Optimal investment for insurers with correlation risk: risk aversion and investment horizon

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This article investigates the optimal investment for insurers with correlation risk, with the variance–covariance matrix among risky financial assets evolving as a stochastic positive definite matrix process. Using the Wishart diffusion matrix process, we formulate the insurer’s investment problem as the maximization of the expected constant relative risk-averse utility function subject to stochastic correlation, stochastic volatilities, and Poisson shocks. We obtain the explicit closed-form investment strategy and optimal expected utility through the Hamilton–Jacobi–Bellman framework. A verification theorem is derived to prove the uniform integrability of a tight upper bound for the objective function. The economic implication is that a long-term stable optimal investment policy requires the insurer to maintain a high risk-aversion level when the financial market contains stochastic volatility and/or stochastic correlation.

Keywords: Wishart process; utility theory; verification theorem; correlation risk.

1. Introduction

The post-2008 crisis market environment, which is characterized by low-interest rates, forces insurers to pursue investment strategies that focus on creating value for both policyholders and shareholders. Insurers must then participate in the financial market. Unlike a hedge fund that benchmarks against a financial index, an insurer has to meet insurance liabilities. Therefore, insurers’ investment problem has become an important topic in financial economics.

The classic optimal investment problem for an insurer begins with an investment strategy for a single risky financial asset (or index) subject to insurance liability. Hipp & Plum (2000) were the first to propose an optimal investment strategy for an index from the perspective of an insurer. Yang & Zhang (2005) then introduced a compound Poisson process to model insurance liability in such a way that the insurer’s wealth followed a jump-diffusion model.

Once a financial asset appears in the insurer’s portfolio, the stochastic dynamics of that asset must be sufficiently realistic to capture market phenomena. A typical stylized feature of financial indices is stochastic volatility (SV). Although they did not take insurance liability into account, Chacko & Viceira (2005) and Cerny & Kallsen (2008) addressed optimal investment problems using the Heston (1993) SV model from the perspective of a trader. More recently, Jung & Kim (2012) applied the constant...
elasticity of variance (CEV) model to an index to solve the optimal investment problem for insurers, Li et al. (2016) considered the optimal insurance and reinsurance within the CEV model and Badaoui & Fernández (2013) successfully solved the SV case.

The assumption that insurers invest only in financial indices is far from realistic. Accordingly, Chiu & Li (2006) investigated the optimal asset-liability management (ALM) problem with multiple risky assets, where assets and liability jointly follow the multivariate Black-Scholes model. Chiu & Wong (2013) generalized this analysis to accommodate cointegrated risky assets and compound Poisson insurance liability. Shen & Siu (2015) investigated the investment consumption within the cointegration market. Although cointegration allows for a dependent structure among risky assets beyond constant correlations, it is unable to explain the stochastic correlations among those assets.


Buraschi et al. (2010) considered the expected utility maximization with the Wishart process for a hedge fund showing that correlation risk had significant implications for market demand for index futures. Chiu & Wong (2014a) investigated the mean–variance (MV) portfolio selection problem, and deduced a condition for a stable solution related to the degree of market leverage. Chiu & Wong (2014b) then generalized the MV objective to the optimal ALM problem with insurance liability.

Insurers’ risk appetite is very different from that of traders. Insurers are supposed to be highly risk averse because of their social responsibility. Therefore, the expected utility maximization problem is relevant because the effect of an insurer’s risk aversion can be incorporated into the optimal investment strategy.

This article contributes primarily to investigations of the optimal insurer’s investment decision with stochastic correlations. We derive the explicit closed-form solution to the optimal investment strategy using the Hamilton–Jacobi–Bellman (HJB) framework. Our derivation approach is very different from those of Chiu & Wong (2014a,b). A comprehensive HJB framework requires the establishment of a verification theorem for a stable (non-blowup) solution subject to the insurance liability of Poisson shocks. As the verification theorem developed in this article is general enough to cover the optimal investment problem for any Poisson intensity, it is also applicable to the case considered by Burasci et al. (2010) if the intensity is set to zero.

We further investigate the effects of risk aversion and the investment horizon in an optimal investment policy with correlation risk. By carefully examining the aforementioned verification theorem, we show that the greater the risk aversion of the insurer, the greater the stability of the optimal investment policy when the correlations are stochastic. To maintain a stable investment rule, a riskier insurer is forced to shorten the investment horizon. Therefore, an insurer with a high level of risk aversion is able to manage a longer investment plan. Chiu & Wong (2013) already showed that insurers have to be highly risk averse in a financial market with cointegration. This article further confirms the importance of a high risk-aversion level for insurers participating in a financial market with correlation and volatility risks. Therefore, regardless of whether the market is predictable (stationary with normal innovation, corresponding to the case of cointegration) or highly unpredictable (containing SV and correlation risk), insurers should bear the importance of a high level of risk aversion in mind. A stable optimal investment strategy for all levels of risk aversion probably exists only in the Black–Scholes model.
The remainder of this article is organized as follows. Section 2 describes the problem formulation including the financial market setting and the model for insurance liabilities. Section 3 derives the optimal investment strategy and proves the verification theorem with conditions for the stable solution. We also prove the trade-off between risk aversion and investment horizon for a stable investment strategy. Section 4 provides the numerical examples to illustrate the optimal investment decision with correlation risks and Section 5 concludes the paper.

2. Problem formulation

2.1. Financial market with correlation risk

Consider a financial market in which $n + 1$ assets are traded continuously within a given the time horizon $[0, T]$. We label these assets $S_i$, $i = 0, 1, 2 \ldots, n$, with $S_0$ indicating a risk-free asset. The process of a risk-free asset satisfies the following differential equation.

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = s_0 > 0,$$

where $r(t)$ is the continuous deterministic risk-free rate. The remaining $n$ risky asset price processes form a vector of risky asset prices, $S(t) = (S_1(t), \ldots, S_n(t))'$, which satisfies the following vector-valued stochastic differential equation (SDE).

$$dS(t) = I_S(t) \left[ \tilde{\mu}(\Sigma(t), t) dt + \Sigma^{1/2}(t) dB_t \right], \quad t \in [0, T], \quad 1 \leq i, j \leq n, \quad (1)$$

where $I_S(t)$ is a diagonal matrix with diagonal elements equal to $S(t)$; $B_t = (B^1_t, \ldots, B^n_t)'$ is a standard $n$-dimensional Brownian motion with cov$(B^i_t, B^j_t) = 0$ for $i \neq j$; function $\tilde{\mu}(\Sigma(t), t)$ is a vector of possibly state-dependent appreciation rates, and $\Sigma^{1/2}(t)$ is the positive square root of covariance matrix $\Sigma(t)$.

Note that when volatility matrix $\Sigma^{1/2}(t)$ is a deterministic matrix-value function, then the risky asset price dynamic in (1) becomes the classical Black–Scholes model. Here, $\Sigma^{1/2}(t)$ is assumed to be a stochastic matrix process and thus so too is covariance matrix $\Sigma(t)$. Gourieroux et al. (2009) and Gourieroux & Sufana (2010) considered the Wishart diffusion process proposed by Bru (1991) as the stochastic process of a covariance matrix and applied it to evaluate option prices. They concluded that the Heston (1993) SV model is a special case of the Wishart diffusion process. Buraschi et al. (2010) adopted the Wishart diffusion process as a covariance matrix process for solving the utility maximization problem. They used $\tilde{\mu}(\Sigma(t), t) = rI + \Sigma(t)\beta(t)$ as the setting for the appreciation rate of risky assets in solving that problem, which is equivalent to assuming a deterministic market price of covariance risk (DMPCR). These assumptions are consistent with the analysis of Cerny & Kallsen (2008) in the case of a single risky asset.

Another approach considered by Buraschi et al. (2010) assumes the precision matrix $\Sigma^{-1}(t)$ to follow a Wishart process and $\tilde{\mu}(\Sigma(t), t) = rI + \mu(t)$, where $\mu(t)$ is deterministic, which is equivalent to a market with a deterministic risk premium (DRP).

Consider $g : \mathbb{R}^{n \times n}_+ \rightarrow \mathbb{R}^{n \times n}_+$ to be a bijection onto the space of $n$-dimensional positive definite matrices $\mathbb{R}^{n \times n}_+$. Covariance matrix $\Sigma(t)$ satisfies the matrix-valued SDE:

$$\Sigma(t) = g(\Lambda(t)), \quad (2)$$

$$d\Lambda(t) = \left[ \eta Q'Q + M\Lambda(t) + \Lambda(t)M' \right] dt + \Lambda^{1/2}(t)dW_tQ + Q'dW_t'\Lambda^{1/2}(t), \quad \Lambda(0) = \Lambda_0,$$
where $\eta$ is a positive constant, $Q$ is an $n \times n$ invertible constant matrix, $M$ is an $n \times n$ diagonalizable constant matrix with non-negative eigenvalues, and $W_t$ is a standard $n \times n$ matrix Brownian motion such that

$$B_t = W_t \rho + Z_t \sqrt{1 - \rho' \rho}. \quad (3)$$

Here, $Z_t$ is an $n$-dimensional Wiener process independent of $W_t$, and $\rho = (\rho_1, \cdots, \rho_n)'$ is a correlation column vector such that $\|\rho\|_2 < 1$. All of the Wiener processes, $B_t, Z_t$, and $W^{k+}$, are defined on a fixed-filter complete probability space $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{F}_{\geq 0})$.

The expression in (3) aims to model the correlation between the asset returns process and the covariance matrix process through the vector $\rho$. The Brownian motion matrix $W_t$ is associated with the covariance matrix process $\{\Lambda(t)\}_{t \geq 0}$. As $B_t$ is the Brownian motion vector for the asset returns, (3) shows that $B_t$ contains information from the covariance matrix for a non-zero $\rho$ and $Z_t$ represents the idiosyncratic risk components. When there is only one risk asset, the expression of (3) resembles the correlation between the asset return and its volatility in the Heston stochastic volatility model.

The matrix stochastic process, $\{\Lambda(t)\}$, in (2) is a Wishart process. If $g(\Lambda(t)) = \Lambda(t)$, then the covariance matrix itself follows a Wishart process; otherwise, that is, if $g(\Lambda(t)) = \Lambda^{-1}(t)$, then the precision matrix is modelled as a Wishart process. Constant matrix $M$ with non-negative eigenvalues characterizes the persistence, or, approximately speaking, the mean reversion, to ensure the stationarity of $\Lambda(t)$. When $M$ is singular, process (2) resembles the continuous-time cointegration model of Chiu & Wong (2011), with cointegration existing among the elements of the covariance matrix, which accounts for the common volatility in major stock index futures markets found by Booth et al. (1996). Invertible matrix $Q$ determines the long-term equilibrium of Wishart process $\Lambda(t)$ as $\eta QQ'$ and controls the covariance among the elements in $\Lambda(t)$. A constant $\eta > n - 1$ is assumed to ensure that $\Lambda(t)$ is positive and follows a Wishart distribution for all $t \in (0, T]$ (Bru, 1991).

Using the function $g(\cdot)$, we can properly describe the stochastic dynamics of a financial market with correlation risk. We consider the situations of Buraschi et al. (2010) and Chiu & Wong (2014a).

1. DMPCR: The parameters in (1) and (2) are defined as follows.

$$\tilde{\mu}(\Sigma(t), t) = r(t)\mathbf{1} + \Sigma(t)\beta(t), \quad g(\Lambda) = \Sigma, \quad (4)$$

where $\beta(t)$ is a time deterministic function.

2. DRP: The parameters in (1) and (2) are defined as follows.

$$\tilde{\mu}(\Sigma(t), t) = r(t)\mathbf{1} + \mu(t), \quad g(\Lambda) = \Sigma^{-1}, \quad (5)$$

where $\mu(t)$ is a time deterministic function.

The specifications of DMPCR and DRP enable us to derive a closed-form solution to the optimal trading strategy. When the covariance matrix is directly modelled by the Wishart process, we must take the DMPCR so that the drift term of the asset return is the sum of interest rate and a factor proportional to the covariance matrix. It is consistent with the investment problem with the Heston SV model considered by Kraft (2005) and Cerny & Kallsen (2008). Otherwise, no closed-form solution can be obtained. If we insist on having a deterministic drift term in the asset returns, one possible way is to set the inverse of the volatility as the Heston model (Chacko & Viceira, 2005). As the Wishart process is essentially
a multivariate extension of the Heston model, our model should be consistent with its one-dimensional setting. Buraschi et al. (2010) found that DRP fit their dataset better. Note that DMPCR and DRP settings are equivalent for a constant covariance matrix. Therefore, our model setting is consistent with classical models with a constant covariance matrix, and the aforementioned two single-factor SV models.

2.2. Wealth process and insurance liability

The classical risk process assumes that an insurer’s wealth evolves as

\[ Y(t) = Y_0 + C(t) - L(t), \]

where \( Y_0 \) is the initial wealth, non-negative deterministic function \( C(t) \) is the accumulated insurance premium, and \( L(t) \) represents the accumulated random payments for insurance claims. In this article, we adopt the insurance liability model used by Chiu & Wong (2013):

\[ L(t) = \int_0^t Y(s) \left( 1 - e^{-\delta s} \right) dN_s, \quad L(0) = 0, \]

where \( \{N_s, 0 \leq t \leq T \} \) is a Poisson process with intensity \( \nu(t) \), where the moment-generating function of \( \int_{\tau_1}^{\tau_2} \nu(t) \, dt \) exists for \( 0 \leq \tau_1 \leq \tau_2 \leq T \), \( 1 - e^{-\delta s} \) is an insurance claim portion at time \( t \), and \( z \) is a non-negative random variable that is independent of \( N_t, W_t, \) and \( Z \) and has a well-defined moment-generating function. Risk model (6) ignores that insurance companies usually participate in the financial market (see the comments in Hipp & Plum, 2000). Consider an insurer that invests in the financial market with correlation risk. Assume for the moment that the insurance premium \( C(t) \equiv 0 \) to simplify the mathematical setup. The situation of a positive premium is fully addressed in Section 3.5.

Let \( u_i(t) \) be the cash amount invested in asset \( i \), and \( N_i(t) \) be the number of units of asset \( i \) in the insurer’s portfolio. The insurer’s wealth level at time \( t \) is

\[ Y(t) = \sum_{i=0}^{m} u_i(t) = \sum_{i=0}^{m} N_i(t) S_i(t), \quad Y(0) = Y_0. \]

The portfolio \( u(t) = (u_1(t), u_2(t), \ldots, u_m(t))^t \) is said to be admissible if \( u(t) \) is a non-anticipating and \( \mathcal{F}_t \)-adapted process that satisfies \( \mathbb{E}\left[ \int_0^T u(\tau)' u(\tau) \, d\tau \right] < \infty \). Applying Itô’s Lemma to (8) with respect to (1) gives us

\[ dY(t) = \left[ r(t) Y(t) + u(t)' \mu(\Sigma(t), t) \right] dt + u(t)' \Sigma^{1/2}(t) dB_t - zdN_t, \quad Y(0) = Y_0, \]

where \( \mu(\Sigma(t), t) = \tilde{\mu}(\Sigma(t), t) - r(t) \mathbf{1} \) and \( z \) has the same distribution as \( z_i \). Define \( \mathbb{H} := \{ \mathcal{H}_t \}_{t \geq 0} \), the filtration generated by \( N_t \) augmented by \( \mathcal{P} \)-null sets. Let \( \mathbb{G} \) be filtration \( \{ \mathcal{G}_t \}_{t \geq 0} \), where \( \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t \), the smallest filtration containing \( \mathbb{F} \) and \( \mathbb{H} \). Note that \( \mathcal{G}_t \) can be regarded as information available to the investor at time \( t \).

2.3. Optimal investment for an insurer

The formal research problem is given as follows.

\[
\max_{u(\cdot)} \mathbb{E}[U(Y(T))] \quad \text{s.t.} \quad (1), (9), (2) \quad \text{and} \quad u(\cdot) \in \Pi,
\]
where $\Pi$ is the set of squarely integrable functions with respect to the space of $(\Omega, \mathcal{G}, \mathcal{P})$, and

$$U(y) = \frac{y^{1-\gamma} - 1}{1-\gamma} \quad \text{for} \quad \gamma \in (0, 1), \quad \text{and} \quad U(y) = \ln y \quad \text{for} \quad \gamma = 1. \quad (11)$$

The function in (11) is known as the constant relative risk-averse (CRRA) utility function.

If $\Sigma(t)$ is a time-deterministic function and the Poisson process is absent in (9), then the problem is reduced to the classic Merton problem. As $\Sigma(t)$ is essentially a stochastic matrix that evolves according to SDE (2) in this article, wealth process (9) resembles a diffusion model with stochastic drift and volatility for a given control $u(t)$. The $\Lambda(t)$ in (2) is generally unbounded, such that the drift and volatility of $Y(t)$ are also generally unbounded, which induces mathematical sophistication in deriving the optimal investment strategy and proving the condition necessary for a stable investment policy with the verification theorem.

3. The optimal solution

We divide our derivation into two cases. In the first case, $\gamma < 1$ to ensure that the utility function is proportional to a power function. This refers to a power utility. In the second case, we take the limit of $\gamma \to 1$ on the CRRA utility. As the limit is a logarithmic function, it refers to the logarithmic utility or Bernoulli’s utility function.

3.1. Power utility

In our research problem, the optimal decision is not affected by adding or subtracting a constant to the objective function. The power utility maximization problem can be reduced to

$$\max E \left[ \frac{Y(T)^{1-\gamma} - 1}{1-\gamma} Y(t) \right], \quad (12)$$

where the insurer’s wealth follows SDE (9).

**Theorem 3.1** If the insurer’s wealth process follows SDE (9) with a stochastic covariance matrix process (2) and insurance liability process (7), then the expected utility maximization problem (12) has the optimal control (investment policy):

$$u^*(t, Y(t), \Lambda) = \begin{cases} \frac{1}{\gamma} \left[ b(t) + 2K(t, T, b)Q' \rho \right] Y(t), & \text{for DMPCR;} \\ \frac{\Lambda}{\gamma} \left[ b(t) + 2K(t, T, b)Q' \rho \right] Y(t), & \text{for DRP;} \end{cases} \quad (13)$$

where

$$b(t) = \begin{cases} \beta(t), & \text{for DMPCR;} \\ \mu(t), & \text{for DRP;} \end{cases} \quad \Lambda(t) = \begin{cases} \Sigma(t), & \text{for DMPCR;} \\ \Sigma^{-1}(t), & \text{for DRP;} \end{cases} \quad (14)$$

$$K(t, b, T) = R_2(t, b)R_1(t, b)^{-1} \quad \text{with} \quad \tau = T - t. \quad (15)$$
\( R = \left( R_1(\tau, b) \right) \) is the solution of the linear system of ordinary differential equations (ODEs) in the interval \([0, T]\):

\[
\frac{dR}{d\tau} = \left( \begin{array}{ccc}
-M - \frac{1-\gamma}{\gamma} Q \rho b' & -2Q' \left( I + \frac{1-\gamma}{\gamma} \rho \rho' \right) \rho & R \end{array} \right), \quad R(0, b) = \left( \begin{array}{c}
1
\end{array} \right); 
\]

and the optimal value of objective function

\[
E \left[ \frac{Y(T)^{1-\gamma}}{1-\gamma} | G_0 \right] \mid_{u=u^*} = \frac{Y_0^{1-\gamma}}{1-\gamma} e^{A(0,T,b)+tr(K(0,T,b)\Lambda(0))}, 
\]

where

\[
A(t, T, b) = \int_t^T (1-\gamma) r(s) + E \left[ (e^{-z(1-\gamma)} - 1) v(s) \right] + \eta \, tr \left( QQ'K(s, T, b) \right) ds. \tag{16}
\]

**Proof.** The proof is based on the classic HJB framework. For a fixed terminal time \( T \), the corresponding HJB equation is

\[
V_t + E[(V(t, Y - Y(1 - e^{-z}), \Lambda) - V)] + \mathcal{L}_V V + \sup_u \{ \mathcal{L}_Y V + \mathcal{L}_{\Lambda} V \} = 0, \tag{17}
\]

where

\[
\mathcal{L}_V V = \frac{\partial V}{\partial Y} (rY + u' \mu) + \frac{1}{2} \frac{\partial^2 V}{\partial Y^2} u' \Sigma u; \\
\mathcal{L}_V V = tr \left[ (\eta QQ' + M \Lambda + \Lambda M') DV + 2\Lambda DQ' QDV \right]; \\
\mathcal{L}_{\Lambda} V = \begin{cases}
2tr (DQ' \rho \Lambda) \frac{\partial}{\partial \Lambda} V, & \text{for DMPCR;} \\
2tr (DQ' \rho u') \frac{\partial}{\partial \Lambda} V, & \text{for DRP}
\end{cases}
\]

and \( D_{ij} := \frac{\partial}{\partial \Lambda_{ij}} \), with \( V(T, Y, \Lambda) = \frac{Y^{1-\gamma}}{1-\gamma} \). Thus, the optimal feedback control, \( u^* \), maximizes

\[
\frac{\partial V}{\partial Y} u' \mu + \frac{1}{2} \frac{\partial^2 V}{\partial Y^2} u' \Sigma u + \mathcal{L}_V V. \tag{18}
\]

If \( \frac{\partial^2 V}{\partial Y^2} < 0 \), then differentiating (18) with respect to \( u \) and setting the differential to zero results in

\[
u^* = \begin{cases}
-\Sigma^{-1} \frac{\partial V}{\partial Y} \mu + 2\Lambda DQ' \rho \frac{\partial}{\partial \Lambda} V, & \text{for DMPCR;} \\
-\Sigma^{-1} \frac{\partial V}{\partial Y} \mu + 2DQ' \rho \frac{\partial}{\partial \Lambda} V, & \text{for DRP.}
\end{cases}
\]
Otherwise, if \( \frac{\partial^2 V}{\partial y^2} \geq 0 \), then the optimization has no solution. Substituting \( u^* \) into HJB equation (17) renders the partial differential equation (PDE) of \( V \):

\[
V_t + E[(V(t, Y - Y(1 - e^{-\gamma}), \Lambda) - V)v] + \text{tr} \left[ (\rho Q' + M \Lambda + \Lambda M')DV + 2\Lambda DQ'QDV \right]
\]

\[
- \frac{1}{2} \frac{\partial^2 V}{\partial Y^2} b'b' - \frac{\partial V}{\partial Y} b' \Lambda DQ' \rho \frac{\partial}{\partial Y} V - \frac{\partial^2 V}{\partial Y^2} \rho' Q'D' \Lambda b \frac{\partial}{\partial Y} V - \frac{2}{\partial^2 Y} \left[ \left( \frac{\partial}{\partial Y} \right) \rho' Q'D' \right] \Lambda \left[ DQ' \rho \frac{\partial}{\partial Y} V \right]
\]

\[= 0,
\]

with terminal condition \( V(T, Y, \Lambda) = U(Y) \). As \( U(Y) = \frac{Y^{1-\gamma}}{1-\gamma} \), consider an exponential affine form of \( V \):

\[
V(t, Y, \Lambda) = \frac{Y^{1-\gamma}}{1-\gamma} \exp \left[ A(t, T, b) + \text{tr} \left( K(t, T, b) \Lambda \right) \right],
\]

where \( K \) and \( A \) are deterministic (matrix) functions of \( t \) of the forms presented in (15) and (16), respectively. According to Radon’s Lemma, the representation of \( K \) in (15) is a solution of the following matrix Riccati equation:

\[
\dot{K} + K \left( M + \frac{1}{\gamma} Q' \rho b' \right) + \left( M + \frac{1}{\gamma} Q' \rho b' \right)' K + 2KQ' \left( I + \frac{1}{\gamma} \rho \rho' \right) QK + \frac{(1 - \gamma)bb'}{2\gamma} = 0
\]

with \( K(T, T, b) = 0_{n \times n} \). Furthermore, the structure of the matrix Riccati equation shows the symmetric property of matrix function \( K \). Clearly, the terminal value of the function in (20) satisfies the terminal condition in (19) and \( \frac{\partial^2 V}{\partial y^2} < 0 \). Taking partial derivatives to the affine form of \( V \) with respect to \( t, Y \), and \( \Lambda \), we have:

\[
V_t = (\dot{A} + \text{tr}(\dot{K} \Lambda)) V; \quad \frac{\partial V}{\partial Y} = \frac{1-\gamma}{Y} V; \quad \frac{\partial^2 V}{\partial Y^2} = -\frac{\gamma(1-\gamma)}{Y^2} V; \quad DV = KV;
\]

\[
E \left[ (V(t, Y - Y(1 - e^{-\gamma}), \Lambda) - V)v \right] = E \left[ (e^{-\gamma(1-\gamma)} - 1) v \right] V.
\]

Once these expressions are substituted into the left-hand side of (19), simple but tedious calculation easily verifies that the proposed solution form satisfies PDE (19). Thus, the solution form in (20) is actually a solution of PDE (19).

Although the value function is twice continuously differentiable, the parameters are not uniformly bounded and predictable. Thus, the classical verification theorem of Fleming & Soner (1993) (III, Theorem 8.1) cannot be used to show that the proposed affine form of the value function in (20) and control \( u^* \) in (13) are the optimal value function and optimal feedback control, respectively. The solution is essentially a Nirvana optimal solution (Kim & Omberg (1996)) for which we need to establish a verification theorem with appropriate conditions for a stable optimal solution.

3.2. Verification theorem

The following two propositions together serve as a verification theorem for the solution of HJB equation (17). The results and proofs are classical. To smooth the proofs of the propositions, the following
notation is introduced.

\[ \mathcal{A}^uJ = \frac{\partial J}{\partial t} + \mathcal{L}_J + \mathcal{L}_A J + \mathcal{L}_{A^u} J + \mathbb{E}\left[ \left( J(t, y - y(1 - e^{-\gamma}), \Lambda) - J(t, y, \Lambda) \right) v \right], \]

where

\[ \mathcal{L}_J V = \frac{\partial V}{\partial y} (ry + u\mu) + \frac{1}{2} \frac{\partial^2 V}{\partial y^2} u\Sigma u; \]
\[ \mathcal{L}_A V = \text{tr} \left[ (\eta QQ' + MA + \Lambda M')DV + 2\Lambda DQ'QDV \right]; \]
\[ \mathcal{L}_{A^u} V = \begin{cases} 2\text{tr} (DQ'\rho u'\Lambda) \frac{\partial V}{\partial y}, & \text{for DMPCR;} \\ 2\text{tr} (DQ'\rho u) \frac{\partial V}{\partial y}, & \text{for DRP.} \end{cases} \]

Clearly, HJB equation (17) can be rewritten as \( \sup_u \mathcal{A}^u V = 0. \)

**Proposition 3.1** It is assumed that \( J \in C^2([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+^{n \times n}) \) is a non-negative function such that \( \mathcal{A}^u J \leq 0, J(T, y, \Lambda) = U(y) \) and \( J(t, 0, \Lambda) = 0 \) for every admissible control \( u. \) Then,

\[ J(t, y, \Lambda) \geq \mathbb{E} \left[ U(Y^u(T)|\mathcal{G}_t) \right]. \]

**Proof.** It is assumed that \( u \) is an admissible control and that \( \{\tau_n\}_n \) is a localizing sequence of stopping times for the local semi-martingale \( (Y^u(s), \Lambda(s)) \) starting with \( (y, \Lambda) \) at time \( t. \) Applying Itô’s Lemma to \( J \) with respect to (9), we have

\[ dJ = \left\{ \frac{\partial J}{\partial t} + \mathcal{L}_J J + \mathcal{L}_A J + \mathcal{L}_{A^u} J \right\} dt + \left( J(t, y - y(1 - e^{-\gamma}), \Lambda) - J(t, y, \Lambda) \right) dN_t \]
\[ + \frac{\partial J}{\partial y} u\Sigma^{1/2} dB_t + \text{tr} \left( Q' dW_t\Lambda^{1/2} DJ \right) + \text{tr} \left( Q' dW_t Q DJ \right) \]
\[ = \left\{ \frac{\partial J}{\partial t} + \mathcal{L}_J J + \mathcal{L}_A J + \mathcal{L}_{A^u} J + \left( J(t, y - y(1 - e^{-\gamma}), \Lambda) - J(t, y, \Lambda) \right) v(t) \right\} dt \]
\[ + \frac{\partial J}{\partial y} u\Sigma^{1/2} dB_t + \text{tr} \left( Q' dW_t\Lambda^{1/2} DJ \right) + \text{tr} \left( Q' dW_t Q DJ \right). \]

Hence, we have

\[ \mathbb{E} \left[ J(T \wedge \tau_n, Y^u(T \wedge \tau_n), \Lambda(T \wedge \tau_n)) | \mathcal{G}_t \right] = J(t, y, \Lambda) + \int_t^{T \wedge \tau_n} \mathcal{A}^u J(s, Y^u(s), \Lambda(s)) ds \leq J(t, y, \Lambda). \]

Due to the non-negativity of \( J, \{J(T \wedge \tau_n, Y^u(T \wedge \tau_n), \Lambda(T \wedge \tau_n))\}_n \) is a sequence of non-negative measurable functions. As

\[ \lim_{n \to \infty} J(T \wedge \tau_n, Y^u(T \wedge \tau_n), \Lambda(T \wedge \tau_n)) = J(T, Y^u(T), \Lambda(T)) = U(Y^u(T)), \quad a.s., \]
Fatou’s Lemma yields
\[
E \left[ U(Y^u(T)) \mid \mathcal{G}_t \right] = E \left[ \lim \inf_{n \to \infty} J(T \wedge \tau_n, Y^u(T \wedge \tau_n), \Lambda(T \wedge \tau_n)) \mid \mathcal{G}_t \right] \\
\leq \lim \inf_{n \to \infty} E \left[ J(T \wedge \tau_n, Y^u(T \wedge \tau_n), \Lambda(T \wedge \tau_n)) \mid \mathcal{G}_t \right] \leq J(t, y, \Lambda). \quad \square
\]

**Proposition 3.2** Let \( J \in C^2([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+^{n \times n}) \) be a non-negative function such that the family of random variables \( \{J(\tau, Y^u(\tau), \Lambda(\tau))\}_\tau \) is uniformly integrable, where \( u^* \) is an admissible control with the property \( \mathcal{A}u^* J = 0 \), and \( \tau \in [t, T) \) is a stopping time for the process \( (Y^u(s), \Lambda(s)) \) starting with \( (y, \Lambda) \) at time \( t \). Furthermore, if \( J(T, y, \Lambda) = U(y), J(t, 0, \Lambda) = 0 \), and \( \mathcal{A}u^* J \leq 0 \) for all admissible controls \( u \), then
\[
J(t, y, \Lambda) = V(t, y, \Lambda) \quad \forall (t, y, \Lambda) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+^{n \times n}.
\]

**Proof.** Owing to the uniform integrability of the family \( \{J(\tau, Y^{u^*}(\tau), \Lambda(\tau))\}_\tau \), we have
\[
E \left[ U(Y^{u^*}(T)) \mid \mathcal{G}_t \right] = E \left[ \lim \inf_{n \to \infty} J(T \wedge \tau_n, Y^{u^*}(T \wedge \tau_n), \Lambda(T \wedge \tau_n)) \mid \mathcal{G}_t \right] \\
= \lim \inf_{n \to \infty} E \left[ J(T \wedge \tau_n, Y^{u^*}(T \wedge \tau_n), \Lambda(T \wedge \tau_n)) \mid \mathcal{G}_t \right] \leq J(t, y, \Lambda).
\]
Furthermore, \( \mathcal{A}u^* J = 0 \) induces that \( E \left[ U(Y^{u^*}(T)) \mid \mathcal{G}_t \right] = J(t, y, \Lambda) \). Hence, by Proposition 3.1, \( J(t, y, \Lambda) = V(t, y, \Lambda) \) for all \( (t, y, \Lambda) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+^{n \times n} \).

The foregoing propositions are classical and valid for some positive utility functions, such as the exponential utility function. To verify the uniform integrability of \( \{J(\tau, Y^{u^*}(\tau), \Lambda(\tau))\}_\tau \) associated with the CRRA utility, we need to calculate \( E[J(\tau, Y^{u^*}(\tau), \Lambda(\tau)) \mid \mathcal{G}_t] \), where
\[
J(\tau, Y(\tau), \Lambda(\tau)) = V(\tau, Y(\tau), \Lambda(\tau)) \quad \text{with} \quad U(y) = \frac{y^{1-\gamma}}{1-\gamma};
\]
\( V(t, Y(t), \Lambda(\tau)) \) is as given in (20). The next step is to calculate the expected value of \( J(\tau, Y^{u^*}(\tau), \Lambda(\tau)) \) at filtration \( \mathcal{G}_t \), where \( 0 \leq t \leq \tau \).

**Lemma 3.1** Given that \( 0 \leq t \leq \tau \leq T \) and using the notations in Theorem 3.1,
\[
E \left[ V(\tau, Y^{u^*}(\tau), \Lambda(\tau)) \mid \mathcal{G}_t \right] = E \left[ \frac{(Y^{u^*}(\tau))^{1-\gamma}}{1-\gamma} e^{\mathcal{A}(\tau,t,b)+t(\mathcal{K}(\tau,t,b)\Lambda(\tau))} \right] \mid \mathcal{G}_t \\
= \frac{(Y(\tau))^{1-\gamma}}{1-\gamma} e^{\mathcal{A}(\tau,t,b)+t(\mathcal{K}(\tau,t,b)\Lambda(\tau))},
\]
where $K(s, T, b)$ is as defined in (15); 

$$
\tilde{A}(t, \tau, b) = \int_t^T (1 - \gamma) r(s) + E[\exp(-z(1 - \gamma)) - 1] \nu(s) \, ds 
+ \int_t^T tr(\eta QQ'K(s, T, b)) \, ds + \int_t^T tr(\eta QQ'\tilde{K}(s, \tau, b)) \, ds;
$$

(21)

$\tilde{K}(s, \tau, b)$ satisfies matrix ODE

$$
\begin{cases}
\dot{\tilde{K}} + \tilde{K} \tilde{M} + \tilde{M}' \tilde{K} + 2\tilde{K} Q Q' \tilde{K} + \frac{1 - \gamma}{2\gamma} [bb' - 4(KQ')'(KQ')'] = 0 \\
\tilde{K}(\tau, \tau, b) = K(\tau, T, b);
\end{cases}
$$

(22)

and $\tilde{M} = M + \frac{1 - \gamma}{\gamma} Q' \rho (b' + 2 \rho QK)$.

Proof. Let 

$$
\tilde{v}(t, Y, \Lambda) = \frac{(Y(t))^{1-\gamma}}{1 - \gamma} e^{\tilde{A}(t, T, b) + tr(\tilde{K}(t, \tau, b)\Lambda(t))},
$$

where $0 \leq t \leq \tau \leq T$, $K(s, T, b)$ and $\tilde{A}(t, \tau, b)$ are as defined in (15) and (21), respectively, and $\tilde{K}(t, \tau, b)$ is the solution of matrix ODE (22). Clearly, as $t = \tau$, $\tilde{v}(\tau, Y, \Lambda) = V(\tau, Y, \Lambda)$.

After adopting optimal control $u^*$ (for both DMPCR and DRP) in Theorem 3.1, the wealth process of the insurer, $Y^*$, becomes

$$
\frac{dY^*}{Y^*} = r^* dt + \sigma^* dB_t - (1 - e^{-z}) dN_t,
$$

where

$$
r^* = r + \frac{1}{\gamma} (b + 2KQ')' \Lambda b,
$$

$$\sigma^* = \frac{1}{\gamma} (b + 2KQ')' \Lambda^{1/2}.
$$

According to the Feynman–Kac formula, the expectation in this lemma satisfies the following partial integro-differential equation.

$$
\frac{\partial \tilde{v}}{\partial t} + E[\tilde{v}(t, Y^* - Y^*(1 - e^{-z}), \Lambda) - \tilde{v}(t, Y^*, \Lambda)] \nu] + \mathcal{L}_{Y^*} \tilde{v} + \mathcal{L}_\Lambda \tilde{v} + \mathcal{L}_{M^*} \tilde{v} = 0,
$$

(23)

where

$$
\mathcal{L}_{Y^*} \tilde{v} = \frac{\partial \tilde{v}}{\partial Y^*}(r^* Y^*) + \frac{1}{2} \frac{\partial^2 \tilde{v}}{\partial Y^{*2}} \sigma^* \sigma^{**} Y^*;
$$

$$\mathcal{L}_\Lambda \tilde{v} = tr \left[ (\eta QQ' + M \Lambda + \Lambda M') \mathcal{D} \tilde{v} + 2 \Lambda DQ' Q \mathcal{D} \tilde{v} \right];
$$

$$\mathcal{L}_{M^*} \tilde{v} = 2Y^* \sigma^* \Lambda^{1/2} \mathcal{D} \rho \frac{\partial}{\partial Y^*} \tilde{v}.$$
Taking the partial derivative on $\tilde{v}$, we have
\[ \frac{\partial \tilde{v}}{\partial t} = (\hat{A} + tr(\hat{K} \Lambda)) \tilde{v} \]
with
\[ \hat{A} = -(1 - \gamma) r - E[exp(-z(1-\gamma)) - 1] v(s) - tr(\eta QQ' \tilde{K}(t, \tau, b)) \]
\[ \tilde{K} = -\tilde{K}M - M'\tilde{K} - 2\tilde{K}Q\tilde{K} - \frac{1 - \gamma}{2\gamma} [bb' - 4(KQ'\rho)(KQ'\rho)'] \]
\[ \tilde{M} = M + \frac{1 - \gamma}{\gamma} Q'\rho (b' + 2\rho'QK) \]
\[ \frac{\partial \tilde{v}}{\partial Y} = \frac{1 - \gamma}{Y} \tilde{v} \]
\[ \frac{\partial^2 \tilde{v}}{\partial Y^2} = -\frac{\gamma (1 - \gamma)}{Y^2} \tilde{v} \]
\[ D\tilde{v} = \tilde{K}\tilde{v} \]

After substituting the foregoing expressions into the left-hand side of (23), we have
\[ \left\{ \hat{A} + tr(\hat{K} \Lambda) + E[e^{z(1-\gamma)} - 1] v + (1 - \gamma) r + \frac{1 - \gamma}{\gamma} b' \Lambda (b + 2KQ' \rho) \right. \]
\[ - \frac{1 - \gamma}{2\gamma} tr ((b + 2KQ' \rho)(b + 2KQ' \rho)') \Lambda + tr ([\eta QQ' + M \Lambda + \Lambda M'] \tilde{K} + 2\Lambda \tilde{K}Q\tilde{K}) \]
\[ + \frac{1 - \gamma}{\gamma} tr (\tilde{K}Q' \rho (b + 2KQ' \rho)') \Lambda \right\} \tilde{v} \]
\[ = \left\{ \left[ \hat{A} + (1 - \gamma) r + E[exp(-z(1-\gamma)) - 1] v(s) + tr(\eta QQ' \tilde{K}(t, \tau, b))] \right. \]
\[ + tr (\tilde{K} + \tilde{K} \left( M + \frac{1 - \gamma}{\gamma} Q' \rho (b' + 2\rho'QK) \right) + \left( M + \frac{1 - \gamma}{\gamma} Q' \rho (b' + 2\rho'QK) \right)' \tilde{K} \]
\[ + 2\tilde{K}Q\tilde{K} + \frac{1 - \gamma}{2\gamma} [bb' - 4(KQ'\rho)(KQ'\rho)'] \right\} \tilde{v} = 0, \]

which is the same as the right-hand side of (23). Hence,
\[ \tilde{v}(t, Y^u, \Lambda) = E \left[ \left. \frac{Y^u(\tau)}{1 - \gamma} e^{\Lambda(t, T, b) + tr(K(t,T,b)\Lambda(t))} \right| G_t \right] = E \left[ \left. V(\tau, Y^u(\tau), \Lambda(\tau)) \right| G_t \right]. \]

Showing the uniform integrability of $\{V(\tau, Y^u(\tau), \Lambda(\tau))\}$, is equivalent to showing the boundedness of function value $\tilde{v}(t, Y, \Lambda)$ by Lemma 3.1. Hence, the boundedness of the solution of stochastic Riccati differential equation (SRDE) (22) induces the boundedness of function value $\tilde{v}(t, Y, \Lambda)$. The following proposition is useful for showing the boundedness of $\tilde{K}(t, \tau)$. The details can be found in a study by Abou-Kandil et al. (2003).
Proposition 3.3  For \( i = 1, 2 \), let \( K_i \) be the solution of
\[
\dot{K}_i = -A_i'(t)K_i - K_iA_i(t) - Q_i(t) + K_iS_i(t)K_i
\]
on some interval \( \mathcal{I} \). If for some \( t_f \in \mathcal{I} \), \( K_1(t_f) \preceq K_2(t_f) \), and if
\[
\left( \begin{array}{cc}
Q_2 & A_2' \\
A_2 & -S_2
\end{array} \right) (t) - \left( \begin{array}{cc}
Q_1 & A_1' \\
A_1 & -S_1
\end{array} \right) (t) \preceq 0 \quad \text{for } t \in \mathcal{I},
\]
where 0 is a matrix with zero-valued entries, then \( K_1(t) \preceq K_2(t) \) for all \( t \in \mathcal{I} \cap (-\infty, t_f] \).

Lemma 3.2  Given that \( 0 \leq t \leq \tau \leq T \) and using the notations in Theorem 3.1 and Lemma 3.1,
\[
\tilde{K}(t, \tau, b) \preceq K(t, T, b),
\]
for any stopping time \( \tau \in [0, T] \) and \( t \in [0, \tau] \).

Proof.  Consider a stopping time \( \tau \) such that \( \tau \in [0, T] \) and \( t \in [0, \tau] \), and choose
\[
A_1 = M; \quad A_2 = M + \frac{1 - \gamma}{\gamma} Q' \rho b';
\]
\[
S_1 = -2Q'Q; \quad S_2 = - \left( 2Q'Q + \frac{2(1 - \gamma)}{\gamma} Q' \rho' Q \right);
\]
\[
Q_1 = \frac{1 - \gamma}{2\gamma} [bb' - 4(KQ'\rho)(KQ'\rho')]; Q_2 = \frac{1 - \gamma}{2\gamma} bb',
\]
where \( \tilde{M} = M + \frac{1 - \gamma}{\gamma} Q' \rho (b' + 2\rho'QK) \). It is then obvious that \( \tilde{K}(t, \tau, b) \) is the solution of
\[
\dot{K}_1 = -A_1'K_1 - K_1A_1'(t) - Q_1(t) + K_1S_1(t)K_1, \quad K_1(\tau) = \tilde{K}(\tau, \tau, b)
\]
and that \( K(t, T, b), t \in [0, \tau] \), is the solution of
\[
\dot{K}_2 = -A_2'(t)K_2 - K_2A_2(t) - Q_2(t) + K_2S_2(t)K_2, \quad K_2(\tau) = K(\tau, T, b).
\]
Therefore, we have
\[
\left( \begin{array}{cc}
Q_2 & A_2' \\
A_2 & -S_2
\end{array} \right) (t) - \left( \begin{array}{cc}
Q_1 & A_1' \\
A_1 & -S_1
\end{array} \right) (t) \preceq 0 \quad \text{for } t \in [0, \tau],
\]
\[
= \frac{2(1 - \gamma)}{\gamma} \left( \begin{array}{cc}
KQ'\rho\rho'QK & -KQ'\rho\rho'Q \\
-Q\rho\rho'QK & Q'\rho\rho'Q
\end{array} \right) (t)
\]
\[
= \frac{2(1 - \gamma)}{\gamma} \left( \begin{array}{cc}
K & 0_{n \times n} \\
0_{n \times n} & I_{n \times n}
\end{array} \right) \left( \begin{array}{cc}
Q'\rho & -\rho'Q \\
-Q'\rho & \rho'Q
\end{array} \right) \left( \begin{array}{cc}
K & 0_{n \times n} \\
0_{n \times n} & I_{n \times n}
\end{array} \right) (t)
\]
\[
\preceq 0_{2n \times 2n}.
\]
According to Proposition 3.3, \( \tilde{K}(t, \tau, b) = K_1(t) \preceq K_2(t) = K(t, T, b) \) for any stopping time \( \tau \in [0, T] \) and \( t \in [0, \tau] \).

Note that \( K(t, T, b) - \tilde{K}(t, \tau, b) \succeq 0_{n \times n} \) induces

\[
Q' \left[ K(t, T, b) - \tilde{K}(t, \tau, b) \right] Q \succeq 0_{n \times n} \Rightarrow tr \left( Q' \left[ K(t, T, b) - \tilde{K}(t, \tau, b) \right] Q \right) \geq 0
\]

\[
\Leftrightarrow tr \left( QQ' \tilde{K}(t, \tau, b) \right) \geq tr \left( QQ' \tilde{K}(t, \tau, b) \right).
\]

(24)

Similarly, we have

\[
tr \left( K(t, T, b) \Lambda(t) \right) \geq tr \left( \tilde{K}(t, \tau, b) \Lambda(t) \right).
\]

(25)

**Theorem 3.2** Given the \( \tilde{A}(t, \tau, b) \) and \( \tilde{K}(t, \tau, B) \) defined in Lemma 3.1, the function

\[
\tilde{v}(t, Y, \Lambda) = \frac{(Y(t))^{1-\gamma}}{1-\gamma} \frac{e^{\tilde{A}(t,T,b)+tr(\tilde{K}(t,\tau,b)\Lambda(t))}}{\tilde{K}(t,\tau,b)\Lambda(t)}
\]

is bounded for all \( 0 \leq t \leq \tau \leq T \) if \( K(t, \tau, \Lambda) \) has no finite escape time on \([0, T]\).

**Proof.** Suppose that \( K(t, \tau, \Lambda) \) has no finite escape time on \([0, T]\), which implies that \( \tilde{K}(t, \tau, \Lambda) \preceq K(t, T, b) \) for \( 0 \leq t \leq \tau \leq T \) by Lemma 3.2. Also, \( E[\exp(-z(1-\gamma))] \) is a moment-generating function of random variable \( z \), which means that

\[
E[\exp(-z(1-\gamma))] = E[(\exp(-z(1-\gamma)) - 1)]E[z(s)] < \infty.
\]

Hence, we have

\[
\tilde{v}(t, Y, \Lambda)
= \frac{Y(t)^{1-\gamma}}{1-\gamma} \frac{e^{\tilde{A}(t,T,b)+tr(\tilde{K}(t,\tau,b)\Lambda(t))}}{\tilde{K}(t,\tau,b)\Lambda(t)}
\]

\[
= \frac{Y(t)^{1-\gamma}}{1-\gamma} \frac{e^{\int_{t}^{T} (1-\gamma) \sigma(s) \left[ (\exp[-z(1-\gamma)] - 1) \right] v(s) ds + \int_{t}^{T} tr(\eta QQ' \tilde{K}(s,\tau,b)) ds + \int_{t}^{T} tr(\eta QQ' \tilde{K}(s,\tau,b) ds + tr(\tilde{K}(t,\tau,b)\Lambda(t))}}{\tilde{K}(t,\tau,b)\Lambda(t)}
\]

\[
\leq \frac{Y(t)^{1-\gamma}}{1-\gamma} \frac{e^{\int_{t}^{T} (1-\gamma) \sigma(s) \left[ (\exp[-z(1-\gamma)] - 1) \right] v(s) ds + \int_{t}^{T} tr(\eta QQ' \tilde{K}(s,\tau,b)) ds + \int_{t}^{T} tr(\eta QQ' \tilde{K}(s,\tau,b) ds + tr(\tilde{K}(t,\tau,b)\Lambda(t))}}{\tilde{K}(t,\tau,b)\Lambda(t)}
\]

\[
< \infty.
\]

Theorem 3.2 provides a sufficient condition for the solution \( u^* \) obtained in Theorem 3.1 to be the optimal control for the problem in (10). More precisely, it asserts that the solution is optimal once function \( K(t, T, b) \) has no finite escape time. The theorem allows us to rank the stability of the optimal investment policy according to the insurer’s risk-aversion level. We observe a similar result for the single-factor Heston SV model in Kraft (2005). However, the extension to the Wishart process is highly non-trivial because the scalar Riccati equation associated with the Heston model blows up when the denominator of the solution is zero. Our solution is based on the matrix Riccati equation which blow-up condition requires appropriate matrix comparisons. In addition, we take the insurance liability into account to allow jumps in the wealth process.
3.3. The trade-off between risk aversion and investment horizon

**Lemma 3.3** Define \( K_\gamma(t, \tau, b) = K|_\gamma(t, \tau, b) \); if \( 0 < \gamma_1 \leq \gamma_2 < 1 \), then

\[
K_\gamma_1(t, T, b) \succ K_\gamma_2(t, T, b)
\]

for any stopping time \( t \in [0, T] \).

**Proof.** According to this definition, \( K_\gamma_i(t, T, b) \) is the solution of the following SRDE for \( i = 1, 2 \).

\[
\begin{align*}
\dot{K}_\gamma_i &= -A_i'K_\gamma_i - K_\gamma_i A_i'(t) - Q_i(t) + K_\gamma_i S_i(t)K_\gamma_i, \quad K_\gamma_i(T, T, b) = 0_{n \times n},
\end{align*}
\]

where

\[
\begin{align*}
A_1 &= M + \frac{1 - \gamma_i}{\gamma_i} Q \rho b'; \\
S_i &= -\left(2Q'Q + \frac{2(1 - \gamma_i)}{\gamma_i} Q' \rho \rho' Q\right); \\
Q_i &= \frac{1 - \gamma_i}{2\gamma_i} bb',
\end{align*}
\]

for \( i = 1, 2 \). Consider

\[
\left(\begin{array}{cc}
Q_1 & A_1' \\
A_1 & -S_1
\end{array}\right)(t) - \left(\begin{array}{cc}
Q_2 & A_2' \\
A_2 & -S_2
\end{array}\right)(t)
= \left(\frac{1 - \gamma_1}{2\gamma_1} - \frac{1 - \gamma_2}{2\gamma_2}\right)\left(\begin{array}{cc}
bb' & 2b \rho' Q \\
2Q' \rho b' & 4Q' \rho \rho' Q
\end{array}\right)(t)
= \frac{\gamma_2 - \gamma_1}{2\gamma_1 \gamma_2} \left(\begin{array}{cc}
bb' & 2b \rho' Q \\
2Q' \rho b' & 4Q' \rho \rho' Q
\end{array}\right)(t)
\geq 0_{2n \times 2n}.
\]

According to Proposition 3.3, \( K_\gamma_1(t, T, b) \succ K_\gamma_2(t, T, b) \) for any stopping time \( t \in [0, T] \). □

Lemma 3.3 implies that the lower the degree of risk-aversion, the larger the upper bound of the function \( K \) that characterizes the uniform integrability (stability) of the solution. As the optimal value function is an exponential function of \( K \), the boundedness of \( K_\gamma_1(t, T, b) \) ensures that \( K_\gamma_2(t, T, b) \) is also bounded, although the converse is not true. Therefore, an insurer A with a risk-aversion coefficient of \( \gamma_2 \) is more likely to have a stable optimal investment strategy than an insurer B with \( \gamma_1 \) for \( \gamma_1 \leq \gamma_2 \).

Let \( \tau_1 \) and \( \tau_2 \) be the first escape times of \( K_\gamma_1(t, T, b) \) and \( K_\gamma_2(t, T, b) \), respectively. As \( K_\gamma_1(t, T, b) \succ K_\gamma_2(t, T, b) \), \( K_\gamma_1(t, T, b) \) blows up before \( K_\gamma_2(t, T, b) \). In other words, \( \tau_1 \leq \tau_2 \). Hence, an insurer A with \( \gamma_2 \) can adopt a longer investment horizon, \( T \), than an insurer B with \( \gamma_1 \) because insurer A can choose any \( T < \tau_2 \), whereas insurer B can only choose any \( T < \tau_1 \).
Remarks:

1. Under the constant covariance matrix assumption, there is no finite escape time and hence no trade-off between risk-aversion and the investment horizon. Therefore, our result shows that the stochastic covariance matrix prevents insurers from planning their investment for a long time. An insurer should strike the best balance between the investment horizon and its risk-aversion level.

2. A similar conclusion could be made for the single-factor SV models, although studies have not explicitly mentioned it. However, we offer a unified analysis for both DMPCR and DRP, and incorporate the insurance liability into the analysis. Hence, Lemma 3.3 covers a wider class of models with stochastic variance matrices.

3.4. Logarithmic utility

When $\gamma$ approaches 1, the power utility function tends toward a logarithmic function. The corresponding investment policy and objective value require separate analysis. We now concentrate on maximizing the expected logarithmic utility function: $E[U(Y(T))] = E[\ln Y(T)]$. Applying Itô’s Lemma to the log-wealth process with respect to $\mu$, we have

$$d(\ln Y) = \left( r + \pi' \mu - \frac{1}{2} \pi \Sigma \pi \right) dt + \pi' \Sigma^{1/2} dB_t + \left( \ln(Y - Y \left( 1 - e^{-z} \right)) \right) - \ln Y \right) d\mathcal{N}_t$$

$$= \left[ r + \pi' \mu - \frac{1}{2} \pi \Sigma \pi - vz \right] dt + \pi' \Sigma^{1/2} dW_t - zd\mathcal{M}_t,$$

where $\pi(t) = \frac{\pi(t)}{\pi(t)}$ is the wealth portion invested in risky assets at time $t$ and $\mathcal{M}_t := \mathcal{N}_t - \int_0^T \nu(s) ds$ is a $\mathbb{G}$-martingale. Hence, we have

$$\max_{\pi(t)} E[\ln Y(T)]$$

$$= \ln Y(0) + \max_{\pi(t)} E \left[ \int_0^T \left( r - vz + \frac{1}{2} \mu' \Sigma^{-1} \mu - \frac{1}{2} \left( \pi - \Sigma^{-1} \mu \right)' \Sigma \left( \pi - \Sigma^{-1} \mu \right) \right) dt \right].$$

It is thus clear that the expected final utility attains its maximum value at

$$\pi^* = \Sigma^{-1} \mu \iff u^*(t, Y, \Lambda) = \Sigma^{-1} \mu Y(t)$$

$$= \begin{cases} b(t) Y, & \text{for DMPCR} \\ \Lambda b(t) Y, & \text{for DRP}, \end{cases}$$

where $b$ is defined as in (14), and the maximum objective value for log utility is

$$\ln Y(0) + \int_0^T r(t) dt - E[z] \int_0^T E[v(t)] dt + \frac{1}{2} E \left[ \int_0^T \mu' \Sigma^{-1}(t) \mu dt \right].$$

(27)
Theorem 3.3 If the insurer’s wealth process follows SDE (9), and insurance liability follows (7), then the utility maximization problem (10) with log utility following stochastic correlation risk models (4) and (5) has the following optimal solution (investment policy).

\[
\begin{cases} 
  b(t)Y, & \text{for DMPCR} \\
  \Lambda b(t)Y, & \text{for DRP}
\end{cases}
\]  

(28)

where \( b \) is as defined in (14), and the optimal value function, \( E[\ln Y(T)]|_{u=u^*} \), equals

\[
\ln Y_0 + \int_0^T r(s)ds - E[z] \int_0^T E[\nu(s)]ds + \frac{1}{2} \tilde{\Lambda}(0, T, b) + \frac{1}{2} \text{tr} (\tilde{\mathbf{K}}(0, T, b) \Lambda(0)),
\]

(29)

where

\[
\tilde{\mathbf{K}}(t, T, b) = \int_t^T \mathbf{e}^{\mathbf{M}(s-t)}b(s)b(s)' \mathbf{e}^{\mathbf{M}(s-t)}ds,
\]

(30)

\[
\tilde{\Lambda}(t, T, b) = \text{tr} \left[ \eta \mathbf{Q}Q' \int_t^T \mathbf{K}(s, T, b(s))ds \right]
\]

\[
= \text{tr} \left[ \eta \mathbf{Q}Q' \int_t^T \mathbf{e}^{-\mathbf{M}s} \left( \int_s^T \mathbf{e}^{\mathbf{M}\tau}b(\tau)b(\tau)' \mathbf{e}^{\mathbf{M}\tau}d\tau \right) \mathbf{e}^{-\mathbf{M}s}ds \right].
\]

(31)

Proof. In the analysis preceding this theorem, we show that (27) is the optimal value function for log utility. It remains for us to prove that

\[
\widehat{V}(t, T, \Lambda(t)) := \tilde{\Lambda}(t, T, b) + \text{tr} (\tilde{\mathbf{K}}(t, T, b) \Lambda(0))
\]

(32)

for the both correlation risk models, that is, (4) and (5). Note that

\[
\mu' \Sigma^{-1} \mu = \begin{cases} 
  (\Lambda \beta(t))' \Lambda^{-1} (\Lambda \beta(t)), & \text{for DMPCR} \\
  \mu(t) \Lambda \mu(t), & \text{for DRP}
\end{cases}
\]

\[
= b(t)' \Lambda b(t)
\]

by the definition of \( b(t) \). Thus, \( E \left[ \int_t^T \mu' \Sigma^{-1} \mu dt \right] \) can be rewritten as \( E \left[ \int_t^T b(t)' \Lambda(t)b(t) dt \right] \) with deterministic function \( b(t) \).

Application of the Feynman–Kac formula to \( E \left[ \int_t^T b(t)' \Lambda(t)b(t) dt \right] \) with respect to (2) yields

\[
\frac{\partial \widehat{V}}{\partial t} + L_\Lambda \widehat{V} + b' \Lambda b = 0 \quad \text{with} \quad \widehat{V}(T, T, b) = 0,
\]

(33)

where

\[
L_\Lambda \widehat{V} = \text{tr} \left[ (\eta \mathbf{Q}Q' + \mathbf{M} \Lambda + \Lambda \mathbf{M}') D\widehat{V} + 2 \Lambda D\mathbf{Q}Q' D\widehat{V} \right]
\]
if \( \hat{V}(t, T, \Lambda) := \mathbb{E}\left[ \int_t^T b(t') \Lambda(t) b(t') dt' \right] \). Substituting the partial derivative of \( \hat{V} \) into the left-hand side of (33) gives as

\[
\hat{A} + \text{tr}(\hat{K} \Lambda) + \text{tr} \left[ \eta Q Q \hat{K} + M \hat{K} + \Lambda M' \hat{K} + b' \Lambda b \right] = \hat{A} + \text{tr} \left( \eta Q Q \hat{K} \right) + \text{tr} \left[ \left( \hat{K} + \hat{K} M + M' \hat{K} + b b' \right) \Lambda \right] = 0 + \text{tr} \left( \mathbf{0}_{n \times n} \Lambda \right) = 0,
\]

which is equal to the right-hand side of (33). Also, \( \hat{V}(T, T, \Lambda) = 0 \) by the definition of \( \hat{V} \), which means that function \( \hat{V}(t, T, \Lambda(\tau)) := \hat{A}(t, b) + \text{tr} \left( \hat{K}(t, T, b) \Lambda(0) \right) = \hat{v}(t, T, \Lambda) := \mathbb{E}\left[ \int_t^T b(t') \Lambda(t) b(t') dt' \right] \). \( \Box \)

3.5. **Positive insurance premium**

When a positive insurance premium is allowed in (8), we need only to revise the wealth process to \( Y(t) = \sum_{j=0}^m u_j(t) + C(t) - L(t), Y(0) = Y_0 \). Let \( \hat{Y}(t) = Y(t) + \int_t^T e^{-\int_s^t r(\tau) d\tau} dC(s) \). By Itô’s Lemma, the SDE of \( \hat{Y}(t) \) becomes \( \mathrm{d}\hat{Y}(t) = \left[ r(t) \hat{Y}(t) + u(t)' \beta(t) \right] dt + u(t)' \sigma(t) dW_t - z dN_t \), with initial condition \( \hat{Y}(0) = Y_0 + \int_0^T e^{-\int_0^s r(\tau) d\tau} dC(s) \), which is exactly the same as (9) except for the initial condition. Therefore, all of the results, including the optimal ALM policy and efficient frontier, are applicable once \( Y_0 \) is replaced by \( \hat{Y}(0) \). The solution is equivalent to the case in which the insurance company collects a lump-sum premium at the beginning of the investment that has an amount equal to the present value of all future premiums.

4. **Numerical examples**

After deriving the optimal utility policies for the cases of both DRP and DMPCR, we now demonstrate the efficient frontiers for both numerically. We assume that the initial liability value is zero and that the initial wealth level is 10. Random variable \( z \), which affects the size of the insurance claim, follows an exponential distribution with mean \( \bar{z} \). The Poisson process, \( N_t \), has intensity \( \nu = 5 \). The investment horizon is 6 months \( (T = 1/2) \). To investigate the effect of different market beliefs on the rates of return, we consider the optimal asset allocation among the S&P 500 Index, 30-year Treasury bond futures, and a risk-free bank deposit. We use the parameters estimated by Buraschi et al. (2010), who applied the general method of moments (GMM) to S&P 500 Index and 30-year Treasury bond futures returns, sampled monthly from January 1990 to October 2003.

Table 1 presents the average realized volatilities and correlation from the GMM estimation. Mathematically, the numbers imply that

\[
\frac{1}{n} \sum_{j=1}^n \hat{\Sigma}(t_j) = \begin{pmatrix}
0.0474^2 & 0.0412 \times 0.0474 \times 0.0312 \\
0.0412 \times 0.0474 \times 0.0312 & 0.0312^2
\end{pmatrix}, \tag{34}
\]

where \( t_j \) is the \( j \)-th sampling month and \( n \) is the total number of months. Our numerical example thus takes \( \hat{\Sigma}(0) = \frac{1}{n} \sum_{j=1}^n \hat{\Sigma}(t_j) \), where 0 refers to the initial investment time. For the two-asset model, several parameters are required to define Wishart process \( \Lambda \).
RISK AVERSION AND INVESTMENT HORIZON

Table 1 Average realized volatilities and correlation

<table>
<thead>
<tr>
<th></th>
<th>Volatility of returns</th>
<th>Volatility of correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>0.0474</td>
<td>0.0412</td>
</tr>
<tr>
<td>Treasury</td>
<td>0.0312</td>
<td></td>
</tr>
</tbody>
</table>

Table 2 GMM estimates for the model under DRP

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>Q</th>
<th>(\tilde{\mu})</th>
<th>(\rho)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.149</td>
<td>0.114</td>
<td>0.706</td>
<td>0.494</td>
<td>0.0616</td>
</tr>
<tr>
<td>0.070</td>
<td>-0.112</td>
<td>0.806</td>
<td>0.641</td>
<td>0.0114</td>
</tr>
</tbody>
</table>

Table 3 GMM estimates for the model under DMPCR

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>Q</th>
<th>(\beta)</th>
<th>(\rho)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.122</td>
<td>0.747</td>
<td>0.160</td>
<td>0.083</td>
<td>4.612</td>
</tr>
<tr>
<td>0.884</td>
<td>-0.888</td>
<td>-0.021</td>
<td>0.009</td>
<td>2.891</td>
</tr>
</tbody>
</table>

For the model under DRP, the parameters of the Wishart processes are matrices \(M\) and \(Q\), and vectors \(\tilde{\mu}\) and \(\rho\), where \(\Sigma = \Lambda^{-1}\) and \(\tilde{\mu} = r\mathbf{1} + \mu\). Table 2 summarizes the corresponding GMM estimates in Buraschi et al. (2010), and we follow them by setting \(\eta = 10\). Basically, the Wishart process only requires \(\eta > n - 1\) for a positive definite matrix \(\Lambda(t)\) for all \(t > 0\), where \(n\) is the number of risky assets. Therefore, we only need \(\eta > 1\) for this empirical analysis. Perhaps, Buraschi et al. (2010) want to ensure a stronger positivity in the matrix and set a large number of \(\eta\). In fact, the \(\eta\) should better be estimated as well. As we do not want to estimate the parameters again, we use their estimated parameters. The estimation is out of the scope of this paper. The parameters for the model under DMPCR are matrices \(M\) and \(Q\), and the vectors \(\beta\) and \(\rho\), where \(\Sigma = \Lambda\) and \(\tilde{\mu} = r\mathbf{1} + \Sigma\beta\). Table 3 reports the GMM estimates in Buraschi et al. (2010).

We assume that the risk-free asset is available and offers a risk-free interest rate of 3%, that the initial liability value is zero, and that the initial wealth level is 10. Random claim payment \(z\) follows an exponential distribution with mean \(\frac{1}{2}\). Poisson process \(N^t\) has an intensity \(\nu = 5\).

After substituting the estimated parameters into the corresponding models, the initial investment amounts on the S&P 500 and Treasury bond futures can be found. Figure 1 plots the optimal investment capital ratio between the S&P 500 Index and the Treasury bond futures by varying \(\epsilon\) with the volatility of the covariance/precision matrix equaling \(\epsilon Q\). Both market assumptions have a similar feature: the investment amount in the stock index decreases with risk aversion. This is intuitive as the bank account and the default-free bond are often regarded as less risky than the stock index. As insurers have to ensure that they pay insurance claims, the optimal ALM strategy reduces the risky position. In addition, an increase in the volatility of the covariance matrix leads to an increase in stock index investment over that in bond futures.
5. Conclusion

By maximizing the expected CRRA utility, we produce the optimal investment for insurers when the financial market features a stochastic variance–covariance matrix. By establishing a verification theorem, we show that highly risk-averse insurers are able to plan a long-term investment strategy. Our numerical examples illustrate that when insurers’ degree of relative risk aversion is insufficiently high, they are interested in short-term investment in the stock market rather than investment in the bond market. When the uncertainty of the variance-covariance matrix is greater, riskier insurers participate more in the stock market.

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