \\ 0 & -0.01 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad D = 0.1, \quad D_1 = 1.2, \quad L = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix},$$

$$G_1 = G_2 = G_5 = G_6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_3 = G_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We consider the case of $d = 0$. The aim is to calculate the maximum allowable time delay $h$ such that the system can be stabilized with disturbance attenuation level $\gamma$. Based on the criterion in Li et al. (2009) and Corollaries 1 and 2, the computation results for the allowable upper bound of $h$ are shown in Table 3. Obviously, our results are less conservative than those in the existing reference and, furthermore, it can be seen that increasing $N$ may yield the criteria with less conservativeness.

Suppose that $B_d = \Delta B_d(t) = F = F_d = C_d = D_d = 0$. In this case, the system becomes the one in Chen et al. (2004). Table 4 lists the results of the maximum allowable time delay derived from methods in Chen et al. (2004) and Corollaries 1 and 2 in this paper. It can be seen from Table 4 that our methods
Table 3  Comparison of maximum allowed delay $h$ for a given $\gamma$

<table>
<thead>
<tr>
<th>$\gamma = 1$</th>
<th>Maximum $h$</th>
<th>Controller law $K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Li et al. (2009)</td>
<td>0.344</td>
<td>[0.408 – 1.365]</td>
</tr>
<tr>
<td>Corollary 3</td>
<td>0.459</td>
<td>[0.488 – 1.122]</td>
</tr>
<tr>
<td>Corollary 4</td>
<td>0.475</td>
<td>[0.046 – 0.983]</td>
</tr>
</tbody>
</table>

Table 4  Comparison of maximum allowed delay $h$ for a given $\gamma$

<table>
<thead>
<tr>
<th>$\gamma = 4.86$</th>
<th>$h$</th>
<th>$K$</th>
<th>$\gamma = 1.65$</th>
<th>$h$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chen et al. (2004)</td>
<td>0.390</td>
<td>[0.474 – 2.701]</td>
<td>Chen et al. (2004)</td>
<td>0.3</td>
<td>[−0.060 – 2.959]</td>
</tr>
<tr>
<td>Corollary 3</td>
<td>0.986</td>
<td>[−0.126 – 2.301]</td>
<td>Corollary 3</td>
<td>0.864</td>
<td>[−0.056 – 2.218]</td>
</tr>
<tr>
<td>Corollary 4</td>
<td>1.189</td>
<td>[−0.146 – 2.449]</td>
<td>Corollary 4</td>
<td>1.005</td>
<td>[−0.068 – 2.368]</td>
</tr>
</tbody>
</table>

in this paper provide an $H_\infty$ controller achieving much smaller $\gamma$ for much bigger $h$, which implies that the proposed method is much less conservative than the existing results.

5. Conclusion

The robust stochastic stabilization and robust $H_\infty$ control for uncertain stochastic systems with interval time-varying delays have been investigated by using a Lyapunov–Krasovskii approach. We focus on the $H_\infty$ state-feedback controller design and proposed delay-dependent results. By dividing the time-varying delay range into multiple subintervals, the variation of the Lyapunov–Krasovskii functional is checked in different subintervals, respectively. Each subinterval decreases and less conservative criteria can be obtained. Sufficient conditions are obtained in terms of delay-dependent LMIs which can be efficiently solved with available computational software. Numerical examples show that the derived criteria can lead to less conservative results than those obtained based on the existing methods.

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References


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