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Abstract: In a shallow water environment, wavenumbers can be estimated by computing time and spatial Fourier transforms of horizontal array measurements. The frequency-wavenumber representation allows wide band estimation but a sufficient number of hydrophones are required for accurate wavenumber resolution. This paper presents the application of an autoregressive (AR) model to compute the high resolution wavenumber spectrum. The smallest number of required sensors for the AR model is found using a stabilization diagram. The method is validated on simulated and experimental data. The wavenumbers are accurately estimated over a wide frequency band using fewer sensors than are needed for the spatial Fourier Transform.

1. Introduction
At low frequency, the shallow water zone is equivalent to a dispersive waveguide. The propagative horizontal wavenumbers, $k_l$, depend on the water column properties and on the seabed composition. They are commonly used as input data of inversion algorithms to retrieve the seafloor characteristics.

For a Horizontal Line Array (HLA), wavenumber estimation is equivalent to a spectrum analysis. Indeed, $k_l$ values can be obtained by performing a two-dimensional Fourier transform (2D-FT) on the array measurements: The 2D-FT is computed along time and range. For dispersive and multimodal media, the frequency wavenumber ($f - k$) diagram allows a wavenumber identification on a wide frequency band.1 It has been applied in underwater acoustics domain by Nicolas et al.2 Note that because of FT properties, the resolution in the $k$ dimension of the $f - k$ diagram depends on the HLA hydrophone number.

High resolution methods were developed to perform spectral analysis without the FT limitations. The subspace decomposition methods ESPRIT and MUSIC (Ref. 3) and parametric models, such as the Prony algorithm4 and the autoregressive (AR) models,5 have been proposed for modal identification in shallow water simulations and measurements in a narrow band context. In Ref. 6, the Prony method is combined to a holographic processing for a vertical array; several acquisitions of the noise from a moving ship are realized to represent the wavenumbers on histograms. The Prony algorithm was later applied to ultrasound measurements in a tank7 with HLA measurements using several inter-sensor spaces for the computation of high resolution $f - k$ diagrams.

Nonetheless those high resolution methods are sensitive to noise; i.e., for AR models, the order and number of elements in the HLA have to be sufficient to provide suitable results. This article presents the application of an AR model in order to reduce the number of HLA sensors without downgrading the wavenumber resolution on $f - k$
diagrams. The main idea is to compute a simple simulation with approximated values of the environment parameters to predict the measurement setup. The document is organized as follows. In Sec. 2, the calculation of the \( f - k \) diagram by FT is presented for a Pekeris waveguide. Then, the AR modeling for spectral estimation is introduced in Sec. 3. The AR coefficients are computed using the modified covariance algorithm (MCA) as used in Ref. 5. The point is to identify the number of required sensors to provide an accurate wavenumber separation. We propose an \textit{a posteriori} analysis of the stabilization diagram to determine the minimum number of sensors to achieve an accurate modal separation on the \( f - k \) diagram. Finally, the method is applied on simulated and experimental data from North Sea measurements.

2. Frequency-wavenumber representation: \( f - k \) diagram

In shallow water, considering a wideband source \( S \) located at depth \( z_s \) and an array composed of \( M \) hydrophones spaced from \( \Delta_r \), the pressure received at the frequency \( f \) on a hydrophone at \( [r, z] \) is expressed as a modal summation,

\[
P(f, r) = QS(f) \sum_{l=1}^{L} \Psi_I(z_s,f) \Psi_I(z_r,f) \frac{e^{-i k_f^{{\text{true}}}(f)} r}{\sqrt{k_f^{{\text{true}}}(f) r}},
\]

where \( Q \) is a constant, \( \Psi_I \) and \( k_I \) are the modal depth function and the horizontal wavenumber for the \( l \)-th mode respectively, and \( L \) is the number of propagating modes.

Before going to the frequency-wavenumber \( f - k \) domain, the received signal can be time-shifted so that the direct wave impinges on all sensors at the same time.\(^2\) Considering that the direct wave travels as a plane wave with speed \( c_w \), the corrected pressure field becomes \( P^{\text{corr}}(f, r) = P(f, r) e^{2 \pi i f / c_w} \). The \( k_I \) observed after correction are linked to the true waveguide wavenumber \( k_f^{{\text{true}}} \) through \( k_f^{{\text{true}}} = k_I - f / c_w \). This correction allows a better readability of the wavenumbers in the \( f - k \) domain since the \( k_I \) values tend to 0 when the frequency increases. Note that this operation can easily be performed on the received signal in the time-distance domain and is a classical seismic processing.\(^1\)

A frequency-wavenumber representation of the received signal, i.e., the \( f - k \) diagram, can then be calculated by taking the modulus of the spatial FT of Eq. (1),\(^1\)

\[
P^{\text{FT}}(f, k) = \left| P^{\text{corr}}(f, r) e^{2 \pi i k r} dr \right|.
\]

As an example, Fig. 1(a) presents a simulated \( f - k \) diagram for a Pekeris waveguide (the Pekeris model parameters are given in Sec. 3.2). This diagram allows a good representation of the dispersion relationship between wavenumber and frequency over a wide frequency band.

In the \( f - k \) domain, the resolution of the wavenumber spectrum is given by classical FT properties \( \Delta_k^{\text{FT}} = 1/(M \Delta_r) \), where \( M \) is the number of hydrophones in the HLA. For a specified inter-sensor space, a large number of hydrophones are thus required to achieve a good wavenumber separation. However, high-resolution methods have been developed to overcome this FT limit. In Sec. 3, we propose the application of an AR model for the \( f - k \) diagram computation.

3. AR model of the shallow water waveguide

Modeling the array signal as an AR process consists in estimating the \( m \)-th spatial sample as a summation of the \( N_{\text{AR}} \) previous samples and an additive, zero-mean Gaussian noise \( w \).\(^9\) Considering equally spaced sensors and a given frequency \( f_0 \), the pressure received on the \( m \)-th sensor at range \( r = m \Delta_r \) becomes
where \( a_0[n] \) are the AR coefficients and \( N_{\text{AR}} \) is the order of the AR model. The signal at frequency \( f_0 \) is now described by the AR coefficients and the variance \( \sigma_{f_0}^2 \) of the noise \( w_{f_0} \). The AR model could be interpreted as the prediction of the \( m \)th sample value from a linear combination of the \( N_{\text{AR}} \) previous ones.

The computation of Eq. (3) autocorrelation leads to the Yule Walker equations; their resolution provides AR parameters \( a_0[n] \) and \( \sigma_{f_0}^2 \). Several algorithms were introduced for solving this system: Levinson, Burg, etc. The MCA proposed by Marple minimizes in the least square sense the backward and forward prediction errors. It is the most appropriate algorithm for spectral estimation and will be used in this study. It is known that the MCA algorithm could be unstable and necessitates that \( N_{\text{AR}} \leq 2/3M \).

### 3.1 Spectral density of an AR model

The power spectral density (PSD) corresponding to Eq. (3) is given by

\[
\hat{P}[f_0, m] = -\sum_{n=1}^{N_{\text{AR}}} a_0[n] P^{\text{corr}}[f_0, m-n] + w_{f_0}[m],
\]

(3)

Equation (4) is an all pole filter which is adapted for describing modal behavior. It possesses peaks at the wavenumber values \( k_i \), but it is not accurate for the estimation of the mode amplitudes.

The theoretical resolution of the PSD has been calculated by Marple for the case of two sines with the same amplitude,

\[
\Delta_{k_i}^{\text{AR}} = \frac{1.03}{\Delta_{i}\sqrt{N_{\text{AR}}[x(1 + N_{\text{AR}})]^{0.31}}},
\]

(5)
where $z$ is the linear expression of the signal-to-noise ratio (SNR). In general, spectrum resolution is better with the AR model than with FT: $\Delta_k^{\text{AR}} < \Delta_k^{\text{FT}}$, but, for some cases of low SNR and low AR order, the FT presents a better spectrum resolution than the AR model.

Because of Eq. (4), an intuitive approach is to choose the pole number $N_{\text{AR}}$ equal to the mode number $L$, so that $P^{\text{AR}}(f_0, k)$ possesses $N_{\text{AR}} = L$ peaks (in the $k$ dimension) that correspond to the $L$ wavenumbers $k(f_0)$. However, in a noisy context, additional poles are required to reduce the model errors due to noise. Then, poles can be separated into two sets: Some of them are physical poles and correspond to the waveguide modes, while the other poles are mathematical poles reducing the noise influence on the estimation. Several a posteriori criteria have been proposed for finding the optimal model order, but they are generally ineffective with real noisy data. Empirical approaches are often preferred. For narrowband signals, the order can be selected when two close wavenumbers are well resolved for several SNR. For a wideband analysis, as mode number increases with the frequency, both physical and mathematical pole numbers increase.

The separation of an unknown number of wavenumbers $L$ has to be considered. The pole number $N_{\text{AR}}$ (which comprises both physical and mathematical poles) must thus be big enough to describe the corresponding noisy spectrum. The pole number $N_{\text{AR}}$ and the hydrophone number $M$ must also be linked through $N_{\text{AR}} \leq 2M/3$. To avoid unstable behavior of the MCA, $N_{\text{AR}}$ is usually chosen between $M/2$ and $2M/3$. In order to have enough poles to describe the wavenumber spectrum while at the same time minimizing the hydrophone number $M$, we propose to fix the order $N_{\text{AR}}$ as the floor of $2M/3$ such as

$$N_{\text{AR}} = \left\lfloor \frac{2}{3} M \right\rfloor.$$  

Our aim is to minimize the sensor number $M$, and Eq. (6) allows to jointly minimize $M$ and $N_{\text{AR}}$. In the following, we show how to choose the minimal pole number, and thus the minimal number of sensors to ensure a stable MCA.

### 3.2 Stabilization diagram for finding the smallest number of sensors

The stabilization diagram has been introduced to help identify the physical poles from the mathematical poles. The main idea is to plot the wavenumber spectrum for increasing values of $N_{\text{AR}}$. The minimal order $N_{\text{AR}}$ is selected when the spectrum presents a sufficiently good modal separation. In our case, as the order is linked to the number of hydrophones by Eq. (6), the operation is equivalent to finding the minimum number of hydrophones. Thus, the stabilization diagrams will be plotted as a function of hydrophone number $M$.

Considering the multimodal behavior of the waveguide and the wideband analysis of the $f - k$ diagram, the stabilization diagram has to be computed at the highest frequency, which is the worst case frequency. Indeed, the highest mode number and the closest spaced wavenumbers are related to the highest frequency. If the modes are separated at this frequency, the separation for lower frequencies is straightforward.

To illustrate the operation, the stabilization diagram is applied to a simulated Pekeris waveguide with the following parameters: Water sound speed $c_w = 1520$ m/s, density $\rho_w = 1$ kg/m$^3$, depth $D = 130$ m; and seabed sound speed $c_s = 1876$ m/s and density $\rho_s = 3$ kg/m$^3$. An impulsive source is simulated, and the impulse response of the waveguide is computed up to 60 Hz. A zero-mean Gaussian noise is added on each sensor of the array signal with a 10 dB SNR. Sampling frequency is 250 Hz while hydrophones are separated by $D = 25$ m. The stabilization diagram associated with the simulation is represented in Fig. 1(d) for the highest considered frequency $f = 60$ Hz. The red lines are added at the theoretical values of the wavenumbers for the Pekeris waveguide. Five modes are sufficiently separated for $M \approx 80$ hydrophones (corresponding model order $N_{\text{AR}} \approx 53$): Adding more
poles does not change wavenumber values: $-6.3, -4.3, -3, -2$, and $-1\, km^{-1}$. The two first modes correspond to their theoretical values for $M \approx 200$ sensors; in fact, their amplitudes are really low at 60 Hz as shown in Fig. 1(a). Without the a priori knowledge about the mode number and values, $N_{AR} = 53$ and $M = 80$ seems to be a sufficient condition to achieve an adequate modal separation.

3.3 Comparison with classical $f$ - $k$ diagrams

To demonstrate that chosen model order $N$ and sensor numbers $M$ are relevant, $f$ - $k$ diagrams computed using FT and AR methods are compared. Diagrams in Figs. 1(a) and 1(b) are respectively obtained by FT on 240 and 80 sensors. The $f$ - $k$ diagram obtained with an AR model using $N_{AR} = 53$ and $M = 80$ is plotted in Fig. 1(c). Visually, the wavenumber resolution is better than the one obtained by the FT on 240 sensors [Fig. 1(a)]. For this case, the theoretical resolutions are calculated: $\Delta_k^{FT} = 0.17\, km^{-1}$ with 240 sensors, $\Delta_k^{FT} = 0.50\, km^{-1}$ with 80 sensors, and $\Delta_k^{AR} = 0.16\, km^{-1}$. Because of its low resolution, the 80-sensor FT does not allow resolution of nearby modes. As an example, modes 1 and 2 overlap for frequencies higher than 25 Hz and modes 2 and 3 for frequencies higher than 45 Hz [Fig. 1(b)]. Note that the first two modes were not separated at 60 Hz in the stabilization diagram for $M = 80$ [Fig. 1(d)]. However, modes 1 and 2 are suitably resolved with the AR model for lower frequencies [Fig. 1(e)]. The use of the AR model thus allows to reduce greatly the sensor number and to enhance the wavenumber resolution.

As stated in Sec. 1, high resolution methods are relatively sensitive to noise. Moreover, the $\Delta_k^{AR}$ decreases when the noise increases. As a result, if the data are too noisy, the proposed AR methodology becomes less reliable. However, the theoretical $\Delta_k^{AR}$ is smaller than the 80-sensor $\Delta_k^{FT}$ while SNR $> -23$ dB. This theoretical limit seems to be unreachable in practice. As an illustration, Fig. 2 presents $f$ - $k$ diagrams computed using 80-sensor FT and AR methods. For SNR $= 0$ dB and SNR $= 5$ dB, the proposed AR methodology provides a better resolution than the classical FT. However, for SNR $= -5$ dB, the $f$ - $k$ diagram computed using the AR method is unreadable while the classical FT methodology still provides some modal resolution.

![Fig. 2](http://dx.doi.org/10.1121/1.4869821)

Fig. 2. (Color online) Simulated data (Pekeris waveguide) with various SNR: $f$ - $k$ diagrams obtained by FT on 80 sensors with (a) SNR $= -5$ dB, (b) SNR $= 0$ dB, and (c) SNR $= 5$ dB; $f$ - $k$ diagrams obtained by AR model with 53 poles and 80 sensors with (d) SNR $= -5$ dB, (e) SNR $= 0$ dB, and (f) SNR $= 5$ dB.
4. Application on shallow water measurements

The method is applied on experimental data recorded in the North Sea. The HLA is composed of 240 hydrophones spaced 25 m and the sampling frequency is 250 Hz. The array lies on the seabed at 130 m depth. The source is an air gun with a relatively flat spectrum over 5 to 50 Hz. The SNR is estimated to be around 10 dB. The environment has been estimated from the HLA measurement in Ref. 2. It can be approximated by a Pekeris waveguide with the parameters presented in Sec. 3.2.

The classical \( f - k \) diagram obtained by FT are plotted in Figs. 3(a) and 3(b), using, respectively, 240 and 70 sensors of the HLA. A full AR study is performed as described in Sec. 2. The stabilization diagram is presented in Fig. 3(d). The wavenumber values estimated in the 240 sensor FT at 60 Hz are plotted in red lines. The minimum number of hydrophones is estimated to be 70 which leads to an order of 46. Note that when \( M > 150 \), the AR model is over-parameterized and the PSD presents double-peaks around the true wavenumber values.

The high resolution \( f - k \) plot using 70 hydrophones is represented in Fig. 3(c). It enables resolution of nine modes, while only seven are identified using classical FT [Figs. 3(a) and 3(b)]. Moreover, the AR \( f - k \) diagram (computed using 70 sensors) presents a better resolution in the wavenumber dimension than the \( f - k \) diagram obtained using the FT on 240 sensors.

5. Conclusion

In this paper, we show that high-resolution \( f - k \) diagrams can be computed using an AR model and a reduced number of hydrophones. An \textit{a posteriori} study based on a stabilization diagram is proposed to identify the lowest number of sensors that allows accurate wavenumber estimation. For simulated and measured data, the method provides better resolution than classical Fourier analysis and the required hydrophone number is reduced by about a factor of 3. Moreover, the optimal number of sensors is of the same order of magnitude for both simulation and experimental data. As a consequence, the stabilization diagram can be computed for simulated data to estimate the optimal number of sensors before an experiment.

A key point of the methodology is that it is possible to define a frequency for which the (unknown) mode number is the highest and the wavenumber separation is
the smallest. For a Pekeris waveguide, or more generally for a reflection-dominated waveguide, this frequency is usually the highest considered frequency. If one considers a waveguide with mode crossing in the $f - k$ domain, the proposed methodology cannot be applied.

Future improvements could be envisaged by introducing wideband physical a priori information. Tracking algorithms in the frequency dimension would then enhance the performances of the wavenumber estimation.

References and links