JUNE 05 2017

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J. Acoust. Soc. Am. 141, EL507–EL512 (2017)
https://doi.org/10.1121/1.4984043

Semi-analytical modeling of ground/plate interaction for general elastic boundary conditions

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Abstract: This letter introduces ground/plate interaction using the concept of ground cross-modal impedance in the case of general elastic boundary conditions. Navier equations are generally used to account for ground vibration with two propagating waves, the dilatational wave and the shear wave. The plate equation of motion follows the Kirchhoff-Love hypothesis, where shear and rotational inertia are neglected in the plate thickness. The general elastic boundary conditions are expressed analytically through a two-dimensional Fourier series in the plate displacement solution. This study shows that the plate general boundary conditions have a small influence on the plate velocity. However, two categories of boundary conditions could be implemented, especially at low frequency.

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Date Received: November 2, 2016 **Date Accepted:** May 8, 2017

1. Introduction

Generally speaking, in the railway sector, ground/structure interaction often constitutes a problem, especially in the case of tramways and high speed train infrastructures.[1,2] An important requirement that has to be taken into account by railway construction managers is that the construction design must not generate nuisances for residents caused by ground borne noise or by ground borne vibration.[3] In order to respond to this requirement, modeling can be used to perform railway risk assessments and to design solutions to reduce annoyance.

Several models have been developed over the last decade and their quality has improved due to their greater accuracy.[4,5] Recently, an analytical model called SIPROVIB (Ref. 6) was proposed by the authors in order to perform a vibration risk assessment for a new tramway line. This analytical model is based on a plate flexural equation following the Kirchhoff-Love hypothesis. The plate boundary conditions which are usually used are simply supported and guided due to the complexity induced by the addition of other boundary conditions. It was shown in a previous paper[6] that guided boundary conditions are a good approximation for plate vibrations coupled to the ground. This demonstration was performed on the basis of a comparison with measurements carried out on a tramway slab.

In this letter, the authors propose an extension of SIPROVIB for the case of general boundary conditions. These general boundary conditions are based on work performed by Li *et al.*[7] which introduced a solution that perfectly fits the ground/plate interaction resolution performed in SIPROVIB. A comparison between boundary conditions is made to show the influence of such boundary conditions.

2. Problem formulation

The problem under consideration is a plate of dimension (L_x, L_y) with general elastic boundary conditions. The plate is coupled to the ground along its surface.

2.1 Plate equation of motion for general elastic boundary condition

2.1.1 Plate equation of motion

The plate is modeled according to the Kirchhoff-Love hypothesis which is a good approximation in the railway domain where the frequency range of interest is

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commonly [4; 250] Hz. The plate is excited with a point force on one face and the whole of the other face is coupled to the ground. The plate equation of motion is given by the following equation:

$$D\nabla^4 w(x,y) - \rho h w(x,y) = F(x,y) + \sigma_p(x,y), \tag{1}$$

where $F(x,y) = F_e\delta(x-x_e)\delta(y-y_e)$ and $\sigma_p(x,y)$ are the amplitude of the point force F_e located at (x_e, y_e) and the normal stress applied by the ground on the plate, respectively.

It has been shown that the plate equation of motion represents a good approximation of the tramway concrete slab coupled to the ground up to 250 Hz and for a slab thickness of 0.6 m.

2.1.2 Solution for general elastic boundary conditions

The solution of the plate equation of motion (1) is difficult to obtain for general boundary conditions. When considering the analytical solution, most studies consider simply supported or guided boundary conditions due to their modal development using sines and cosines. Although they do not represent most real boundary conditions, these boundary conditions are used because they allow explaining some of the physical phenomena under consideration (acoustic radiation, ground radiation, etc.). However, in some cases, it can be necessary to obtain the real boundary conditions in order to make the model more predictable.

Li *et al.*[7] introduced a mathematical method for solving the plate equation of motion under general elastic boundary conditions. This method is used in this study because it allows general boundary conditions and the resolution fits with the resolution of the ground/plate interaction.

The solution for the plate equation of motion (1) is considered in the following form:

$$w(x,y) = \sum_{nm} a_{nm}\cos(\lambda_n x)\cos(\lambda_m y) + \sum_{l=1}^{4}\left(\xi_l(y)\sum_m c_{lm}\cos(\lambda_n x) + \xi_l(x)\sum_n d_{ln}\cos(\lambda_m y)\right), \tag{2}$$

where a_{nm}, c_{lm}, and d_{ln} correspond to the unknown modal amplitude of the plate mode, $\lambda_n = n\pi/L_x$ and $\lambda_m = m\pi/L_y$.

Li *et al.*[7] showed that additional functions need to be taken into account if the general boundary condition is required. These functions, called $\xi_i(y)$, are given in Ref. 7. In fact, these functions can have different forms. They have been considered because they are sufficiently smooth and respect the following boundary conditions $\xi_1'(y = L \,\text{else}\, L = 0) = \xi_2'(y = L\,\text{else}\,L = 0) = \xi_3'''(y = L\,\text{else}\,L = 0) = \xi_4'''(y = L\,\text{else}\,L = 0) = 0$. For more details on these functions, readers can find information in Li *et al.*[7] These additional functions may present some difficulties especially when it comes to eliminating the sum over the modes in Eq. (2). In order to get around this, a modal decomposition of functions $\xi_i(y)$ on a cosine basis allows reducing the problem. It is also necessary to develop the force on a modal base of the plate. For a point force, the decomposition is given in Ref. 6. The general boundary conditions for a plate are given by Q_x and Q_y which are the shear forces in directions x and y, respectively, and by M_x and M_y which are the moments in directions x and y, respectively.

For the sake of clarity in Sec. 2, the notations used are similar to those used in Li *et al.*[7]

By introducing Eq. (2) into the shear force and moment at the plate boundary, we obtain a relation between a and p such that

$$H \cdot p = Q \cdot a, \tag{3}$$

where $p = \{c_1^1, \ldots, c_N^1, \ldots, c_1^4, \ldots, c_N^4, d_1^1, \ldots, d_M^1, \ldots, d_M^1, \ldots, d_1^4, \ldots, d_M^4\}$ and $a = \{a_{00}, \ldots, a_{N0}, a_{01}, \ldots, a_{NM}\}$.

The system of Eq. (3) provides a relation between the modal amplitude of the main function and the additional functions. This leads to a set of $4(M+N)$ equations to which the system of equations for the plate equation of motion is added.

By replacing function ξ_i by their cosine development, the stress applied by the ground to the plate takes the same form as the case of the guided boundary conditions.[6]

Using Eq. (3), it is possible to give an expression of the mechanical impedance

$$\left[Z_{nmpq}^{\text{plate}}\right] = D(k_{nm}^4 + [B][H]^{-1}[Q]) - \rho h\omega^2(1 + [F][H]^{-1}[Q]). \tag{4}$$

In the general case, the mechanical impedance has intermodal modes while in the simply supported and guided cases it vanishes. Thus, for the guided boundary conditions, terms $[B][H]^{-1}[Q]$ and $[F][H]^{-1}[Q]$ would be null and $[Z_{nmpq}^{\text{plate}}]$ would be a diagonal matrix.[3]

The ground equation of motion is now introduced.

2.2 Ground equation of motion

The Navier equation of motion is a good approximation of ground vibration for small displacements.[8] The ground is assumed to be homogeneous, isotropic, and linear. The unknowns of the Navier equations are the displacement vector $\boldsymbol{u}^T = \{u_x, u_y, u_z\}$ where each term is linked by a differential operator. The Helmholtz decomposition allows uncoupling the terms in the equation of motion. In the system of equations we obtain two waves propagating into the ground: the dilatational wave c_p and the shear wave c_s.

For the ground equation of motion, the ground boundary conditions are required at the top surface. The ground top surface is free except under the plate coupled to the ground. A trivial solution of such an equation associated with the ground boundary conditions gives a relation in the wavenumber domain between the normal displacement and the normal stress

$$\tilde{u}_{zz}(k_x, k_y, 0) = N(k_x, k_y)\sigma_p(k_x, k_y, 0), \tag{5}$$

where function $N(k_x, k_y)$ has an analytical expression for a homogeneous ground.

It is noteworthy that the unknown of the problem is the modal plate amplitude a. The relation between the ground and the plate is given through the continuity of displacement.

2.3 Plate and ground solution

In the displacement condition of continuity, we replace the plate displacement by its expression (2) and the ground displacement in the wavenumber domain. Expression $\tilde{u}_{zz}(k_x, k_y, 0)$ given in Eq. (5) is injected into the relation of displacement continuity between the plate and the ground. Stress $\sigma_p(k_x, k_y)$ is given by a two-dimensional (2D) Spatial Fourier Transform. An orthogonalization over the cosine base gives the following system of equations:

$$\sum_{nm}(Z_{nmpq}^{\text{plate}} a_{nm} - F_{nm})\gamma_{nmpq} = S_{pq}\sum_{nm}\left(a_{nm} + \sum_{l=1}^{4}(\beta_{ln}c_{lm} + d_{ln}\alpha_{lm})\right). \tag{6}$$

Equation (6) can be written in matrix format depending on only a. After replacing p by its relation depending on a, we obtain the following system:

$$\left[Z_{nmpq}^{\text{plate}}\right][\gamma_{nmpq}]\{a_{nm}\} - [S_{pq}]\left[1 + [F][H]^{-1}[Q]\right]\{a_{nm}\} = [\gamma_{nmpq}]\{F_{nm}\}. \tag{7}$$

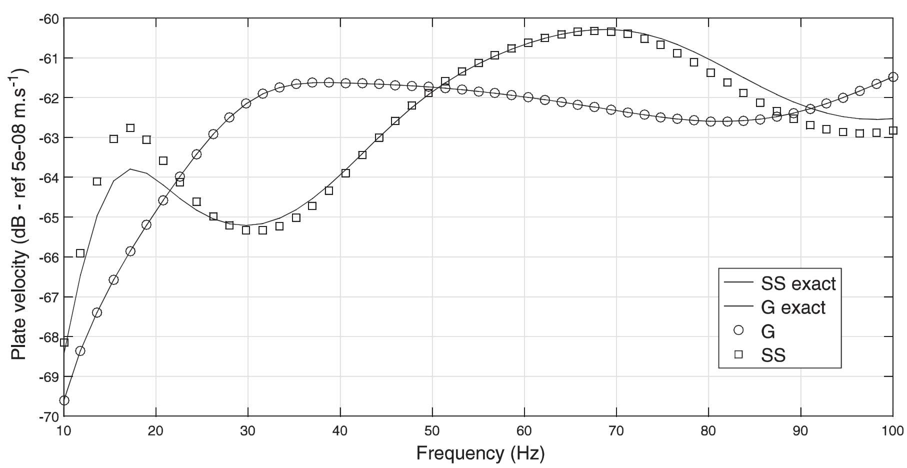

Fig. 1. Comparison of the surface averaged plate velocity between the exact formulation and the approximated formulation for SS and G boundary conditions.

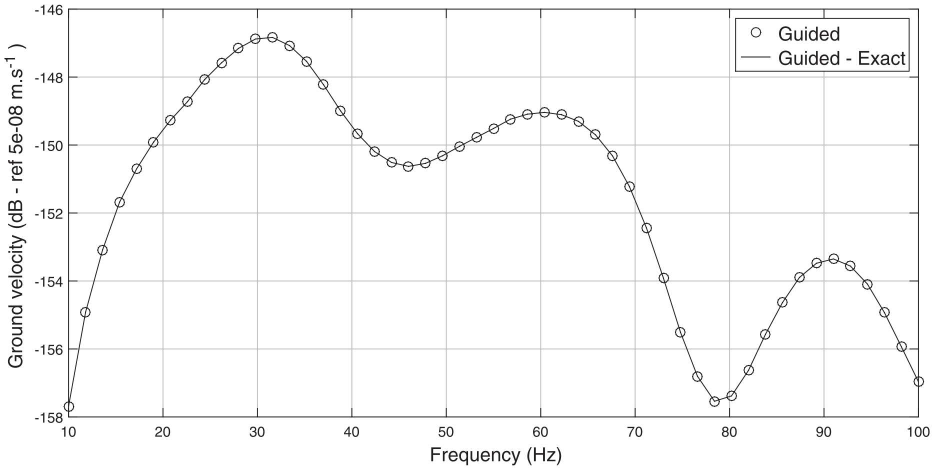

Fig. 2. Comparison of the surface ground velocity at the top surface between the exact formulation and the approximate formulation.

Using an inverse 2D Spatial Fourier Transform of Eq. (5), it is now possible to obtain the ground displacement at the top surface

$$u_{zz}(x, y, 0) = \sum_{nm}(Z^{\text{plate}}_{nmpq} a_{nm} - F_{nm}) T_{nm}(x, y). \quad (8)$$

3. Numerical results

3.1 Model validation

In this section, we show that the above model gives the same results as for the reference formulation which is the Simply Supported (SS) and Guided (G) boundary conditions. The ground under consideration is a soft homogeneous ground with the following mechanical characteristics: $c_s = 160\,\text{m}\,\text{s}^{-1}$, $c_p = 600\,\text{m}\,\text{s}^{-1}$, $\eta_p = 3\%$, $\eta_s = 3\%$, and $\rho = 1200\,\text{kg}\,\text{m}^{-3}$.

The plate under consideration is similar to that in Ref. 6, that is to say, $L_x = 6\,\text{m}$, $L_y = 6\,\text{m}$, $h = 0.2\,\text{m}$, $E_p = 2.5 \times 10^{10}\text{Pa}$, $\nu = 0.3$, $\rho = 2500\,\text{kg}\,\text{m}^{-3}$ and $\eta_p = 5\%$.

Let us first consider the surface averaged plate velocity as defined in Ref. 6. Figure 1 shows the plate velocity for SS (round curve) and G (square curve) boundary conditions using the formulation presented above. In Fig. 1 we also plot the plate velocity using an exact formulation (line curve) for both boundary conditions. It can be seen that there is a good match between the exact formulation and the formulation presented in this letter.

Figure 2 shows a comparison of both formulations for the surface ground displacement at the top surface. Only the G boundary conditions are considered here for the sake of clarity. The surface under consideration for the ground displacement is 5×5 m^2

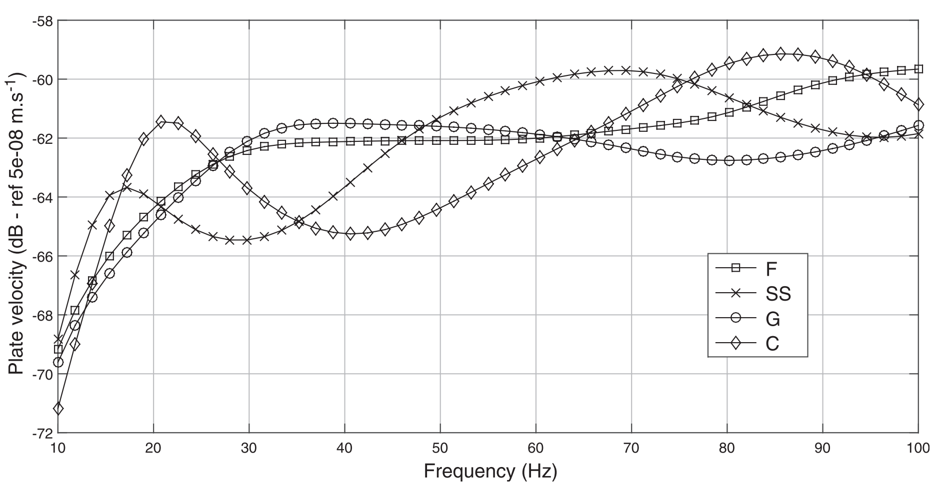

Fig. 3. Comparison of the surface averaged plate velocity for four different plate boundary conditions.

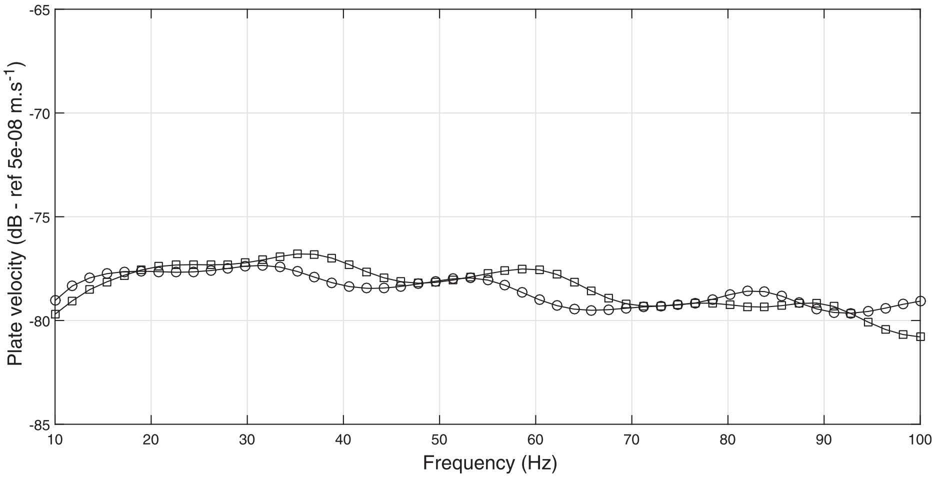

Fig. 4. Comparison of the surface averaged plate velocity for a tramway slab between G and F boundary conditions.

located 2 m from the plate edge. This surface is centered on the plate. A good match can be seen between both formulations of the ground velocity at the top surface. These results validate the formulation for considering general boundary conditions proposed in this letter.

A study of the influence of the boundary conditions is now introduced by considering the plate velocity and the ground velocity at the top surface.

3.2 Influence of plate boundary conditions

The authors have shown that the plate boundary conditions do not have a significant influence on the plate velocity and the ground velocity when it is coupled to the ground. This demonstration was performed by an experimental comparison which shows a good match with the simulation. The following results provide more details on the influence of boundary conditions with the simulation comparison.

The plate under consideration is a $6 \times 6\,\text{m}^2$ square 0.2 m thick. The material characteristics of the plate as well as the ground characteristics are the same as in Sec. 3.1. Figure 3 shows the surface averaged plate velocity in the case of four different boundary conditions, i.e., Free (F), SS, G, and Clamped (C). It can be seen that in the case of F and F boundary conditions, the first modes are very close to each other which can be explained by the body mode taken into account in both cases. However, there is a difference with the SS and C boundary conditions, where the curves are shifted to higher frequencies.

For a plate with a low modal density, as is the case here, the difference can be seen as a function of the boundary conditions, mainly due to the body shape. However, this difference remains small, no more than 4 dB.

We now consider the case of a tramway plate 28 m long, 6.6 m wide, and 0.6 m thick. The plate is excited in the middle and Fig. 4 shows the difference between the F and F boundary conditions. The F boundary conditions were used in Ref. 6 for the experimental validation of the model, so it is interesting to look at the difference with the F boundary conditions. As the modal density is high, the difference between the boundary conditions is small. Indeed the free boundary conditions represent the most physical boundary conditions when modeling a tramway slab. Thus this comparison shows that the G boundary conditions are good approximations when modeling a tramway slab, as assumed by the authors.

This section showed that boundary conditions can be divided into two parts for a low modal density and that the plate boundary conditions for a tramway slab are of the second order.

4. Conclusion

This purpose of this letter was to introduce general plate boundary conditions for a plate coupled to the ground via an analytical model. The model was based on a recent analytical model which can account for ground/plate interaction and on another model which considered general elastic plate boundary conditions. The model was validated with an exact solution. It was shown that the plate boundary conditions were of the second order when coupled to the ground.

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