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On Timescale Separation in Networked Systems With Intermittent Communication

This paper studies the multiple timescale behavior that is induced by dynamic coupling between continuous-time and discrete-time systems, and that arises naturally in distributed networked systems. An order reduction method is proposed that establishes a mathematically rigorous separation principle between the fast evolution of the continuous-time dynamics and the slow updates of the discrete-time dynamics. Quantitative conditions on the discrete update rate are then derived that ensure the stability of the coupled dynamics based on the behavior of the isolated systems. The results are illustrated for a distributed network of satellites whose attitudes evolve continuously while communicating intermittently over the network. [DOI: 10.1115/1.4038096]

1 Introduction

In multi-agent systems, distributed graph-based protocols between agents are often coupled to dynamic weight evolution and tracking of the agents' own nonlinear dynamics. One approach to analyzing these types of complex systems is to assume that the agents' network communication occurs very quickly, leading to a set of coupled, continuous-time dynamics [1]. These purely continuous-time dynamics can then be collectively designed to achieve desirable behavior [2,3]. However, this approach can break down as the time between communication updates becomes too large, leading to instability [4,5]. An alternative approach is to view the updates over the network as occurring only intermittently. This produces an inherently hybrid-time character in the coupled dynamics, as shown in Fig. 1. Intuitively, if the agents move quickly relative to the slow discrete updates over the network, then they will reach their immediate goal, and the network dynamics will evolve as if the agents are always at their state-dependent equilibrium trajectory. This implies that, depending on the rate of updates, a decoupling is possible between the fast, continuous-time agent dynamics and the slow, discrete-time network dynamics that is based on the different characteristic timescales over which the subsystems evolve. Such a separation between the subsystem dynamics is useful because it has the potential to decrease the complexity of analyzing the coupled hybrid-time system, permitting the networked decision dynamics to be designed separately from the agents' dynamics. The question, then, is under what conditions is this separation valid?

Concepts from singular perturbation theory provide an effective approach for understanding such multiple timescale systems [6]. For purely continuous-time systems, singular perturbation theory describes how slow and fast behaviors are induced when a small parameter multiplies the time derivatives of a subset of the system states. The Tikhonov–Levinson theorem then gives conditions under which such a purely continuous-time system can be analyzed based on the properties of its isolated slow and fast dynamics [6,7]. Application of these conditions to networked dynamical systems, for example, has yielded graph-topological stability bounds for the consensus-tracking and state-dependent graph problems [8]. The success of the Tikhonov–Levinson conditions for simplified analysis of continuous-time systems with slow and fast behaviors has led to several extensions in the literature for different classes of systems. These include: (1) discrete-time systems [9], (2) differential inclusions [10], (3) impulsive differential equations [11,12], and (4) hybrid systems where the fast dynamics are constrained to evolve on a compact set [13,14]. However, by focusing on the role of a small parameter in inducing slow and fast behaviors, these extensions do not consider how the update rate can affect the behavior of coupled hybrid-time dynamics and allow their analysis to be simplified.

This paper addresses the described research gap. In particular, this work studies the role of the discrete update period on the properties of coupled continuous-time and discrete-time dynamics. The paper makes two main contributions. The first is a novel reduced-order modeling technique, based on concepts of perturbation and asymptotic theory, that provides mathematical rigor to the intuitive notion of a separation principle in such hybrid-time systems as the time between discrete updates grows. In particular, conditions are derived under which the design of the discrete-time dynamics can be separated from the behavior of the continuous-time dynamics, and asymptotic bounds on the error that this decoupled design introduces are explicitly computed in terms of the update period. In the context of networked systems, these conditions give insight into when a designer can choose the network-based update algorithm independently from the behavior of the individual agents. The second contribution is a set of quantitative conditions that give a lower bound on the update rate under which the coupled hybrid-time system is guaranteed to be stable, and which are based on stability properties of the decoupled systems. These techniques can be practically applied to understand the

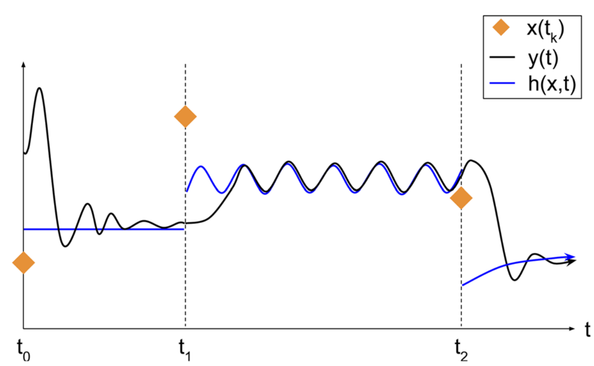

Fig. 1 Evolution of the coupled discrete-time network states *x* and continuous-time agent states *y* as the agents evolve toward their state-dependent equilibrium *h*(*x*, *t*) over each interval

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effects of implemented communication rates on the behavior of networked dynamical systems.

The structure of the paper is as follows: In Sec. 2, the hybrid-time system under study is defined. Section 3 then develops reduced-order models for the hybrid-time system and proves their efficacy under appropriate conditions. Next, Sec. 4 presents the method for finding quantitative stability bounds for the coupled hybrid-time dynamics. An application to a distributed network of satellites is illustrated in Sec. 5, and conclusions are presented in Sec. 6.

2 Problem Formulation

Consider the class of systems defined by

$$\begin{aligned} x\left(t_k^+\right) &= f\left(x(t_k^-), y(t_k), k; \mu\right),\ x(t_0) = x_0(\mu) \\ \dot{y}(t) &= g\left(x\left(t_k^+\right), y(t), t; \mu\right),\ y(t_0) = y_0(\mu) \end{aligned} \tag{1}$$

where the vector of discrete-time states is $x \in D_x \subseteq \mathbb{R}^{n_x}$, the vector of continuous-time states is $y \in D_y \subseteq \mathbb{R}^{n_y}$, updates of the discrete-time system occur at distinct times t_1, t_2,..., differences between these distinct times are lower-bounded by $t_k – t_{k–1} \geq \tau > 0$ for $k \in \{1, 2, \ldots\}$, and $\mu = 1/\tau$ measures the corresponding fastest update rate. In the context of networked dynamical systems, for example, y describes the agent states while x encodes the network states that update at time t_k. Here, the dependence on μ in the last argument of f and g captures possible changes in the vector fields due to a change in the update rate. Similarly, the dependence of x_0 and y_0 on μ is included to capture possible changes in the initial conditions due to a different update rate being used. Of course, g, f, x_0, and y_0 may not change with the update rate in a particular system, but including this potential dependency on the update rate allows the ensuing results to be applied to a wider class of systems. To pose a well-defined problem, assume that:

Assumption 1. *The functions g, f, x_0, and y_0 are $\mathcal{O}(1)$ as $\tau \to \infty$ in the domains D_y and D_x.*

Assumption 2. *The function g is continuously differentiable in all its arguments and the functions f, x_0, and y_0 are Lipschitz in their arguments over the domains D_y and D_x.*

3 Reduced-Order Modeling

In this section, two reduced-order models are formulated for the hybrid-time system (1). The validity of these models in approximating the behavior of Eq. (1) is then rigorously proven, justifying the separate design of the continuous-time agent dynamics from the discrete-time network decision dynamics. The reduced-order models are characterized as follows:

Decision System. Define the equilibrium trajectory of the isolated continuous-time agent dynamics as a known function $h : D_x \times \mathbb{R}_+ \to D_y$, which satisfies $\dot{h}(p, t) = g(p, h(p, t), t; 0)$ for $p \in D_x$ and $t \in R_+$. The decision system is then defined as

$$\bar{x}\left(t_k^+\right) = f\left(\bar{x}(t_k^-), h(\bar{x}(t_k^-), t_k), k; 0\right) \tag{2}$$

subject to $\bar{x}(t_0) = x_0(0)$.

Interval Correction System. Define the kth time interval between discrete-time updates as $\mathcal{I}_k \triangleq \{t \in \mathbb{R} \mid t_k \leq t < t_{k+1}\}$ and the elapsed time within this interval as $\eta \triangleq t - t_k$. The interval correction system is then defined separately for each interval $\mathcal{I}_k$ as

$$\begin{aligned} \dot{\hat{y}}_k(\eta) = g&\left(\bar{x}\left(t_k^+\right), \hat{y}_k(\eta) + h\left(\bar{x}\left(t_k^+\right), \eta + t_k\right), \eta + t_k; 0\right) \\ &- \frac{\partial h\left(\bar{x}\left(t_k^+\right), \eta + t_k\right)}{\partial t} \end{aligned} \tag{3}$$

subject to $\hat{y}_0(0) = y_0(0) - h(\bar{x}(t_0), t_0)$ for the first interval and $\hat{y}_k(0) = h(\bar{x}(t_k^-), t_k) - h\left(\bar{x}\left(t_k^+\right), t_k\right)$ otherwise, and where $\bar{x}$ is the state vector of the decision system defined in Eq. (2).

Remark 1. The decision system (2) describes the reduced-order behavior of the isolated discrete-time network dynamics with the continuous-time agent dynamics always at their state-dependent equilibrium. Note that the system is purely discrete.

Remark 2. The interval correction system (3) describes the evolution of the isolated continuous agent dynamics toward the equilibrium trajectory between each set of consecutive discrete updates. The initial conditions are based on the state vector $\bar{x}$ of the decision system alone. They are independent of the state of the interval correction system on any previous intervals. Note that the interval correction system describes purely continuous-time dynamics. The new time variable η is introduced here to make clear that solutions of Eq. (3) are dependent on the elapsed time and are defined only for a particular interval $\mathcal{I}_k$.

Remark 3. The last argument of the vector fields f and g is set to zero in Eqs. (2) and (3) because the reduced-order models describe the dynamics in the limit of τ growing very large so that the discrete network updates always occur with the agents at their state-dependent equilibrium.

Remark 4. While the original dynamics (1) are fully coupled, the reduced-order models (2) and (3) obtain their triangular structure by exploiting the equilibrium trajectory $h(\bar{x}, t)$.

With the reduced-order models defined in Eqs. (2) and (3), the next logical question is under what conditions these models provide an accurate description of the original coupled dynamics (1). To this end, the following lemma first provides conditions to ensure that the approximation provided by the reduced-order continuous-time interval dynamics remains close to the true continuous-time dynamics over the time interval $\mathcal{I}_k$, given that the initial conditions of Eq. (3) and the state vector $\bar{x}$ of the decision system (2) are both close to the true values of Eq. (1) at the start of the interval.

Lemma 1. *For the dynamics (1) under Assumptions 1 and 2, further assume that the interval correction system (3) is uniformly asymptotically stable for all points $\bar{x} \in D_x$ and the corresponding known trajectories $h\left(\bar{x}\left(t_k^+\right), t\right)$. Then, if $y(t_k) - \hat{y}_k(0) - h\left(\bar{x}\left(t_k^+\right), t_k\right) = o(1)$ and $x\left(t_k^+\right) - \bar{x}\left(t_k^+\right) = o(1)$ in the limit $\tau \to \infty$, τ can be chosen large enough that*

$$y(t) - \hat{y}_k(t - t_k) - h\left(\bar{x}\left(t_k^+\right), t\right) = o(1)$$

for all t in the time interval $\mathcal{I}_k = \{t \in \mathbb{R} \mid t_k \leq t < t_{k+1}\}$. Further, there is a t_j with $t_k < t_j \leq t_{k+1}$ such that

$$y(t) - h\left(\bar{x}\left(t_k^+\right), t\right) = o(1)$$

holds for all $t \in [t_j, t_{k+1})$.

Proof. The proof of the lemma follows three steps:

(1) Formulate the dynamics of the continuous-state approximation error over an interval as a perturbed version of the interval correction dynamics.
(2) Use the stability of the interval correction system and a converse Lyapunov theorem to obtain a Lyapunov function with certain desirable bounding properties for the interval correction dynamics.
(3) Use the interval correction dynamics' Lyapunov function as well as the form of the perturbation in dynamics of the continuous-state approximation error to obtain bounds on the norm of this approximation error.

The steps are detailed as follows:

Step 1. In the following, all instances of $x\left(t_k^+\right)$ will be written as x and all instances of $\bar{x}\left(t_k^+\right)$ as $\bar{x}$ for conciseness since both $x\left(t_k^+\right)$ and $\bar{x}\left(t_k^+\right)$ are constant over the interval $\mathcal{I}_k$ where the analysis is performed.

Define the error in the approximation over the interval $\mathcal{I}_k$ as $v(\eta) \triangleq y(\eta + t_k) - h(\bar{x}, \eta + t_k) - \hat{y}_k(\eta)$. The error dynamics within the interval are then written as

$$\frac{dv}{d\eta} = \frac{d}{d\eta}\left[y(\eta+t_k) - h(\bar{x},\eta+t_k) - \hat{y}_k(\eta)\right]$$
$$= \dot{y}(\eta+t_k) - \frac{\partial h(\bar{x},\eta+t_k)}{\partial t} - \dot{\hat{y}}_k(\eta)$$

Now, with $\dot{y}$ defined in Eq. (1) and $\dot{\hat{y}}_k$ defined in Eq. (3), the error dynamics are

$$\begin{aligned}
\frac{dv}{d\eta} &= g(x, y(\eta+t_k), \eta+t_k;\mu) - \frac{\partial h(\bar{x},\eta+t_k)}{\partial t} \\
&\quad -\left\{ g\left(\bar{x}, \hat{y}_k(\eta) + h(\bar{x},\eta+t_k), \eta+t_k; 0\right) - \frac{\partial h(\bar{x},\eta+t_k)}{\partial t}\right\} \\
&= g(x,y,\eta+t_k;\mu) - g\left(\bar{x}, \hat{y}_k + h(\bar{x},\eta+t_k), \eta+t_k; 0\right) \\
&\quad +\left\{ g(\bar{x},y,\eta+t_k;\mu) - g(\bar{x},y,\eta+t_k;\mu)\right\} \\
&\quad +\left\{ g(\bar{x},y,\eta+t_k;0) - g(\bar{x},y,\eta+t_k;0)\right\} \\
&\quad +\left\{ g(\bar{x}, v+h(\bar{x},\eta+t_k), \eta+t_k; 0) \right. \\
&\quad \left. -g(\bar{x}, v+h(\bar{x},\eta+t_k), \eta+t_k; 0)\right\} \\
&\quad +\left\{ g(\bar{x}, h(\bar{x},\eta+t_k), \eta+t_k; 0) - \frac{\partial h(\bar{x},\eta+t_k)}{\partial t}\right\} \\
&= g(\bar{x}, v+h(\bar{x},\eta+t_k), \eta+t_k; 0) - \frac{\partial h(\bar{x},\eta+t_k)}{\partial t} \\
&\quad +\{ g\left(\bar{x}, v+\hat{y}_k + h(\bar{x},\eta+t_k), \eta+t_k; 0\right) \\
&\quad -g(\bar{x}, v+h(\bar{x},\eta+t_k), \eta+t_k; 0) \\
&\quad \underbrace{-g\left(\bar{x}, \hat{y}_k + h(\bar{x},\eta+t_k), \eta+t_k; 0\right) +g(\bar{x}, h(\bar{x},\eta+t_k), \eta+t_k; 0)\}}_{\Delta_1} \\
&\quad +\underbrace{\left\{ g(\bar{x},y,\eta+t_k;\mu) - g(\bar{x},y,\eta+t_k;0)\right\}}_{\Delta_2} \\
&\quad +\underbrace{\left\{ g(x,y,\eta+t_k;\mu) - g(\bar{x},y,\eta+t_k;\mu)\right\}}_{\Delta_3}
\end{aligned}$$

using $y(\eta+t_k) = v(\eta) + h(\bar{x},\eta+t_k) + \hat{y}_k(\eta)$ by definition of the approximation error, and $g(p, h(p,\eta+t_k), \eta+t_k; 0) - \partial h(p,\eta+t_k)/\partial t = 0$ for $p \in D_x$ by definition of the equilibrium trajectory. That is, the error dynamics may be written as

$$\frac{dv}{d\eta} = g(\bar{x}, v+h(\bar{x},\eta+t_k), \eta+t_k; 0) - \frac{\partial h(\bar{x},\eta+t_k)}{\partial t} + \Delta G \quad (4)$$

a perturbed version of the interval correction dynamics (3), where the perturbation is $\Delta G = \Delta_1 + \Delta_2 + \Delta_3$. Now, the first component of the perturbation satisfies

$$\begin{aligned}
\|\Delta_1\| &= \| g\left(\bar{x}, v+\hat{y}_k + h(\bar{x},\eta+t_k), \eta+t_k; 0\right) \\
&\quad -g(\bar{x}, v+h(\bar{x},\eta+t_k), \eta+t_k; 0) \\
&\quad -g\left(\bar{x}, \hat{y}_k + h(\bar{x},\eta+t_k), \eta+t_k; 0\right) \\
&\quad +g(\bar{x}, h(\bar{x},\eta+t_k), \eta+t_k; 0)\| \\
&= \left\| \int_{h(\bar{x},\eta+t_k)}^{\hat{y}_k + h(\bar{x},\eta+t_k)} \left(\frac{\partial g(\bar{x}, s+v, \eta; 0)}{\partial y} - \frac{\partial g(\bar{x}, s, \eta; 0)}{\partial y}\right) \mathrm{d}s \right\| \\
&\le \left\| \int_{h(\bar{x},\eta+t_k)}^{\hat{y}_k + h(\bar{x},\eta+t_k)} M\|v\| \mathrm{d}s \right\| \\
&\le M\|v\|\|\hat{y}_k\|
\end{aligned}$$

over the domain for some $M \ge 0$ since g is continuously differentiable. The second component has the bound

$$\|\Delta_2\| = \|g(\bar{x},y,\eta+t_k;\mu) - g(\bar{x},y,\eta+t_k;0)\| \le \mu L_4$$

over the domain for some $L_4 \ge 0$ since g is Lipschitz in its last argument. The third component is then similarly bounded by

$$\|\Delta_3\| = \|g(x,y,\eta+t_k;\mu) - g(\bar{x},y,\eta+t_k;\mu)\| \quad \le L_1\|x-\bar{x}\|$$

over the domain for some $L_1 \ge 0$ since g is Lipschitz in its first argument. Therefore, the overall perturbation bounds

$$\|\Delta G\| \le \left(M\|\hat{y}_k\|\right)\|v\| + \{L_1\|x-\bar{x}\| + \mu L_4\} \quad (5)$$

hold over the domain.

Step 2. By assumption, the interval correction system (3) is uniformly asymptotically stable for each $\bar{x} \in D_x$. Therefore, by the converse Lyapunov theorem [7, Theorem 4.16], there exists a Lyapunov function V over some domain $D_V = \{v \in \mathbb{R}^n \mid \|v\|_2 < r\}$ for the dynamics that satisfies the inequalities $\alpha_1(\|v\|) \le V(\eta, v) \le \alpha_2(\|v\|)$

$$\frac{\partial V}{\partial \eta} + \frac{\partial V}{\partial v}\left\{ g(\bar{x}, v+h(\bar{x},\eta+t_k), \eta+t_k; 0) - \frac{\partial h(\bar{x},\eta+t_k)}{\partial t}\right\}$$
$$\le -\alpha_3(\|v\|)$$

and $\|\partial V/\partial v\| \le \alpha_4(\|v\|)$ for class $\mathcal{K}$ functions α_i defined on $[0, r]$.

Step 3. In Eq. (5), the perturbation ΔG in the error dynamics (4) has been shown to satisfy

$$\|\Delta G\| \le \gamma(\eta)\|v\| + d$$

with $\gamma(\eta) = \left(M\|\hat{y}_k\|\right)\|v\|$ and $d = L_1\|x-\bar{x}\| + \mu L_4$. Due to the asymptotic stability of the interval correction dynamics

$$\|\hat{y}_k\| \le \beta_1\left(\|\hat{y}_k(0)\|, \eta\right), \ \forall\, 0 \le \eta < t_{k+1} - t_k \quad (6)$$

for some class $\mathcal{KL}$ function β_1. Now, $\gamma(\eta) = M\beta_1\left(\|\hat{y}_k(0)\|, \eta\right)$ and d are both non-negative, continuous, and bounded for all $0 \le \eta \le t_{k+1} - t_k$. Since $\|x-\bar{x}\| = o(1)$ and $\|v(0)\| = o(1)$, μ can be chosen, dependent on r (the same r that defines D_V) and for $0 \le \eta \le t_{k+1} - t_k$, small enough that

$$\begin{aligned}
\gamma(\eta)\|v\| + d &= M\beta_1\left(\|\hat{y}_k(0)\|, \eta\right)\|v\| \\
&\quad + L_1\|x-\bar{x}\| + \mu L_4 \\
&\le \delta < \theta\alpha_3\left(\alpha_2^{-1}(\alpha_1(r))\right)/\alpha_4(r)
\end{aligned}$$

with $\|v\| < r$, $\delta \in \mathbb{R}_+$, and $0 < \theta < 1$. Therefore, by Ref. [7, Lemma 9.3], for all $\|v(0)\| < \alpha_2^{-1}(\alpha_1(r))$, trajectories of the approximation error system (4) satisfy

$$\|v(\eta)\| \le \beta_2\left(\|v(0)\|, \eta\right), \ \forall\, 0 \le \eta < \bar{\eta}$$

and

$$\|v(\eta)\| \le \alpha_5(\delta), \ \forall\, \bar{\eta} \le \eta \le t_{k+1} - t_k$$

for some class $\mathcal{KL}$ function β_2, some finite $\bar{\eta}$, and α_5 the class κ function defined by

$$\alpha_5(\delta) = \alpha_1^{-1}\left(\alpha_2\left(\alpha_3^{-1}\left(\frac{\delta\alpha_4(r)}{\theta}\right)\right)\right)$$

This gives the bound on $y(t) - h(\bar{x}, t) - \hat{y}_k(t - t_k)$ for all $t_k \le t < t_{k+1}$. Since Eq. (6) gives an asymptotically decaying bound on $\|\hat{y}_k\|$, τ can be chosen large enough that there exists a t_j with $t_k < t_j \le t_{k+1}$ such that $\|\hat{y}_k(t - t_k)\| = o(1)$ for $t \ge t_j$ and thus $y(t) - h(\bar{x}, t) = o(1)$ holds for all $t \in [t_j, t_{k+1})$. □

In Lemma 1, the Landau "little-Oh" $o(1)$ bounds on the approximation error of the continuous agent state over an interval can be interpreted as saying that this error decreases asymptotically as τ grows large even though the reduced-order model is not quite the same as the full model and the approximation starts from different initial conditions than the true dynamics. This is a result of the assumption of uniform asymptotic stability for Eq. (3), which allows the continuous dynamics to eventually evolve to a known trajectory when $\bar{x}$ is known. As described by the following lemma, tighter bounds on this approximation error can be found under the more restrictive assumption of exponential stability for Eq. (3).

LEMMA 2. *Under the conditions of Lemma 1, if instead the errors in initial conditions are* $\mathcal{O}(\tau^{-1})$ *and the interval correction system (3) is exponentially stable for all points* $\bar{x} \in D_x$ *and their corresponding known trajectories* $h(\bar{x}(t_k^+), t)$*, then the o(1) bounds on the approximation in Lemma 1 are replaced by* $\mathcal{O}(\tau^{-1})$ *bounds.*

Proof. The proof adopts the same three-step structure as the proof of Lemma 1, with the following changes:

Step 1. Follows identically.

Step 2. By Ref. [7, Theorem 4.14], the conditions on the converse Lyapunov function V are satisfied with $\alpha_i(\|v\|) = c_i\|v\|^2$ for $i = 1, 2, 3$ and $\alpha_4(\|v\|) = c_4\|v\|$ for positive constants c_1–c_4.

Step 3. Since exponential stability holds, then the bound on $\|\hat{y}_k\|$ in Eq. (6) is of the form $\beta_1(\|\hat{y}_k(0)\|, \eta) = k\|\hat{y}_k(0)\|e^{-a\eta}$. Further, $\int_0^\eta \gamma(\tau)d\tau \le 0 \cdot \eta + \omega$ for some non-negative constant ω. Define $a = 1/2c_3/c_2 > 0$ and $p = \exp(c_4\omega/2c_1) \ge 1$, where the c_i comes from the bounds converse Lyapunov function V, and choose τ large enough that $v(0) = y(t_k) - \hat{y}_k(0) - h(\bar{x}(t_k^+), t_k) = \mathcal{O}(\tau^{-1})$ satisfies $\|v(0)\| < r/p/\sqrt{c_1/c_2}$ and small enough that $d < 2c_1ar/c_4p$. By Ref. [7, Lemma 9.4], trajectories of the approximation error system (4) therefore satisfy the norm bound

$$\|v(\eta)\| \le \sqrt{\frac{c_2}{c_1}}p\|v(0)\|e^{-\alpha\eta} + \frac{c_4p}{2c_1}d\int_0^\eta e^{-a(\eta-\tau)}\mathrm{d}\tau$$

whose first term is of $\mathcal{O}(\tau^{-1})$ for all $\eta \ge 0$ if $\|v(0)\| = \mathcal{O}(\tau^{-1})$, and whose second term is of $\mathcal{O}(\tau^{-1})$ because d is and since the integral is bounded for all $\eta \ge 0$. This gives the bound on $y - h(\bar{x}, t) - \hat{y}_k(t - t_k)$ for all $t_k \le t < t_{k+1}$. Since $\|\hat{y}_k\|$ has an exponentially decaying bound in η, τ can be chosen large enough that there exists a t_j with $t_k < t_j \le t_{k+1}$ such that $\|\hat{y}_k(t - t_k)\| = \mathcal{O}(\tau^{-1})$, giving the desired bound on $y(t) - h(\bar{x}, t)$. □

Under the more stringent condition of exponential stability for the interval correction system (3), the corresponding Landau "big-Oh" $\mathcal{O}(\tau^{-1})$ bounds in Lemma 2 give a stricter statement of the rate of the approximation error's decrease as τ grows large. This is possible because exponential stability gives concrete time bounds for the evolution of y toward $h(\bar{x}, t)$ in step 3 of the lemma.

With Lemmas 1 and 2 in place, the validity of approximating the coupled hybrid-time system (1) by dynamics of the reduced-order models (2) and (3) can now be determined.

THEOREM 1. *For the dynamics (1) under Assumptions 1 and 2, further assume that the interval correction system (3) is uniformly asymptotically stable for all points* $\bar{x} \in D_x$ *and the corresponding known trajectories* $h(\bar{x}(t_k^+), t)$*. Then, for the reduced-order models* $\bar{x}$ *and* $\hat{y}$*, respectively, characterized in Eqs. (2) and (3), and any* $t_f \ge t_0$*, there exists a* τ_0*,* $0 < \tau_0 < \infty$*, such that for all* $\tau \ge \tau_0$ *the approximations*

$$x(t_k^+) = \bar{x}(t_k^+) + o(1)$$
$$y(t) = h(\bar{x}(t_k^+), t) + \hat{y}_k(t - t_k) + o(1)$$

are valid for all $t \in [t_0, t_f]$*. Further, for each interval* $\mathcal{I}_k$ *between discrete updates with* $t_k < t_f$*, there is a* t_j *with* $t_k < t_j \le t_{k+1}$ *such that the approximation*

$$y = h(\bar{x}(t_k^+), t) + o(1)$$

holds for all $t \in [t_j, t_{k+1})$*.*

Proof. The proof uses induction to follow the error in the approximation for x over discrete updates, with the error in the y approximation bounded over the continuous-time intervals using Lemma (1).

Define $u(t_k^+; \mu) = x(t_k^+; \mu) - \bar{x}(t_k^+)$. Suppose for some k that $\|x(t_k^-) - \bar{x}(t_k^-)\| = o(1)$ and $\|y(t_k - \tau) - h(\bar{x}(t_k^-), t_k - \tau)\| = o(1)$. Then, under the assumption of uniform asymptotic stability of Eq. (3), τ can be always be chosen large enough that $\|y(t_k) - h(\bar{x}(t_k^-), t_k^-)\| = o(1)$ by Lemma 1. Further, after the transition

$$\begin{aligned}\|u(t_k^+; \mu)\| &= \|f(x(t_k^-), y(t_k), k; \mu) - f(\bar{x}(t_k^-), h(\bar{x}(t_k^-), t_k), k; 0)\| \\ &\le P_1\|x(t_k^-) - \bar{x}(t_k^-)\| + P_2\|y(t_k) - h(\bar{x}(t_k^-), t_k)\| + P_4\mu \\ &= o(1)\end{aligned}$$

using the Lipschitz property of f so that $\|x(t_k^+) - \bar{x}(t_k^+)\| = o(1)$. From the initial conditions and the assumption that x_0 and y_0 are $\mathcal{O}(1)$ and Lipschitz, $\|x(t_1^-) - \bar{x}(t_1^-)\| = \|x(t_0^+) - \bar{x}(t_0^+)\| = \|x_0(\mu) - x_0(0)\| = \mathcal{O}(\mu)$ and $\|y(t_0) - \hat{y}_0(0) - h(\bar{x}(t_0^+), t_0)\| = \|y_0(\mu) - y_0(0)\| = \mathcal{O}(\mu)$, and are thus both $o(1)$. Therefore, τ can be chosen large enough that the approximations

$$x(t_k^+) = \bar{x}(t_k^+) + o(1)$$
$$y(t) = h(\bar{x}(t_k^+), t) + \hat{y}(t - t_k) + o(1)$$

hold by induction for any finite number of discrete updates, and thus are valid for all $t \in [t_0, t_f]$. The bound on y for $t \in [t_j, t_{k+1})$ then comes from application of Lemma 1. □

Again, stricter bounds on the approximation errors may be found by assuming exponential stability of the interval correction system.

COROLLARY 1. *Under the conditions of Theorem 1, if the interval correction system (3) is instead exponentially stable for all points* $\bar{x} \in D_x$ *and their corresponding known trajectories* $h(\bar{x}(t_k^+), t)$*, then the o(1) bounds on the approximation in Theorem 1 are replaced by* $\mathcal{O}(\tau^{-1})$ *bounds.*

Proof. The desired results follow identically to the proof of Theorem 1 by noting that (1) both $\|x_0(\mu) - x_0(0)\| = \mathcal{O}(\tau^{-1})$ and $\|y_0(\mu) - y_0(0)\| = \mathcal{O}(\tau^{-1})$ hold since x_0 and y_0 are Lipschitz, (2) the continuous-interval bounds given by Lemma 2 may be used instead of the bounds given by Lemma 1 since the stricter requirement of Eq. (3) being exponentially stable is met here. □

Theorem 1 provides a certificate that Eqs. (2) and (3) are good approximations of Eq. (1) as τ grows large, with the trajectories of the reduced-order approximations remaining asymptotically close to the trajectories of the full system. Intuitively, the error bounds grow smaller as the update period grows because the continuous-time agent dynamics have more time to reach their equilibrium trajectory, making the reduced-order models more accurate. Corollary 1 then provides stricter bounds on the approximation error by assuming exponential stability instead of asymptotic stability, as in Lemma 2. These results give mathematical rigor to the notion of a separation principle between the continuous-time and discrete-time dynamics, as desired.

4 Stability Bounds on Communication Rate

For the purpose of implementation, quantitative bounds on τ are desirable that guarantee stability of the hybrid-time system (1). If the reduced-order discrete-time decision system (2) is designed assuming that the continuous-time dynamics are always at their state-dependent equilibrium, then too-frequent updates may lead to instability of the coupled dynamics (1). This is because at the time of the next update, the continuous-time states would not have had a chance to reach their new equilibrium and may in fact have initially moved away from this equilibrium due to nonminimum phase behavior [6, Chap. 6]. This section therefore details an approach to find sufficient lower bounds on τ above which the hybrid-time system is guaranteed to be stable under given conditions and using the stability properties of the individual reduced-order models.

To begin, define the trajectory tracking error $\tilde{y}$ as the difference between the continuous agent state vector and its state-dependent equilibrium trajectory, $\tilde{y}(t) = y(t) - h\big(x(t_k^+), t\big)$. After a discrete jump, the equilibrium trajectory changes due to the updated value of the discrete network state vector, and $\tilde{y}$ is thus correspondingly updated as

$$\begin{aligned}\tilde{y}(t_k^+) &= y(t_k) - h\big(x(t_k^+), t_k\big) \\ &= \big\{\tilde{y}(t_k^-) + h(x(t_k^-), t_k)\big\} - h\big(x(t_k^+), t_k\big) \\ &= \tilde{y}(t_k^-) + h(x(t_k^-), t_k) \\ &\quad - h\big(f\big(x(t_k^-), \tilde{y}(t_k^-) + h(x(t_k^-), t_k), t_k; \mu\big), t_k\big)\end{aligned}$$

The hybrid-time system (1) can therefore be rewritten in terms of $\tilde{y}$ as

$$\begin{aligned}x(t_k^+) &= f(x(t_k^-), \tilde{y}(t_k^-) + h(x(t_k^-), t_k; \mu) \\ \tilde{y}(t_k^+) &= \tilde{y}(t_k^-) + h(x(t_k^-), t_k) \\ &\quad - h\big(f\big(x(t_k^-), \tilde{y}(t_k^-) + h(x(t_k^-), t_k), t_k; \mu\big), t_k\big) \\ \dot{\tilde{y}}(t) &= g\big(x(t_k^+), \tilde{y}(t) + h\big(x(t_k^+), t\big), t; \mu\big) - \frac{\partial h\big(x(t_k^+), t\big)}{\partial t}\end{aligned} \tag{7}$$

subject to $\tilde{y}(t_0) = y(t_0) - h(x(t_0), t_0)$ and valid for all time $t \geq t_0$. This collection of states will be denoted in the following by $z \triangleq \big[x^{\mathrm{T}}, \tilde{y}^{\mathrm{T}}\big]^{\mathrm{T}}$.

In order to derive explicit numerical conditions that guarantee stability of Eq. (7), it will be further assumed that

Assumption 3. *The vector field f can be written as $f\big(x, \tilde{y} + h(x,t), t_k; \mu\big) = Kx + \tilde{f}(x, \tilde{y}, t_k; \mu)$, where $K \in \mathbb{R}^{n_x \times n_x}$ and $\tilde{f}$ is bounded over D_x and D_y as*

$$\begin{aligned}\|\tilde{f}\|^2 &\leq z^{\mathrm{T}}\begin{bmatrix} J_{11} & J_{12} \\ J_{12}^T & J_{13}\end{bmatrix} z + \mu z^{\mathrm{T}}\begin{bmatrix} E_{11} & E_{12} \\ E_{12}^T & E_{13}\end{bmatrix} z \\ &\triangleq z^{\mathrm{T}} J_1 z + \mu z^{\mathrm{T}} E_1 z\end{aligned}$$

with $J_{11}, E_{11} \in \mathbb{R}^{n_x \times n_x}$, $J_{12}, E_{12} \in \mathbb{R}^{n_x \times n_y}$, and $J_{13}, E_{13} \in \mathbb{R}^{n_y \times n_y}$

Assumption 4. *The vector field g can be written as $g\big(x, \tilde{y} + h(x,t), t; \mu\big) - \partial h(x,t)/\partial t = A\tilde{y} + \tilde{g}(x, \tilde{y}, t, \mu)$, where $A \in \mathbb{R}^{n_y \times n_y}$ and $\tilde{g}$ is bounded over D_x and D_y as*

$$\|\tilde{g}(x, \tilde{y}, t, \mu)\|^2 \leq \tilde{y}^{\mathrm{T}} R \tilde{y} + \mu z^{\mathrm{T}}\begin{bmatrix} E_{21} & E_{22} \\ E_{22}^{\mathrm{T}} & E_{23}\end{bmatrix} z \triangleq \tilde{y}^{\mathrm{T}} R \tilde{y} + \mu z^{\mathrm{T}} E_2 z$$

with $E_{21} \in \mathbb{R}^{n_x \times n_x}$, $E_{22} \in \mathbb{R}^{n_x \times n_y}$, and R, $E_{23} \in \mathbb{R}^{n_y \times n_y}$

Assumption 5. *The jump in the $\tilde{y}$ states after a discrete update, $\Delta h \triangleq h(x(t_k^-), t_k) - h\big(f\big(x(t_k^-), \tilde{y}(t_k^-) + h(x(t_k^-), t), k; \mu\big), t_k\big)$, is bounded over D_x and D_y as*

$$\|\Delta h\|^2 \leq z^{\mathrm{T}}\begin{bmatrix} J_{21} & J_{22} \\ J_{22}^T & J_{23}\end{bmatrix} z \triangleq z^{\mathrm{T}} J_2 z$$

with $J_{21} \in \mathbb{R}^{n_x \times n_x}$, $J_{22} \in \mathbb{R}^{n_x \times n_y}$, and $J_{23} \in \mathbb{R}^{n_y \times n_y}$.

Assumption 3 implies that the discrete x dynamics have a zero equilibrium when $\tilde{y} = 0$ and $\mu = 0$. This is reasonable as equilibrium of interest can be relocated to zero by a simple coordinate shift [7]. Assumption 4 allows $\tilde{y}$ to have a stable zero equilibrium when $\mu = 0$, while Assumption 5 provides state-dependent bounds on the change in the continuous agent dynamics' state-dependent equilibrium trajectory due to a discrete update. Nonlinearities of these types are standard in the interconnected systems literature and cover a wide class of practical systems including the attitude dynamics of precision-pointing spacecraft [15, Chap. 6] and multimachine power systems [16,17].

With these assumptions in place, the following theorem now gives sufficient conditions under which the error dynamics (7) are stable about the origin. The approach uses the well-known theory of vector Lyapunov functions [18, Chap. 2], which allows stability properties of the isolated subsystems to be leveraged in the analysis by loosening the restrictions on scalar Lyapunov functions for hybrid systems [19].

Theorem 2. *For the dynamics (7) under Assumptions 3–5, if for some $\tau^\star = 1/\mu^\star$ there are matrices $P_1, P_2 > 0$ and positive scalars d_{ij}, β, γ_i, κ_i such that the linear matrix inequalities (LMIs)*

$$\Phi \leq 0,\ \Omega \leq 0,\ \Psi \leq 0$$

hold with

$$\Phi \triangleq \begin{bmatrix} \begin{pmatrix} -d_{11}P_1 + K^{\mathrm{T}}P_1K \\ +\gamma_1 J_{11} + \gamma_1\mu^\star E_{11}\end{pmatrix} & \gamma_1 J_{12} + \gamma_1\mu^\star E_{12} & K^{\mathrm{T}}P_1 \\ * & \gamma_1 J_{13} + \gamma_1 \mu^\star E_{13} - d_{12}P_2 & 0 \\ * & * & -\gamma_1 + P_1\end{bmatrix}$$

$$\Omega \triangleq \begin{bmatrix} -\kappa_1 P_1 + \mu^\star \beta E_{21} & \mu^\star\beta E_{22} & 0 \\ * & \begin{pmatrix} \kappa_2 P_2 + P_2 A + A^{\mathrm{T}}P_2 \\ +\beta R + \mu^\star \beta E_{23}\end{pmatrix} & P_2 \\ * & * & -\beta I\end{bmatrix}$$

and

$$\Psi \triangleq \begin{bmatrix} -d_{21}P_1 + \gamma_2 J_{21} & \gamma_2 J_{22} & 0 \\ * & -d_{22}P_2 + P_2 + \gamma_2 J_{23} & P_2 \\ * & * & -\gamma_2 I + P_2\end{bmatrix}$$

and such that

$$\rho\left(\begin{bmatrix} d_{11} + \dfrac{d_{12}\kappa_1}{\kappa_2}\big(1 - e^{-\kappa_2\tau^\star}\big) & d_{12}e^{-\kappa_2\tau^\star} \\ d_{21} + \dfrac{d_{22}\kappa_1}{\kappa_2}\big(1 - e^{-\kappa_2\tau^\star}\big) & d_{22}e^{-\kappa_2\tau^\star}\end{bmatrix}\right) \leq 1$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix; then the error dynamics (7) are stable about zero for all $\tau \geq \tau^\star$.

Proof. The proof follows by construction of a vector Lyapunov function and analysis of the corresponding comparison system. To begin, assume the vector Lyapunov function $U(t) = [V_1(t), V_2(t)]^{\mathrm{T}}$ (see Ref. [18, Chap. 2], for example) with $V_1 = x^{\mathrm{T}}P_1 x$ and $V_2 = \tilde{y}^{\mathrm{T}}P_2\tilde{y}$ and $P_i > 0$. Further assume that the LMIs are satisfied for some $\mu^\star_{\mathrm{LMI}}$. Since μ comes into the LMIs through a positive semidefinite matrix, the LMIs are also satisfied for all $\mu \leq \mu^\star_{\mathrm{LMI}}$, or equivalently all $\tau \geq 1/\mu^\star_{\mathrm{LMI}} = \tau^\star_{\mathrm{LMI}}$.

For V_1, calculating the derivative for $t \in \mathcal{I}_k$ yields

$$\dot{V}_1 = 2x^{\mathrm{T}}P_1\dot{x} = 0$$

since x only changes at the discrete jumps. Over jumps

$$\begin{aligned}
V_1(t_k^+) &= f(x,\tilde{y}+h,k;\mu)^{\mathrm{T}} P_1 f(x,\tilde{y}+h,k;\mu) \\
&= \left(Kx+\tilde{f}\right)^{\mathrm{T}} P_1 \left(Kx+\tilde{f}\right) \\
&= \begin{bmatrix} x \\ \tilde{f} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} K & I \end{bmatrix}^{\mathrm{T}} P_1 \begin{bmatrix} K & I \end{bmatrix} \begin{bmatrix} x \\ \tilde{f} \end{bmatrix} \\
&\le \begin{bmatrix} x \\ \tilde{f} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} K & I \end{bmatrix}^{\mathrm{T}} P_1 \begin{bmatrix} K & I \end{bmatrix} \begin{bmatrix} x \\ \tilde{f} \end{bmatrix} \\
&\quad + \gamma_1 \left[z^{\mathrm{T}} J_1 z + \mu z^{\mathrm{T}} E_1 z - \|\tilde{f}\|^2 \right]
\end{aligned}$$

where $\gamma_1 \ge 0$, since $0 \le \gamma_1 \left[z^{\mathrm{T}} J_1 z + \mu z^{\mathrm{T}} E_1 z - \|\tilde{f}\|^2 \right]$. Adding and subtracting $d_{11}V_1$ and $d_{12}V_2$, where each $d_{1j} \ge 0$, this can be further arranged as

$$V_1(t_k^+) \le \begin{bmatrix} z \\ \tilde{f} \end{bmatrix}^{\mathrm{T}} \Phi \begin{bmatrix} z \\ \tilde{f} \end{bmatrix} + d_{11}V_1 + d_{12}V_2$$

Since $\Phi \le 0$, the inequality $V_1(t_k^+) \le d_{11}V_1(t_k^-) + d_{12}V_2(t_k^-)$ therefore holds.

For V_2, the derivative along trajectories for $t \in \mathcal{I}_k$ satisfy

$$\begin{aligned}
\dot{V}_2 &= 2\tilde{y}^{\mathrm{T}} P_2 \dot{\tilde{y}} \\
&\le 2\tilde{y}^{\mathrm{T}} P_2 (A\tilde{y} + \tilde{g}(x,\tilde{y},t,\mu)) \\
&\quad + \beta\left(\tilde{y}^{\mathrm{T}} R \tilde{y} + \mu z^{\mathrm{T}} E_2 z - \|\tilde{g}\|^2\right) \\
&\quad + (\kappa_1 V_1 - \kappa_2 V_2) - (\kappa_1 V_1 - \kappa_2 V_2)
\end{aligned}$$

This can be rearranged as

$$\dot{V}_2 \le \begin{bmatrix} z \\ \tilde{g} \end{bmatrix}^{\mathrm{T}} \Omega \begin{bmatrix} z \\ \tilde{g} \end{bmatrix} + \kappa_1 V_1 - \kappa_2 V_2$$

and therefore $\dot{V}_2 \le \kappa_1 V_1 - \kappa_2 V_2$ holds since $\Omega \le 0$. Over jumps, similarly to the $V_1(t_k^+)$ case

$$\begin{aligned}
V_2(t_k^+) &= \{\tilde{y}+\Delta h\}^{\mathrm{T}} P_2 \{\tilde{y}+\Delta h\} \\
&= \begin{bmatrix} \tilde{y} \\ \Delta h \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & I \end{bmatrix}^{\mathrm{T}} P_2 \begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \Delta h \end{bmatrix} \\
&\le \begin{bmatrix} \tilde{y} \\ \Delta h \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & I \end{bmatrix}^{T} P_2 \begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \Delta h \end{bmatrix} + \gamma_2 \left[z^{\mathrm{T}} J_2 z - \|\Delta h\|^2 \right] \\
&= \begin{bmatrix} z \\ \Delta h \end{bmatrix}^{T} \Psi \begin{bmatrix} z \\ \Delta h \end{bmatrix} + d_{21}V_1 + d_{22}V_2
\end{aligned}$$

Thus, $V_2(t_k^+) \le d_{21}V_1(t_k^-) + d_{22}V_2(t_k^-)$ holds since $\Psi \le 0$.

The vector Lyapunov function $U(t)$ has been shown to satisfy

$$\dot{U} \preccurlyeq \underbrace{\begin{bmatrix} 0 & 0 \\ \kappa_1 & -\kappa_2 \end{bmatrix}}_{B} U$$

and

$$U(t_k^+) \preccurlyeq \underbrace{\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}}_{D} U(t_k^-)$$

where $v \preccurlyeq u$ implies $v_i \le u_i$ for all i. A comparison system for the dynamics is therefore

$$\begin{aligned} u(t_k^+) &= \mathrm{Du}(t_k^-) \\ \dot{u} &= \mathrm{Bu} \end{aligned} \tag{8}$$

subject to $u(t_0) = U(t_0)$. It follows from Ref. [18, Theorem 2.11] that the stability properties of the zero solution of Eq. (8) imply the corresponding stability properties of the error dynamics (7).

To analyze stability of Eq. (8), construct the state evolution of u at $t \in \mathcal{I}_k$ as

$$\begin{aligned}
u(t) &= e^{B(t-t_k)} \prod_{i=1}^{k} \mathrm{D} e^{B(t_i - t_{i-1})} u(t_0) \\
&= e^{B(t-t_k)} \prod_{i=1}^{k} \tilde{D}(t_i - t_{i-1}) u(t_0)
\end{aligned}$$

where $\tilde{D}$ is explicitly calculated as

$$\tilde{D}(t_i - t_{i-1}) = \begin{bmatrix} d_{11} + \dfrac{d_{12}\kappa_1}{\kappa_2}\left(1 - e^{-\kappa_2(t_i - t_{i-1})}\right) & d_{12} e^{-\kappa_2(t_i - t_{i-1})} \\ d_{21} + \dfrac{d_{22}\kappa_1}{\kappa_2}\left(1 - e^{-\kappa_2(t_i - t_{i-1})}\right) & d_{22} e^{-\kappa_2(t_i - t_{i-1})} \end{bmatrix}$$

Now, $\|u(t)\|$ is bounded if each $\tilde{D}$ has spectral radius of at most one. Noting that as $\tau \to \infty$

$$\lim_{\tau \to \infty} \lambda\left(\tilde{D}(\tau)\right) = d_{11} + \frac{d_{12}\kappa_1}{\kappa_2}, \; 0$$

the condition is feasible if $d_{11} + d_{12}(\kappa_1/\kappa_2) \le 1$. Therefore, choosing $\tau^\star \ge \tau^\star_{\mathrm{LMI}}$ such that $\max|\lambda(\tilde{D}(\tau^\star))| \le 1$, and noting that $\max|\lambda(\tilde{D}(\tau))|$ is monotonic in τ, the original system (1) is stable for all $\tau \ge \tau^\star$. □

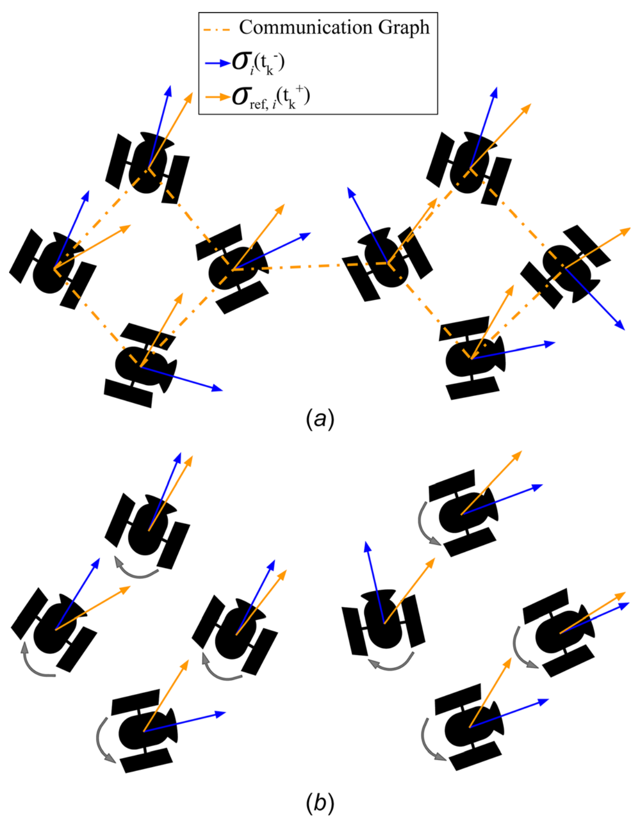

Fig. 2 Distributed attitude consensus for a network of satellites with intermittent communication: (*a*) the set of reference attitudes updates distributively at time t_k over a barbell graph and (*b*) each satellite's attitude evolves toward its reference attitude between discrete updates

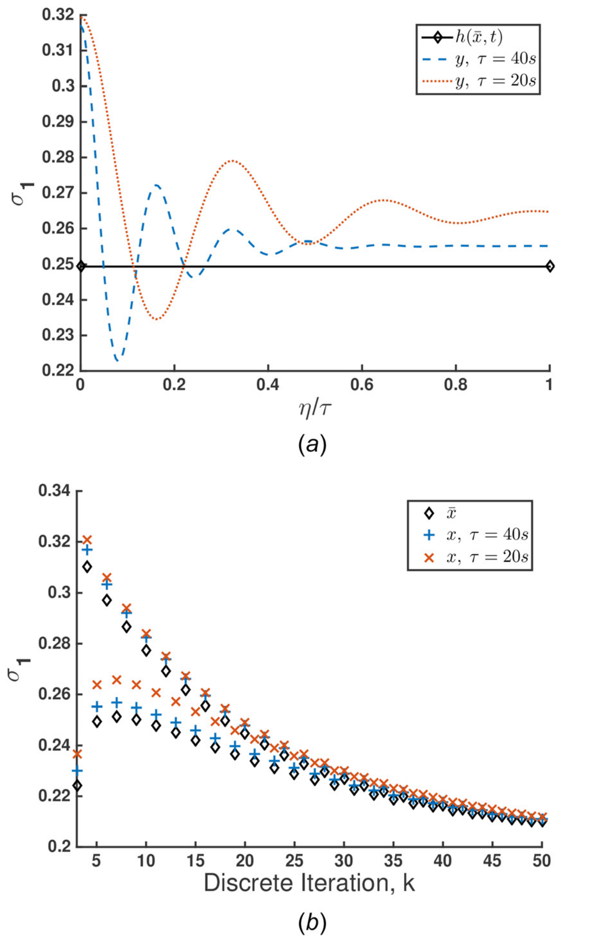

Fig. 3 Comparison of reduced-order models and true evolution of one satellite's state as τ increases: (*a*) continuous evolution of the first MRP within a normalized interval and (*b*) discrete evolution of the first MRP reference commands

The results of Theorem 2 yield stability bounds for the hybrid-time system. The matrices in the LMIs are slack matrices that allow a simpler comparison system to be analyzed in the proof instead of the original dynamics. This comparison system is based on Lyapunov functions for the individual continuous-time and discrete-time dynamics. Of course, the acquired bounds may be conservative since they are based on particular Lyapunov functions. However, the outlined approach can be adapted if more information, such as more appropriate Lyapunov functions, are known for the reduced-order models.

5 Application

This section presents an application of the paper's main results to the analysis of a distributed network of satellites. Using the approach detailed in Sec. 3, reduced-order models are derived that decouple the satellites' discrete-time leader–follower consensus protocol from their individual continuous-time attitude tracking dynamics when the update rate is slow. A bound on the update rate is then found that guarantees stability for the coupled hybrid-time system using the method of Sec. 4, and it is shown how update rates faster than this bound can cause instability.

Consider a group of satellites communicating intermittently over a network to distributively reach consensus on their attitude as illustrated in Fig. 2, where the *i*th satellite's attitude is represented by the modified Rodrigues parameters (MRPs) $\sigma_i \in \mathbb{R}^3$. Further assume that one satellite, the leader, has knowledge of the desired attitude σ_{desired} for the group. A distributed protocol for the *i*th satellite's reference attitude is then provided by the discrete leader–follower dynamics

$$\sigma_{\text{ref},i}(t_k^+) = \begin{cases} \sigma_{\text{desired}}, & i = 1 \\ \sigma_i(t_k^-) + \Delta \sum_{j \in \mathcal{N}(i)} (\sigma_i(t_k^-) - \sigma_j(t_k^-)), & i \neq 1 \end{cases}$$

where Δ is a (fixed) step size, and $j \in \mathcal{N}(i)$ if there is an edge from agent *j* to agent *i* in the satellites' undirected communication graph, $\mathcal{G}$ [20]. Under appropriate closed-loop control [21, Chap. 8], the *i*th satellite's attitude error kinematics are described by

$$\frac{d}{dt}\begin{bmatrix} \hat{\sigma}_i \\ \dot{\sigma}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I_3 \\ -c_{i,1} I_3 & -c_{i,2} I_3 \end{bmatrix} \begin{bmatrix} \hat{\sigma}_i \\ \dot{\sigma}_i \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \tilde{g}_i(\hat{\sigma}_i, \dot{\sigma}_i, t) \end{bmatrix}$$

where $\hat{\sigma}_i = \sigma_i - \sigma_{\text{ref},i}$, and where $\tilde{g}_i$ accounts for the nonlinear effects of imperfect actuation in the control. This is a tracking controller that stabilizes to $h_i(x, t) = \left[\sigma_{\text{ref},i}^{\text{T}},\ \mathbf{0}_{1\times 3}\right]^{\text{T}}$ when $\sigma_{\text{ref},i}$ is fixed.

The discrete evolution of the satellites' reference attitudes, $x = \left[\sigma_{\text{ref},1}^{\text{T}} \cdots \sigma_{\text{ref},n}^{\text{T}}\right]^{\text{T}}$, and continuous evolution of their true states, $y = \left[\sigma_1^{\text{T}}, \ldots, \sigma_n^{\text{T}}, \dot{\sigma}_1^{\text{T}}, \ldots, \dot{\sigma}_n^{\text{T}}\right]^{\text{T}}$, can be equivalently written as the coupled system

$$\begin{aligned} x(t_k^+) &= [K\ \mathbf{0}_{3n\times 3n}]y(t_k) + \begin{bmatrix} \sigma_{\text{desired}} \\ \mathbf{0}_{3(n-1)\times 1} \end{bmatrix} \\ \dot{y}(t) &= A\left(y(t) - \begin{bmatrix} x(t_k^+) \\ \mathbf{0}_{3n\times 1} \end{bmatrix}\right) + \tilde{g}\left(x(t_k^+), y(t), t\right) \end{aligned} \tag{9}$$

where

$$K \triangleq \begin{bmatrix} \mathbf{0}_{1\times n} \\ [I_n - \Delta L(\mathcal{G})]_{2:n,1:n} \end{bmatrix} \otimes I_3$$

$$A \triangleq \begin{bmatrix} \mathbf{0}_{3n\times 3n} & I_{3n} \\ -I_n c_1 \otimes I_3 & -I_n c_2 \otimes I_3 \end{bmatrix}, \text{ and } \tilde{g}(x, y, t) \triangleq \begin{bmatrix} \mathbf{0} \\ \tilde{g}_1(\hat{\sigma}_1, \dot{\sigma}_1, t) \\ \vdots \\ \tilde{g}_n(\hat{\sigma}_n, \dot{\sigma}_n, t) \end{bmatrix}$$

Table 1 Simulation parameters

Parameter	Value
c_1	$0.4 \cdot \mathbf{1}$
c_2	$\mathbf{1}$
Δ	0.4
$\mathcal{G}$	Barbell graph on 8 nodes

Table 2 Values of constants that satisfy Theorem 2

Parameter	Value
P_1	I
P_2	Solution to $P_2 A + A P_2 + I = 0$
d_{11}	0.9750
d_{12}	28.5469
d_{21}	16.1391
d_{22}	10.0091
B	310.9920
γ_1	43.5036
γ_2	3.7445
κ_1	4.0752×10^{-9}
κ_2	0.3016
τ^*	33 s

Here, $\otimes$ represents the Kronecker product, $c_j = [c_{j,1} \cdots c_{j,n}]^{\mathrm{T}}$, $[I_n - L(\mathcal{G})]_{2:n,1:n}$ are the last n–1 rows of the matrix difference, and $L(\mathcal{G})$ is the satellite communication network's graph Laplacian $L(\mathcal{G})$. The dynamics (9) are of the form Eq. (1) and they satisfy the conditions of Theorem 1. Therefore, as shown in Fig. 3, the corresponding reduced-order models

$$\bar{x}\left(t_k^+\right) = [K\,\mathbf{0}_{3n\times 3n}]\bar{x}\left(t_k^-\right) + \begin{bmatrix} \sigma_{\text{desired}} \\ \mathbf{0}_{3(n-1)\times 1} \end{bmatrix}$$

and

$$\dot{\hat{y}}_k(\eta) = A\hat{y}_k(\eta) + \tilde{g}\left(\bar{x}\left(t_k^+\right), \hat{y}_k(\eta) + \bar{x}\left(t_k^+\right), \eta + t_k\right)$$

given by Eqs. (2) and (3), respectively, become good approximations of the true dynamics as the minimum time τ between discrete updates increases. Here, the reduced-order decision dynamics for $\bar{x}$ describe discrete-time leader–follower dynamics whose behavior is linked to the communication graph's topology.

To find a bound on τ that guarantees stability, first rewrite the dynamics (9) in terms of the error states $\tilde{x} = x - \mathbf{1}\otimes\sigma_{\text{desired}}$ and $\tilde{y} = y - \left[(\tilde{x} + \mathbf{1}\otimes\sigma_{\text{desired}})^{\mathrm{T}}, \mathbf{0}\right]^{\mathrm{T}}$. Using $L(\mathcal{G})\mathbf{1} = \mathbf{0}$ (see Ref. [20], for example), the error dynamics are then

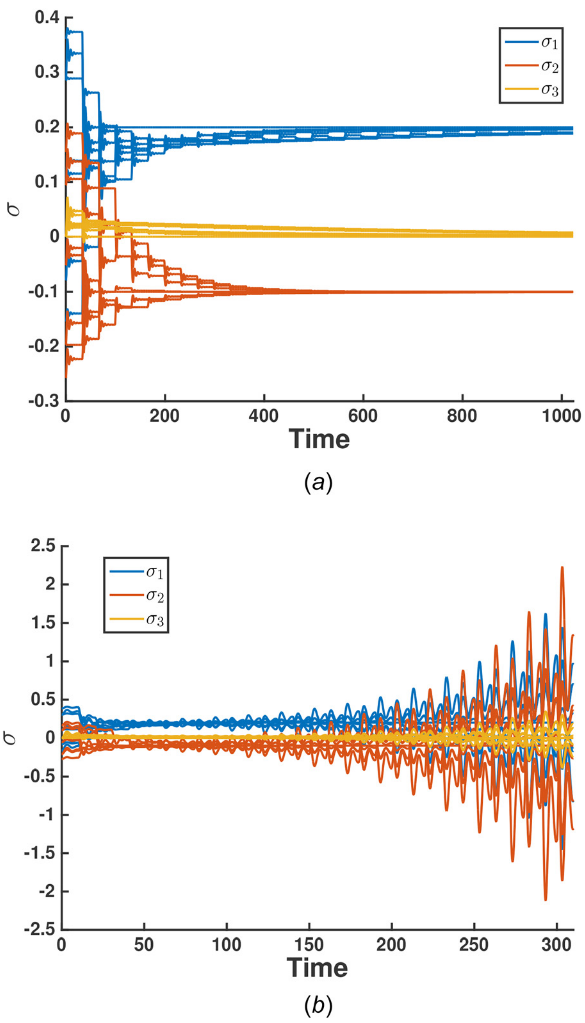

Fig. 4 Evolution of all 8 satellites' attitudes, showing instability for small τ: (*a*) $\tau = 33$ s $= \tau^\star$ and (*b*) $\tau = 10$ s $< \tau^\star$

$$\begin{aligned}
\tilde{x}\left(t_k^+\right) &= K\tilde{x}\left(t_k^-\right) + [K\,\mathbf{0}]\tilde{y}\left(t_k^-\right) \\
\tilde{y}\left(t_k^+\right) &= \tilde{y}\left(t_k^-\right) + \begin{bmatrix} (I-K)\tilde{x}\left(t_k^-\right) - [K\,\mathbf{0}]\tilde{y}\left(t_k^-\right) \\ \mathbf{0} \end{bmatrix} \\
\dot{\tilde{y}}(t) &= A\tilde{y}(t) + \tilde{g}\left(\tilde{x}\left(t_k^+\right) + \mathbf{1}\otimes\sigma_{\text{desired}}\tilde{y}(t)\right. \\
&\quad \left. + \tilde{g}\left(M\tilde{y}(t) + \begin{bmatrix} \tilde{x}\left(t_k^+\right) + \mathbf{1}\otimes\sigma_{\text{desired}} \\ \mathbf{0} \end{bmatrix}, t\right)\right)
\end{aligned} \tag{10}$$

which is of the form Eq. (7). Now, for the satellite and network parameters defined in Table 1, Assumptions 1 and 3 of Theorem 2 are satisfied with

$$J_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & K^{\mathrm{T}}K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \; E_1 = \mathbf{0}$$

and

$$J_2 = \begin{bmatrix} (I_{3n}-K)^{\mathrm{T}}(I_{3n}-K) & -(I_{3n}-K)^{\mathrm{T}}K & \mathbf{0} \\ -K^{\mathrm{T}}(I_{3n}-K) & K^{\mathrm{T}}K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

while Assumption 2 holds by assuming that $R \le (0.01)^2 \otimes I$. The LMIs and spectral radius inequality in Theorem 2 are then satisfied using the constants defined in Table 2, providing the bound $\tau^\star = 33$ s. As expected, Fig. 4(*a*) shows that the hybrid-time system is stable when $\tau = 33$ s. However, the bound given by any particular combination of parameters that satisfy Theorem 2 may be conservative. For these particular initial conditions, Fig. 4(*b*) demonstrates that the satellites exhibit unstable oscillations in their attitudes when $\tau = 10$ s. In this particular case, the instability is due to a resonance phenomenon where the distributed decisions on new reference attitudes are made when satellites have overshot their previous reference attitudes. For large enough τ, however, the satellites' attitudes settle closer to their individual references, which allows the discrete leader–follower consensus protocol to evolve in a stable fashion as designed. This example illustrates the potential danger of too-frequent communication updates even for systems where the discrete network dynamics and continuous agent dynamics are stable when isolated.

6 Conclusion

This paper examined the interactions between continuous-time and discrete-time dynamics often found in networked systems. First, a separation principle between the continuous-time agent dynamics and the discrete-time network dynamics was proven as the period between updates of the discrete system grows. Different asymptotic error bounds for the resulting reduced-order models were found based on the stability properties of the isolated continuous-time agent dynamics. Intuitively, these bounds grow tighter for a slower communication rate because the continuous-time system can evolve closer to its state-dependent equilibrium, yielding a more accurate trajectory in the reduced-order models. In the context of networked dynamical systems, this separation principle rigorously justifies the isolated design of discrete-time network dynamics and continuous-time agent dynamics. Next, a method was described for establishing quantitative upper bounds on the update rate below which the coupled hybrid-time system is guaranteed to be stable. The approach uses separate Lyapunov functions for the continuous-time agent dynamics and the discrete-time network dynamics to form a simpler comparison system that establishes stability of the coupled hybrid-time system when the bound is satisfied. When updates occur faster than this bound, however, the system can become unstable even if the decoupled network and agent dynamics are individually stable. This is because the agents no longer have adequate time to approach their reference trajectories before the next update, and may in fact initially diverge from their reference trajectories due to non-minimum phase behavior. An application where satellites intermittently communicate over a network to establish consensus on their attitudes illustrated these results.

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Nomenclature

Times and Rates

t_k = time at the kth update of the discrete-time system, s
t_k^+ = time immediately following the kth update of the discrete-time system, s
t_k^- = time immediately preceding the kth update of the discrete-time system, s
t_0 = initial time, s
μ = fastest update rate of the discrete-time system, $1/\tau$, Hz
$\mu^\star$ = sufficient upper bound on the update rate to guarantee stability of the coupled system, $1/\tau^\star$, Hz
τ = lower-bound on time between updates of the discrete-time system, s
$\tau^\star$ = sufficient lower-bound on τ to guarantee stability of the coupled system, s

Full Model Symbols

D_x = domain of the discrete-time state vector x
D_y = domain of the continuous-time state vector y
f = vector field of the discrete-time dynamics, mapping $D_x \times D_y \times \mathbb{N} \times \mathbb{R}_+ \to D_x$
g = vector field of the continuous-time dynamics, mapping $D_x \times D_y \times \mathbb{R}_+ \times \mathbb{R}_+ \to D_y$
n_x = dimension of the discrete-time state vector x
n_y = dimension of the continuous-time state vector y
x = discrete-time state vector of the dynamics
$x(t_k^+)$ = value of the discrete-time state vector x at time t_k^+
$x(t_k^-)$ = value of the discrete-time state vector x at time t_k^-
$x_0(\mu)$ = initial conditions of the discrete-time state vector x, a function of the update rate μ
y = continuous-time state vector of the dynamics
$\tilde{y}(t)$ = error between the true continuous-time state vector $y(t)$ and the state-dependent equilibrium trajectory $h(x(t_k^+), t)$, defined for all $t \geq t_0$
$y(t_k)$ = the value of the true continuous-state vector y at the discrete-update at time t_k. Note that $y(t_k) = y(t_k^+) = y(t_k^-)$
$\tilde{y}(t_k^+)$ = value of the error between the true continuous-time state vector y and the state-dependent equilibrium trajectory $h(x(t_k^+), t)$ at time t_k^+
$\tilde{y}(t_k^-)$ = value of the error between the true continuous-time state vector y and the state-dependent equilibrium trajectory $h(x(t_k^-), t)$ at time t_k^-
$y_0(\mu)$ = initial conditions of the continuous-time state vector y, a function of the update rate μ
z = collection of the state vectors x and $\tilde{y}$

Reduced Model Symbols

$h(p, t)$ = equilibrium trajectory of the isolated continuous-time dynamics, mapping $(p, t) \in D_x \times \mathbb{R}_+ \to D_y$
$\mathcal{I}_k$ = time interval between t_k (inclusive) and t_{k+1} (exclusive)
$\bar{x}$ = state vector of the reduced-order discrete-time decision system
$\bar{x}(t_k^+)$ = value of the reduced-order discrete-time decision system state vector $\bar{x}$ at time t_k^+
$\bar{x}(t_k^-)$ = value of the reduced-order decision system state vector $\bar{x}$ at time t_k^-
$\hat{y}_k$ = state vector of the reduced-order continuous-time interval correction system defined on the interval $\mathcal{I}_k$, a function of the elapsed time η within the interval
$\hat{y}_k(0)$ = value of the state vector $\hat{y}_k$ of the kth interval correction system at $\eta = 0$
η = elapsed time within the time interval $\mathcal{I}_k$, $t - t_k$

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