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# Computing Gap Free Pareto Front Approximations with Stochastic Search Algorithms

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## Abstract

Recently, a convergence proof of stochastic search algorithms toward finite size Pareto set approximations of continuous multi-objective optimization problems has been given. The focus was on obtaining a finite approximation that captures the entire solution set in some suitable sense, which was defined by the concept of  $\epsilon$ -dominance. Though bounds on the quality of the limit approximation—which are entirely determined by the archiving strategy and the value of  $\epsilon$ —have been obtained, the strategies do not guarantee to obtain a gap free approximation of the Pareto front. That is, such approximations  $A$  can reveal gaps in the sense that points  $f$  in the Pareto front can exist such that the distance of  $f$  to any image point  $F(a)$ ,  $a \in A$ , is “large.” Since such gap free approximations are desirable in certain applications, and the related archiving strategies can be advantageous when memetic strategies are included in the search process, we are aiming in this work for such methods. We present two novel strategies that accomplish this task in the probabilistic sense and under mild assumptions on the stochastic search algorithm. In addition to the convergence proofs, we give some numerical results to visualize the behavior of the different archiving strategies. Finally, we demonstrate the potential for a possible hybridization of a given stochastic search algorithm with a particular local search strategy—multi-objective continuation methods—by showing that the concept of  $\epsilon$ -dominance can be integrated into this approach in a suitable way.

## 1 Introduction

In a variety of engineering and economic problems, several objectives have to be optimized concurrently. One widely accepted class of algorithms for the approximation

of the solution set (Pareto set) of such multi-objective optimization problems (MOPs) is given by evolutionary strategies. A typical evolutionary multi-objective (EMO) algorithm consists, roughly speaking, of a process to generate new candidate solutions (the generator) and a strategy to store and update a suitable subset of the obtained data according to the given task (the archiver). Under mild assumptions about the generator, the limit approximation set is determined almost entirely by the archiving strategy. By limit behavior we mean the behavior of the sequence of archives  $A_l, l \in \mathbb{N}$ , which is generated by the stochastic search algorithm in the course of the computation for iteration step  $l \rightarrow \infty$ . The task of this work is to develop archivers that aim for finite size and gap free approximations of the solution set, which we implement in the following.

One interesting application of multi-objective optimization and its related tools is the online-optimization of mechatronical systems. One approach to this problem is as follows: first, all relevant (conflicting) objectives of the underlying system are collected and used to formulate a multi-objective optimization problem. This problem is then solved numerically by approximating the Pareto set (denote the approximation by  $\mathcal{P}$ ) offline. This set serves further on as the basis for the online control by providing a repository of reference operating points: the optimal point (or optimal compromise)  $p(\lambda) \in \mathcal{P}$  is determined online, that is, while running the system, according to the current situation or demand  $\lambda$  of the system, and this optimal point is used as the actual *operating point*. Since  $\lambda = \lambda(t)$  varies with the time, this optimal point has to be updated over and over again, according to the sensitivity of the system. (See Pottharst et al., 2004; Witting et al., 2008 for an operating point assignment strategy of a linear drive, and Hestermeyer and Oberschelp, 2003 for an online-adjustment of an active suspension system.) Crucial for the stability of the system is that the switch from one point or system setting  $p(\lambda_1)$  to the next one  $p(\lambda_2)$  cannot be done arbitrarily, but has to be carried out as smoothly as possible. That is, large and abrupt qualitative changes—(among others) in terms of the changes in the influential objective values—have to be avoided. Thus, it is required in these applications, and certainly in others as well, to obtain a gap free (and preferably uniformly spread) Pareto front approximation.<sup>1</sup>

There certainly exist on the other hand scenarios where a smooth changeover of the parameter values over time is of particular interest. The algorithms we consider here are, however, not totally suited—but also not designed—for such cases, since these algorithms are entirely based on the dominance and  $\epsilon$ -dominance relations, which are defined in objective space: consider a point  $f \in F(\mathcal{P})$  of the image of the Pareto set (the Pareto front), where  $F: \mathbb{R}^n \rightarrow \mathbb{R}^k$  is the function of given objectives and consider that  $f$  has several preimages  $x_i \in \mathcal{P}, i = 1, \dots, s$ . Then the archives of the subsequent archiving strategies will probably contain (and retain) one approximate solution of one preimage  $x_i, i \in \{1, \dots, s\}$ , after sufficiently many iterations, but (i) the index  $i$  depends on the order of the incoming solutions and is thus not controllable, and (ii) further preimages  $x_j \in \{1, \dots, s\} \setminus \{i\}$  (or points nearby) will not be accepted further on. For this we refer to archiving strategies that aim for the approximation of the entire Pareto set or a superset of it (e.g., Rudolph et al., 2007; Schütze et al., 2007; Schütze et al., 2008b).

Here we extend the work of Schütze et al. (2008c) and present new archiving strategies for the storage of the essential solutions found by a stochastic search algorithm. The strategy used in Schütze et al. (2008c) is entirely based on the concept of  $\epsilon$ -dominance,

<sup>1</sup>In case the Pareto front falls into different connected components, further techniques (e.g., interpolation strategies among neighboring system settings) in addition have to be considered, but such cases are not part of this work.

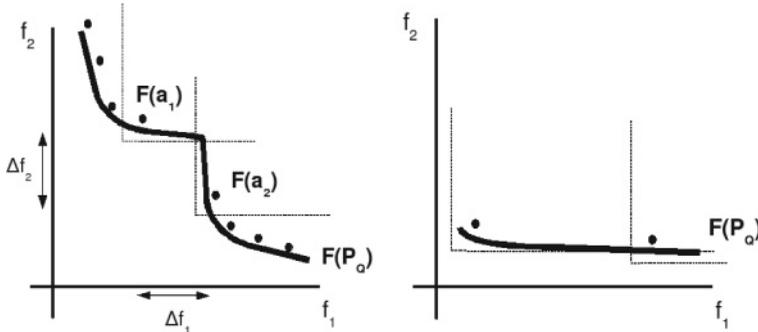


Figure 1: Examples of two  $\epsilon$ -approximate Pareto sets. Dents or flat regions in the Pareto front can lead to (possibly unwanted) gaps in the approximation. In the left figure there is a gap in the approximation between  $F(a_1)$  and  $F(a_2)$  both in the  $f_1$  direction and in the  $f_2$  direction.

which does not consider the distances between solutions in the archive. This can lead to gaps in the approximation set, in particular when some portion of the front is flat or contains a dent (see Figure 1). Though we agree that these phenomena do not occur too often in practice, it is desirable, from a theoretical viewpoint, to have a search algorithm—including a suitable archiving strategy—that can exclude these unwanted gaps in the approximation.

Another important aspect is that the sole usage of the concept of  $\epsilon$ -dominance in the archiving strategy can cause inefficiencies for the resulting search algorithm, in particular when hybridized with a local search procedure. For instance, when using multi-objective continuation methods (see Schütze, 2004 for a combination of this technique with evolutionary strategies), where the underlying idea is to move *along* the efficient set, an unsuited archiving strategy as the one proposed in Schütze et al. (2008c) could lead to difficulties, although these methods are (in principle) very effective locally. For this, consider a Pareto front such as the one displayed in Figure 1 (left) and assume that the archive is given by  $A = \{a_1\}$ . If the continuation method is started with  $\{a_1\}$  and merely the concept of  $\epsilon$ -dominance is used for the archiving strategy, this could lead to the fact that no points  $p$  on the front with  $f_1(p) > f_1(a_1)$  are kept by the archiver. The reason is that there is a relatively large portion of the front near  $F(a_1)$ , where all points are  $\epsilon$ -dominated by  $a_1$ —a barrier which is hard or impossible to overcome by this (or any other) local search strategy. Figure 1 (right) shows a situation that is more extreme.

In this work we propose two different archiving strategies and prove convergence with probability one to gap free (and thus tight) Pareto front approximations. The limit set of the first strategy is a tight  $\epsilon$ -approximate Pareto set which provides a guaranteed uniformity level, while the limit set of the second strategy forms a tight  $\epsilon$ -Pareto set, which, however, lacks uniformity. A previous study of the current work can be found in Schütze et al. (2007). While in Schütze et al. (2007) mainly the limit behavior of the archivers was studied, this work offers in addition more discussion and results that are intended for a better understanding of the effect of the novel strategies.

The remainder of this article is organized as follows: Section 2 states the background required for the understanding of the sequel. In Section 3, we propose the sets of interest, and in Section 4 we give two algorithms that aim for an approximation of the sets.

Section 5 deals with the integration of the archivers into iterative search methods such as multi-objective evolutionary algorithms (MOEAs). In Section 6, we present some numerical results, and finally, we present our conclusions in Section 7.

## 2 Background and Related Work

In the following we consider continuous unconstrained multi-objective optimization problems

$$\min_{x \in \mathbb{R}^n} \{F(x)\}, \tag{MOP}$$

where the function  $F$  is defined as the vector of the objective functions

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^k, \quad F(x) = (f_1(x), \dots, f_k(x)),$$

and where each  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous. Later, we will restrict the search to a compact set  $Q \subset \mathbb{R}^n$ , which the reader may think of as an  $n$ -dimensional box

$$Q = B_{l,u} := \{x \in \mathbb{R}^n : l_i \leq x_i \leq u_i, i = 1, \dots, n\}, \tag{2.1}$$

where  $l, u \in \mathbb{R}^n$  with  $l_i \leq u_i, i = 1, \dots, n$ . In the next definition, we state the classical concept of optimality for MOPs.

**DEFINITION 1:** (a) Let  $v, w \in \mathbb{R}^k$ . Then the vector  $v$  is less than  $w$  ( $v <_p w$ ), if  $v_i < w_i$  for all  $i \in \{1, \dots, k\}$ . The relation  $\leq_p$  is defined analogously.

(b) A vector  $y \in \mathbb{R}^n$  is dominated by a vector  $x \in \mathbb{R}^n$  (in short:  $x < y$ ) with respect to (MOP) if  $F(x) \leq_p F(y)$  and  $F(x) \neq F(y)$  (i.e., there exists a  $j \in \{1, \dots, k\}$  such that  $f_j(x) < f_j(y)$ ).

(c) A point  $x \in \mathbb{R}^n$  is called Pareto optimal or a Pareto point if there is no  $y \in \mathbb{R}^n$  that dominates  $x$ .

Denote by  $P_Q$  the set of Pareto points (or Pareto set) of  $F|_Q$ , where  $Q \subset \mathbb{R}^n$  is the domain. The image  $F(P_Q)$  of the Pareto set is called the Pareto front. Both sets consist typically—that is, under mild regularity assumptions on the objectives—not of finitely many points as for scalar optimization problems, but form  $(k - 1)$ -dimensional objects. In order to guarantee convergence toward  $P_Q, F(P_Q)$ , or other related objects, one often has to assume that there are no weak Pareto points outside  $P_Q$  (see, e.g., Ehrgott, 2005; Schütze et al., 2008a for discussions). Such points are defined as follows.

**DEFINITION 2:** A point  $x \in Q$  is called a weak Pareto point if there exists no point  $y \in Q$  such that  $F(y) <_p F(x)$ .

One important question for an archiving strategy is if it is capable of capturing the incoming data in a suitable way. This holds in particular for multi-objective optimization problems due to the dimensionality of the solution set. An ideal archiver in this sense

is certainly one that stores in the current archive  $A$  for every candidate solution  $x$  that has been found by the generation process an element  $a \in A$  such that  $a$  is equal to  $x$  or dominates it. However, since this leads at least for continuous models to a sequence of archives  $A_i$  with  $|A_i| \rightarrow \infty$  for  $i \rightarrow \infty$ , this demand on an archiver is not adequate for practical use. In the following, we will define a weaker concept of dominance, the so-called (absolute)  $\epsilon$ -dominance (Loridan, 1984), as well as two approximation concepts that will be used for our further studies, since they allow for finite size representations of the solution sets.

DEFINITION 3: Let  $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \mathbb{R}_+^k$  and  $x, y \in \mathbb{R}^n$ .  $x$  is said to  $\epsilon$ -dominate  $y$  (in short:  $x \prec_\epsilon y$ ) with respect to (MOP) if

- (i)  $f_i(x) - \epsilon_i \leq f_i(y) \quad \forall i = 1, \dots, k$ , and
- (ii)  $f_j(x) - \epsilon_j < f_j(y) \quad$  for at least one  $j \in \{1, \dots, k\}$ .

Denote by  $d(\cdot, \cdot)$  any distance and by  $\|\cdot\|$  any norm. Further, let  $B_\delta(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\| < \delta\}$  be the open ball with center  $x_0 \in \mathbb{R}^n$  and radius  $\delta \in \mathbb{R}_+$ .

DEFINITION 4 (Laumanns et al., 2002): (a) Let  $\epsilon \in \mathbb{R}_+^k$ . A set  $A_\epsilon \subset \mathbb{R}^n$  is called an  $\epsilon$ -approximate Pareto set of (MOP) if every point  $x \in \mathbb{R}^n$  is  $\epsilon$ -dominated by at least one  $a \in A_\epsilon$ , that is

$$\forall x \in \mathbb{R}^n : \exists a \in A_\epsilon : a \prec_\epsilon x.$$

- (b) A set  $A_\epsilon^* \subset \mathbb{R}^n$  is called an  $\epsilon$ -Pareto set if  $A_\epsilon^*$  is an  $\epsilon$ -approximate Pareto set and if every point  $a \in A_\epsilon^*$  is a Pareto point of (MOP).

DEFINITION 5 (Sayin, 2000): (a) Let  $\Delta > 0$  and let  $D \subset Z$  be a discrete set.  $D$  is called a  $d_\Delta$ -representation of  $Z$  if for any  $z \in Z$ , there exists  $y \in D$  such that  $d(z, y) \leq \Delta$ .

- (b) Let  $Z \subset \mathbb{R}^n$  be any set and let  $D$  be a  $d_\Delta$ -representation of  $Z$ . Then  $D$  is called a  $\delta$ -uniform  $d_\Delta$ -representation if

$$\min_{x, y \in D, x \neq y} d(x, y) \geq \delta$$

where  $\delta$  is called the uniformity level.

Despite the existence of suitable approximation concepts, investigations on the convergence of particular algorithms toward such approximation sets, that is, their ability to obtain a suitable Pareto set approximation in the limit, have remained rare. Several studies (e.g., Hanne, 1999; Rudolph and Agapie, 2000; Schütze et al., 2008a), consider only the convergence to the entire Pareto set, or to a certain subset, without considering the approximation quality. The only work that deals with the computation of the gap free representation of the Pareto front, to the authors' best knowledge, is Fliege (2004). However, since in that work scalarization methods on convex MOPs are investigated, the approach is not applicable for our purpose.

Finally, the issue of stochastic convergence toward finite-size Pareto set approximations was raised in the area of evolutionary multi-objective optimization, mostly under

the assumption of finite search spaces. One option is to use Markov chain results assuming the underlying search processes to be Markovian (Rudolph, 1998). Another option is to define an order homomorphism of the natural dominance relation of approximation sets into a totally ordered set of quality values, thus enforcing a monotonicity of the sequence of solution sets maintained by an algorithm. As shown in Knowles (2002) and Knowles and Corne (2003), this entails convergence to a subset of the Pareto set as a local optimum of the quality indicator, but no approximation guarantee was given. Knowles and Corne (2003) also analyzed the adaptive grid archiving proposed in Knowles and Corne (2000) and proved that after finite time, even though the solution set itself might permanently oscillate, it will always represent an  $\epsilon$ -approximation whose approximation quality depends on the granularity of the adaptive grid and on the number of allowed solutions. The results depend on the additional assumption that the grid boundaries converge after finite time, which is fulfilled in certain special cases.

In Laumanns et al. (2002), two archiving algorithms were proposed that provably maintain a finite-size approximation of all points ever generated during the search process. As an immediate corollary, these archiving strategies were claimed to ensure convergence to a Pareto set approximation of given quality for any iterative search algorithm that fulfills certain mild assumptions about the process to generate new search points. While this claim holds trivially in the case of discrete (or discretized) search spaces, its extension to the continuous case is not straightforward, and was only recently given in Schütze et al. (2008c). A restriction to discretized models, however, can lead to problems when, for example, memetic strategies are used (metaheuristic search algorithms mixed with local search strategies which use step size control themselves).

Next we define some distances between points as well as between different sets.

**DEFINITION 6:** Let  $u, v \in \mathbb{R}^n$  and  $A, B \subset \mathbb{R}^n$ . The maximum norm distance  $d_\infty$ , the semi-distance  $dist(\cdot, \cdot)$ , and the Hausdorff distance  $d_H(\cdot, \cdot)$  are defined as follows:

- (a)  $d_\infty(u, v) := \max_{i=1, \dots, n} |u_i - v_i|$
- (b)  $dist(u, A) := \inf_{v \in A} d_\infty(u, v)$
- (c)  $dist(B, A) := \sup_{u \in B} dist(u, A)$
- (d)  $d_H(A, B) := \max \{dist(A, B), dist(B, A)\}$

Algorithm 1 gives a framework of a generic stochastic multi-objective optimization algorithm that will be considered in this work. Here,  $Q \subset \mathbb{R}^n$  denotes the domain of the MOP,  $P_j$  the candidate set (or population) of the generation process at iteration step  $j$ , and  $A_j$  the corresponding archive. Algorithms 2 and 3 show two archiving strategies that aim for the computation of  $\epsilon$ -approximate Pareto sets and  $\epsilon$ -Pareto sets, respectively. The difference between these two archivers is the strategy to accept candidate solutions coming from the generation process (for details we refer to Schütze et al., 2008c; or to the discussion below on the difference of Algorithms 4 and 5 which is similar). Convergence results can be found in Theorems 1 and 2, which are closely related to the corresponding results in the present work, however, Algorithms 2 and 3 cannot guarantee that the limit archives do not reveal gaps in the Pareto front.

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**Algorithm 1** Generic stochastic search algorithm.

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1:  $P_0 \subset Q$  drawn at random
2:  $A_0 = \text{ArchiveUpdate}(P_0, \emptyset)$ 
3: for  $j = 0, 1, 2, \dots$  do
4:    $P_{j+1} = \text{Generate}(P_j)$ 
5:    $A_{j+1} = \text{ArchiveUpdate}(P_{j+1}, A_j)$ 
6: end for

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**Algorithm 2**  $A := \text{ArchiveUpdateEps1}(P, A_0)$ .

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1:  $A := A_0$ 
2: for all  $p \in P$  do
3:   if  $\exists a \in A : a \prec_{\Theta_\epsilon} p$  then
4:     CONTINUE ▷ do not execute lines 6 – 11
5:   end if
6:   for all  $a \in A$  do
7:     if  $p \prec a$  then
8:        $A := A \setminus \{a\}$ 
9:     end if
10:  end for
11:   $A := A \cup \{p\}$ 
12: end for

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**Algorithm 3**  $A := \text{ArchiveUpdateEps2}(P, A_0)$

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1:  $A := A_0$ 
2: for all  $p \in P$  do
3:   if  $\nexists a \in A : a \prec_{\Theta_\epsilon} p$  then
4:      $A := A \cup \{p\}$ 
5:   end if
6:   for all  $a \in A$  do
7:     if  $p \prec a$  then
8:        $A := A \cup \{p\} \setminus \{a\}$ 
9:     end if
10:  end for
11: end for

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**THEOREM 1** (Schütze et al., 2008a): *Let an MOP  $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be given, where  $F$  is continuous, let  $Q \subset \mathbb{R}^n$  be a compact set and  $\epsilon \in \mathbb{R}_+^k$ . Further let*

$$\forall x \in Q \text{ and } \forall \delta > 0 : \quad P(\exists l \in \mathbb{N} : P_l \cap B_\delta(x) \cap Q \neq \emptyset) = 1 \quad (2.2)$$

*Then an application of Algorithm 1, where  $\text{ArchiveUpdateEps1}$  is used to update the archive, leads to a sequence of archives  $A_l, l \in \mathbb{N}$ , such that there exists with probability 1 an  $l_0 \in \mathbb{N}$  such that  $A_l$  is an  $\epsilon$ -approximate Pareto set for all  $l \geq l_0$ .*

**THEOREM 2** (Schütze et al., 2008a): *Let (MOP) be given and  $Q \subset \mathbb{R}^n$  be compact, and let there be no weak Pareto points in  $Q \setminus P_Q$ . Further, let  $F$  be injective and*

$$\forall x \in Q \text{ and } \forall \delta > 0 : \quad P(\exists l \in \mathbb{N} : P_l \cap B_\delta(x) \cap Q \neq \emptyset) = 1 \quad (2.3)$$

*Then an application of Algorithm 1, where ArchiveUpdateEps2 is used to update the archive, leads to a sequence of archives  $A_l, l \in \mathbb{N}$ , where the following holds:*

(a) *There exists with probability 1 an  $l_0 \in \mathbb{N}$  such that  $A_l$  is an  $\epsilon$ -approximate Pareto set for all  $l \geq l_0$ .*

(b)

$$\lim_{l \rightarrow \infty} \text{dist}(A_l, P_Q) = 0, \quad \text{with probability 1.}$$

### 3 The Sets of Interest

Motivated by the need for gap free Pareto front approximations and inspired by Definitions 4 and 5, we introduce here several objects.

First we define  $\Delta_M$ -tight  $\epsilon$ -(approximate) Pareto sets. A set  $A_\epsilon \subset \mathbb{R}^n$  is such a set if (i) it is an  $\epsilon$ -(approximate) Pareto set and if (ii) the maximal distance of a point in the Pareto front to the image of an archive element is not larger than a threshold  $\Delta_M$ . Condition (i) refers to the approximation quality of  $A_\epsilon$  in the sense of  $\epsilon$ -dominance (i.e., measured in image space), and condition (ii) refers to the tightness of the representation.

**DEFINITION 7:** *Let  $\epsilon \in \mathbb{R}_+^k$ .*

(a) *A set  $A_\epsilon \subset \mathbb{R}^n$  is called a  $\Delta_M$ -tight  $\epsilon$ -approximate Pareto set of (MOP) if  $A_\epsilon$  is an  $\epsilon$ -approximate Pareto set of (MOP) and*

$$\text{dist}(F(P_Q), F(A_\epsilon)) \leq \Delta_M. \quad (3.1)$$

(b) *A set  $A_\epsilon^* \subset \mathbb{R}^n$  is called a  $\Delta_M$ -tight  $\epsilon$ -Pareto set if  $F_\epsilon^*$  is an  $\epsilon$ -Pareto set of (MOP) and  $d_H(F(P_Q), F(A_\epsilon^*)) \leq \Delta_M$ .*

A more descriptive way to express the condition in Equation (3.1) is as follows: it is  $\text{dist}(F(P_Q), F(A_\epsilon)) \leq \Delta_M$  if for every  $y \in F(P_Q)$  there exists an element  $a \in A_\epsilon$  such that  $d_\infty(y, F(a)) \leq \Delta_M$ . In other words,  $F(P_Q)$  has to be contained in the box collection  $C_{A_\epsilon, \Delta_M}$ , where

$$C_{A, \Delta} := \bigcup_{a \in A} B_\Delta^\infty(F(a)), \quad (3.2)$$

and  $B_\Delta^\infty(x) := \{y \in \mathbb{R}^k : d_\infty(x, y) < \Delta\}$ .

Since we are further interested in uniform approximations of the sets of interest, we add the uniformity level  $\Delta_m$  to the objects in Definition 7.

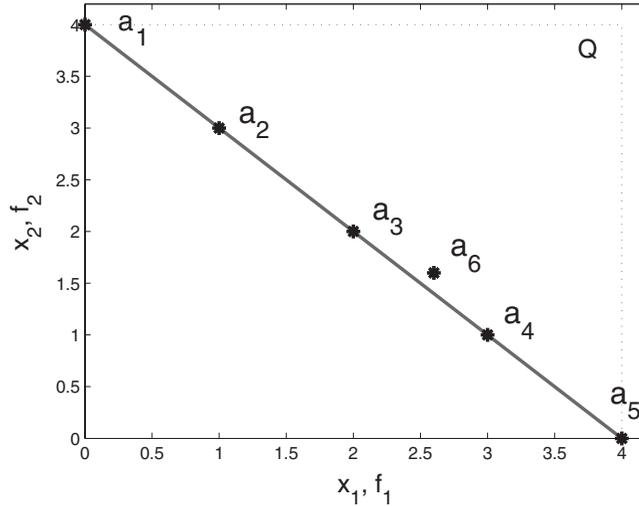


Figure 2: The set  $A_1 := \{a_1, \dots, a_5\}$  forms a  $(1/2, 1)$ -tight  $(1/2, 1/2)$ -Pareto set of MOP (3.3), and the set  $A_2 := A_1 \cup \{a_6\}$  a  $(1/2, 0.6)$ -tight  $(1/2, 1/2)$ -approximate Pareto set.

DEFINITION 8: Let  $\epsilon \in \mathbb{R}_+^k$ .

(a) A set  $A_\epsilon \subset \mathbb{R}^n$  with  $|A_\epsilon| \geq 2$  is called a  $(\Delta_M, \Delta_m)$ -tight  $\epsilon$ -approximate Pareto set if  $A_\epsilon^*$  is a  $\Delta_M$ -tight  $\epsilon$ -approximate Pareto set of (MOP) and

$$\text{dist}(F(a), F(A \setminus \{a\})) \geq \Delta_m, \quad \forall a \in A.$$

(b) A  $(\Delta_M, \Delta_m)$ -tight  $\epsilon$ -Pareto set is defined analogously.

To get an impression about these objects, we consider the following MOP:

$$\begin{aligned} F : Q \subset \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ F(x) &= x \end{aligned} \tag{3.3}$$

where

$$Q = [0, 4]^2 \cap \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 4 - x_2\}. \tag{3.4}$$

The Pareto set is given by  $P_Q = \{(x, 4 - x) \mid x \in [0, 4]\}$ . Let

$$A_1 = \{a_1 = (0, 4), a_2 = (1, 3), a_3 = (2, 2), a_4 = (3, 1), a_5 = (4, 0)\} \tag{3.5}$$

Since all elements  $a_i \in P_Q$  and by distribution of the  $a_i$  the set  $A_1$  forms a  $(1/2, 1/2)$ -Pareto set (see also Figure 2). Since further

$$\begin{aligned} \text{dist}(F(P_Q), F(A_1)) &= \max_{p \in P_Q} \min_{a \in A_1} \|p - a\|_\infty = 1/2, \quad \text{and} \\ \text{dist}(F(a), F(A_1 \setminus \{a\})) &\geq 1, \quad \forall a \in A_1, \end{aligned} \tag{3.6}$$

$A_1$  can be viewed as a  $(1/2, 1)$ -tight  $(1/2, 1/2)$ -Pareto set. Let  $A_2 := A_1 \cup \{a_6 = (2.6, 1.6)\}$  (see Figure 2), then the uniformity level changes to  $\delta = \|a_6 - a_3\|_\infty = 0.6$ , and since  $a_6 \notin P_Q$ ,  $A_2$  forms a  $(1/2, 0.6)$ -tight  $(1/2, 1/2)$ -approximate Pareto set.

Note that not for every value of  $\Delta_m$ ,  $\Delta_M$ , and  $\epsilon$  do the objects defined above exist for a given MOP, depending on the shape of the Pareto front. For instance, the domain  $Q$  of MOP in Equation (3.3) is included in  $[0, 4]^2$ . Hence, if  $\Delta_m > 4$  is chosen, only one solution will be kept in the archive, and thus, no  $(\Delta_M, \Delta_m)$ -tight  $\epsilon$ -Pareto set can be obtained for  $\|\epsilon\|_\infty < 2$  since in this case at least two elements have to be kept. Similarly, no  $(\Delta_M, \Delta_m)$ -tight  $\epsilon$ -Pareto set exists for  $\Delta_M < \Delta_m/2$ , independently of the value of  $\epsilon$ .

Despite these potential problems, however, the values of  $\epsilon$ ,  $\Delta_M$ , and  $\Delta_m$  have to be adjusted a priori.  $\epsilon$  represents the (maximal) tolerable loss of a solution compared to an optimal solution, and is thus relatively easy to identify in a real world application (e.g., Ruhe and Fruhwirt, 1990; White, 1998; Schütze et al., 2008d).  $\Delta_M$  determines the maximal distance between two solutions and thus depends on the search algorithm (e.g., on the step size of the local search procedure) and/or on the preference of the decision maker. In any case,  $\Delta_M \geq \max_{i=1, \dots, k} \epsilon_i$  can be chosen since two solutions which are closer together can, from a practical point of view, be regarded as equal. Finally, the value of the uniformity level  $\Delta_m$  has to be chosen to be less than or equal to  $\min_{i=1, \dots, k} \epsilon_i$ , since otherwise it cannot be guaranteed that the limit archive set forms an  $\epsilon$ -(approximate) Pareto set.

## 4 The Algorithms

In the following, we propose two different strategies for archiving the solutions found by the algorithm and investigate some of their properties. The main focus is on the limit behaviors of the sequence of archives which result by the use of these strategies under certain additional conditions on the generation process. It will be shown that the archives obtained by the first archiver *ArchiveUpdateTight1* form with probability 1 after finitely many steps, and under certain assumptions a  $(\Delta, \Theta\epsilon_m)$ -tight  $\epsilon$ -approximate Pareto set of a given MOP, where  $\Delta$ ,  $\epsilon$ , and  $\Theta$  are given and  $\epsilon_m = \min_{i=1, \dots, k} \epsilon_i$ . In order to maintain the uniformity level  $\Theta\epsilon_m$ , a certain discretization in image space has to be done, which prevents the elements of the archives from converging toward the Pareto set. The second archiver, *ArchiveUpdateTight2*, allows for such a convergent behavior, but, in turn, the uniformity of the archive is lost.

### 4.1 Archiver *ArchiveUpdateTight1*

First, we consider the strategy *ArchiveUpdateTight1*, which is presented in Algorithm 4. Given  $\epsilon \in \mathbb{R}_+^k$  and  $\Delta \in \mathbb{R}_+$ , the archiver accepts a new candidate solution  $p \in Q$  if (i) there exists no element  $a$  of the current archive  $A$  such that  $a$   $\Theta\epsilon$ -dominates  $p$  or (ii) there exists no  $a \in A$  which dominates  $p$  and the distance of  $F(a)$  to  $F(p)$  for all elements  $a \in A$  is larger than a threshold  $\tilde{\Delta}$  (see line 3 of Algorithm 4 or Equation (4.4)). If either (i) or (ii) is true,  $p$  is added to the archive (line 11 of Algorithm 4) and all elements  $a \in A \setminus \{p\}$  that are dominated by  $p$  are discarded from the archive (lines 7–10). Thus, only nondominated solutions are stored. The auxiliary variables  $0 < \Theta < 1$  and  $0 < \tilde{\Delta} < \Delta$  are required to guarantee convergence of the sequence of archives in the limit (see Theorem 4), but in practice they can be set to  $\Theta = 1$  and  $\tilde{\Delta} = \Delta$ .

To prevent the possibility that the archives are unbounded (see discussion in Section 2), Algorithm 4 uses an exclusion strategy that is based on  $\epsilon$ -dominance: new

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**Algorithm 4**  $A := \text{ArchiveUpdateTight1}_{\epsilon, \tilde{\Delta}}(P, A_0)$ .

---

```

1:  $A := A_0$ 
2: for all  $p \in P$  do
3:   if  $(\exists a \in A : a \prec p)$  or  $(\exists a_1 \in A : a_1 \prec_{\Theta\epsilon} p$  and  $\exists a_2 \in A : d_\infty(F(a_2), F(p)) \leq \tilde{\Delta})$ 
   then
4:     CONTINUE  $\triangleright$  do not execute lines 6 – 11
5:   end if
6:   for all  $a \in A$  do
7:     if  $p \prec a$  then
8:        $A := A \setminus \{a\}$ 
9:     end if
10:  end for
11:   $A := A \cup \{p\}$ 
12: end for

```

---

candidate solutions  $x$  can only be added to the archive  $A$  if there does not already exist a solution  $a \in A$  such that  $a \prec_{\Theta\epsilon} x$  (second term in line 3 of Algorithm 4). This strategy allows on the one hand to bound the number of entries of the archive (see Section 4.3) but by this it follows on the other hand that candidates  $x$  can be discarded even if they dominate elements  $a \in A$ . This suboptimality in the selection mechanism of the archiver, however, is restricted to the value of  $\epsilon$  that can be considered to be small. The following result (Theorem 3) shows that the distance of the elements of the archive toward the observed Pareto front (i.e., the set of nondominated solutions found by the generator so far) does not sum up. To be more precise, it will be shown that entries of the resulting archive form a  $\Theta\epsilon$ -approximate Pareto set of the set of points that have been discovered during the run of the algorithm. Thus, monotonicity of the sequence of archives is ensured and cycling or deterioration (Hanne, 1999) cannot occur.

LEMMA 1: *Let  $A_0, P \subset \mathbb{R}^n$  be finite sets,  $\epsilon \in \mathbb{R}_{>0}^k, 0 < \Theta < 1, 0 < \tilde{\Delta} < \Delta$ , and  $A := \text{ArchiveUpdateTight1}(P, A_0)$ . Then the following holds:*

$$\forall x \in P \cup A_0 : \exists a \in A : a \prec_{\Theta\epsilon} x.$$

PROOF: Let  $P = \{p_1, p_2, \dots, p_l\}, l \in \mathbb{N}$ . Without loss of generality we assume that all points  $p_i$  are considered in this ordering (i.e., in the for-loop in line 2 of Algorithm 4). Denote by  $A_i$  the resulting archive after  $p_i$  has been considered, and thus,  $A = A_n$ .

Let  $x \in P \cup A_0$ . There are two cases we have to distinguish.

CASE A:  $x \in A_0$ . In this case, the statement follows since points  $a$  are only discarded from the archive if in turn another point  $p$  with  $p \prec a$  is inserted (see lines 7, 8 and 11 of Algorithm 4), and further, by the transitivity of  $\prec$ .

CASE B:  $x \in P$ , that is, there exists  $j \leq l$  such that  $x = p_j$ , and thus,  $x$  is considered in the  $j$ th step of the algorithm. In case  $x$  is added to  $A_j$  the statement follows analogously to Case A since  $A = \text{ArchiveUpdateTight1}(\{p_{j+1}, \dots, p_l\}, A_j)$ , and  $x \in A_j$ . If  $x$  is not added to  $A_j$ , there are two possibilities (see line 3 of Algorithm 4): (i) there exists an  $a \in A_j$  such that  $a \prec x$ , or (ii) there exists an  $a_1 \in A_j$  with  $a_1 \prec_{\Theta\epsilon} x$ . In both cases  $x$  gets

$\Theta\epsilon$ -dominated by a point  $\tilde{a} \in A_j$ . Thus, again analogous to Case A, a point  $a^* \in A$  exists that is equal to  $\tilde{a}$  or dominates it, and thus  $\Theta\epsilon$ -dominates  $x$ .  $\square$

**THEOREM 3:** *Let  $l \in \mathbb{N}$ ,  $\epsilon \in \mathbb{R}_+^k$ ,  $0 < \Theta < 1$ , and  $0 < \tilde{\Delta} < \Delta$ . Further, let  $A_i, P_i, i = 0, \dots, l$ , be as defined in Algorithm 1, where  $ArchiveUpdateTight1_{\epsilon, \tilde{\Delta}}$  is used to update the archive. Then*

$$\forall x \in \bigcup_{i=1}^l P_i : \exists a \in A_l : a \prec_{\Theta\epsilon} x. \tag{4.1}$$

**PROOF:** The proof is done via induction over  $l$ . For  $l = 1$  we have

$$A_1 = ArchiveUpdateTight1_{\epsilon, \tilde{\Delta}}(P_0, A_0),$$

and the claim follows by Lemma 1. For the induction hypothesis, suppose the claim in Equation (4.1) is right for  $l - 1 > 1$ . Let  $x \in \bigcup_{i=1}^l P_i$ . If  $x \in \bigcup_{i=1}^{l-1} P_i$ , there exists by induction hypothesis an element  $a \in A_{l-1}$  such that  $a \prec_{\Theta\epsilon} x$ . Thus,  $a$  is also a member of the subsequent archive  $A_l$  or is replaced by an element  $p \in P_l$  with  $p \prec a$ . In both cases, there exists an element in  $A_l$  that  $\Theta\epsilon$ -dominates  $x$ . In case  $x \in P_l$ , the claim follows again by Lemma 1 since  $A_l = ArchiveUpdateTight1_{\epsilon, \tilde{\Delta}}(P_l, A_{l-1})$ , and the proof is complete.  $\square$

Next, we are interested in the limit behavior of the sequence  $A_i$  of archives within the use of a stochastic search procedure. To guarantee convergence, we have to make several (mild) assumptions on the model as well as on the process to generate new candidate solutions. Since we primarily address continuous optimization, we assume that  $F$  is continuous and that the domain  $Q$  is compact (e.g., an  $n$ -dimensional box defined by box constraints). By this it follows that the image  $F(Q)$  is bounded, which allows for a finite size Pareto set approximation in the sense of Definitions 4, 7, and 8 for every value of  $\epsilon \in \mathbb{R}_+^k$ . Note that this property is always true for discrete MOPs (i.e.,  $|Q| < \infty$ ). Further, we have to make the following assumption on the generation process (see also Schütze, 2004; Schütze et al., 2008c):

$$\forall x \in Q \text{ and } \forall \delta > 0 : P(\exists l \in \mathbb{N} : P_l \cap B_\delta(x) \cap Q \neq \emptyset) = 1, \tag{4.2}$$

where  $P(A)$  denotes the probability for event  $A$ . The assumption in Equation (4.2) says that every neighborhood  $U \cap Q$  of every point gets visited by  $Generate()$  after finitely many steps with probability 1. The following consideration shows that we cannot assume less: if Equation (4.2) does not hold, there exists with probability 1 a point  $x \in Q$  and a neighborhood  $\tilde{U} = U \cap Q$  of  $x$  such that no candidate solution  $p \in P_l$  lies in  $\tilde{U}$  for all  $l \in \mathbb{N}$ . Thus, no convergence can be guaranteed, since a part of the Pareto set can be contained in  $\tilde{U}$  that is never visited.

We point out that our results also hold for discrete models. In case the MOP is discrete, the assumption in Equation (4.2) reads as

$$\forall x \in Q : P(\exists l \in \mathbb{N} : x \in P_l) = 1, \tag{4.3}$$

which is, for example, fulfilled if  $Generate()$  is a homogeneous finite Markov chain with irreducible transition matrix (Rudolph, 1998; Rudolph and Agapie, 2000).

The next theorem shows that under these assumptions, a generic stochastic search algorithm (Algorithm 1) coupled with the archiver in Algorithm 4 generates a sequence of archives that forms with probability 1 after finitely many steps a  $(\Delta, \Theta\epsilon_m)$ -tight  $\epsilon$ -approximate Pareto set.

**THEOREM 4:** *Let an MOP  $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$  be given, where  $F$  is continuous, let  $Q \subset \mathbb{R}^n$  be a compact set and  $\epsilon \in \mathbb{R}_+^k$ . Let  $\epsilon_m := \min_{i=1,\dots,k} \epsilon_i$ ,  $\epsilon_M := \max_{i=1,\dots,k} \epsilon_i$ , further let  $\Delta, \tilde{\Delta} \in \mathbb{R}_+$  be given such that  $\epsilon_M < \tilde{\Delta} < \Delta$ , let  $0 < \Theta < 1$ , and let the assumption in Equation (4.2) be fulfilled. Then an application of Algorithm 1, where  $\text{ArchiveUpdateTight}_{1,\epsilon,\tilde{\Delta}}$  is used to update the archive, leads to a sequence of archives, such that there exists with probability 1 an  $l_1 \in \mathbb{N}$  such that  $A_l$  is a  $(\Delta, \Theta\epsilon_m)$ -tight  $\epsilon$ -approximate Pareto set for all  $l \geq l_1$ .*

**PROOF:** First, we turn our attention to the question of which elements are added to the archive. The crucial expression  $\mathcal{E}$  (line 3 of Algorithm 4) reads as follows:

$$\underbrace{(\exists a \in A : a < p)}_A \text{ or } \underbrace{(\exists a_1 \in A : a_1 <_{\Theta\epsilon} p)}_{B_1} \text{ and } \underbrace{(\exists a_2 \in A : d_\infty(F(a_2), F(p)) \leq \tilde{\Delta})}_{B_2}$$

Since  $\neg\mathcal{E} = (\neg A \text{ and } \neg B_1) \text{ or } (\neg A \text{ and } \neg B_2)$ , and since  $\neg B_1$  implies  $\neg A$ , it follows that points  $p \in \mathbb{R}^n$  are added to a given archive  $A$  if (and only if) one of the two following expressions is true

$$\begin{aligned} (\mathcal{E}_1) \quad & \exists a \in A : a <_{\Theta\epsilon} p, \text{ or} \\ (\mathcal{E}_2) \quad & \exists a \in A : a < p \text{ and } \forall a \in A : d_\infty(F(a), F(p)) > \tilde{\Delta}. \end{aligned} \tag{4.4}$$

We are now in the position to prove the theorem. By  $\mathcal{E}_1$ , it follows that all points that are added by  $\text{ArchiveUpdateEps1}$  to the archive are also added by  $\text{ArchiveUpdateTight1}$ . Thus, by Theorem 1 it follows that there exists with probability 1 an  $l_0 \in \mathbb{N}$  such that  $A_l$  is an  $\epsilon$ -approximate Pareto set w.r.t.  $F|_Q$  for all  $l \geq l_0$ , since points  $a \in A_l$  are only removed from the archive if in turn another point  $\tilde{a}$  is added that dominates  $a$  (if  $x < y$  and  $y <_\epsilon z$  it follows that  $x <_\epsilon z$ ).

Remaining to be shown is the tightness of the limit archive. The uniformity level  $\epsilon_m$  follows directly by an inductive argument and using the exclusion strategy of Equation (4.4). This and the fact that  $F(Q)$  is bounded is the reason that the size of the archive is bounded above for a given MOP by a number  $n_0 = n_0(\epsilon, F(Q))$ , which will be needed for further consideration (see Section 4.3 for more details).

As mentioned in Section 3 (see Equation (3.2) and related discussion) the claim is right for an archive  $A_l$  if

$$F(P_Q) \subset C_{A_l,\Delta}. \tag{4.5}$$

Assume that  $A_l$  is an  $\epsilon$ -approximate Pareto set for all  $l \geq l_0$  and let  $l \geq l_0$ . By construction of  $\text{ArchiveUpdateTight1}$ , it follows that if  $F(P_Q) \subset C_{A_l,\Delta}$  this inclusion holds for all  $l \geq l_1$ , since in this case no further point will be added to the archive (since the expressions  $\mathcal{E}_1$  and  $\mathcal{E}_2$  in Equation (4.4) will be false for all further candidates). That is, it is sufficient to show the existence of such a number  $l_1$ . In the following, we will do this by contradiction: first we show that by using  $\text{ArchiveUpdateTight1}$  and under the assumptions made above

only finitely many replacements can be done during the run of the algorithm. Then—under the assumption that there exists no number  $l_1$  with the above property—we construct a contradiction by showing that infinitely many replacements have to be done during the run of the algorithm with the given setting.

Let a finite archive  $A_0$  be given. If a point  $p \in \mathbb{R}^n$  replaces a point  $a \in A_0$  (see lines 8 and 11 of Algorithm 4) it follows by construction of *ArchiveUpdateTight1* [see also Equation (4.4)] that

$$\exists i \in 1, \dots, k : f_i(p) < f_i(a) - \Theta\epsilon_i. \tag{4.6}$$

Since the relation  $<$  is transitive, there exists for every  $a \in A$  a history of replaced points  $a_i \in A_i$  where Equation (4.6) holds for  $a_i$  and  $a_{i-1}$ . Since  $F(Q)$  is bounded, there exist  $l_i, u_i \in \mathbb{R}, i = 1, \dots, k$ , such that  $F(Q) \subset [l_1, u_1] \times \dots \times [l_k, u_k]$ . After  $r$  replacements, there exists at least one  $a \in A_{l(r)}$  such that the length  $h$  of the history of  $a$  is at least  $h \geq \lceil r/n_0 \rceil$ , where  $n_0$  is the maximal number of entries in the archive (see above). Denote by  $a_0 \in A_0$  the root of the history. For  $a, a_0$  it follows that

$$\exists i \in 1, \dots, k : f_i(a) < f_i(a_0) - s\Theta\epsilon_i,$$

where  $s \geq \lceil h/k \rceil$ . For  $\tilde{s} > d_{\max} := \Theta^{-1} \max_{i=1, \dots, k} \frac{u_i - l_i}{\epsilon_i}$  (which is given for  $\tilde{r} > n_0 k d_{\max} + n_0 + 1$ ) we obtain a contradiction since in that case  $f_i(a) < l_i$  and thus  $F(a) \notin F(Q)$ . Hence, it follows that there can be done only finitely many such replacements during the run of an algorithm.

Assume that such an  $l_1$  as claimed above does not exist, that is, that  $F(\mathcal{P}_Q) \not\subset C_{A, \Delta}$  for all  $l \in \mathbb{N}$ . Hence, there exists a sequence of image points

$$y_i \in F(\mathcal{P}_Q) \setminus C_{A_i, \Delta} \quad \forall i \in \mathbb{N}. \tag{4.7}$$

Since  $F(Q)$  is compact, there exists an accumulation point  $y^* \in F(\mathcal{P}_Q)$ , that is, there exists a subsequence  $\{i_j\}_{j \in \mathbb{N}}$  with

$$y_{i_j} \rightarrow y^* \text{ for } j \rightarrow \infty. \tag{4.8}$$

Since  $y^* \in F(\mathcal{P}_Q)$ , there exists a neighborhood  $U_1$  of  $y^*$  such that the following holds

$$\forall (y, \tilde{y}) \in F(Q) \times U_1 : y \leq_p \tilde{y} \Rightarrow d_\infty(y, \tilde{y}) \leq \tilde{\Delta} \tag{4.9}$$

Let  $\tilde{U}_1 := U_1 \cap B_{(\Delta - \tilde{\Delta})/2}^\infty(y^*)$ . By Equation (4.2) it follows that there exists with probability 1 an  $l_1 \in \mathbb{N}$  and an  $\tilde{x}_1 \in P_{l_0+l_1}$  with  $\tilde{y}_1 = F(\tilde{x}_1) \in \tilde{U}_1$ . By construction of *ArchiveUpdateTight1*, there exists an element  $a_1 \in A_{l_0+l_1}$  such that  $d_\infty(F(a_1), \tilde{y}_1) < \tilde{\Delta}$  [due to Equation (4.4) there are three possibilities:  $\mathcal{E}_2$  is false and thus there already exists an  $a_1 \in A_{l_0+l_1}$  which (a) dominates  $\tilde{x}$ —in this case the claim follows with Equation (4.9)—or (b) where  $d_\infty(F(a_1), \tilde{y}_1) \leq \tilde{\Delta}$ , or  $\mathcal{E}_2$  is true and thus (c)  $a_1 = \tilde{x}_1$  is added to the archive]. Thus, we have

$$d_\infty(F(a_1), \tilde{y}) \leq d_\infty(F(a_1), \tilde{y}_1) + d_\infty(\tilde{y}_1, \tilde{y}) < \tilde{\Delta} + 2 \frac{\Delta - \tilde{\Delta}}{2} = \Delta \quad \forall \tilde{y} \in \tilde{U}_1. \tag{4.10}$$

By Equations (4.7) and (4.8) there exist  $j_1, \tilde{l}_1 \in \mathbb{N}$  with

$$y_{i_{j_1}} \in \tilde{U}_1 \setminus C_{l_0+l_1+\tilde{l}_1, \Delta}.$$

Since by Equation (4.10) it holds that  $d_\infty(y_{i_{j_1}}, F(a_1)) < \Delta$ , it follows that  $a_1 \notin A_{l_0+l_1+\tilde{l}_1}$ , which is only possible via a replacement in Algorithm 4 (lines 8 and 11).

In an analogous way, a sequence  $\{a_i\}_{i \in \mathbb{N}}$  of elements can be constructed that have to be replaced by other elements. Since this leads to a sequence of infinitely many replacements, this is a contradiction to the assumption, and the proof is complete.  $\square$

Note that the exclusion strategy in Equation (4.4) prevents convergence of the elements of the archives toward the Pareto set. This is due to the fact that for points  $x \in Q$  that are nearly optimal, the set of points in  $Q$  that (i) dominate  $x$  and (ii) are not  $\epsilon$ -dominated by  $x$  can be empty. Such nearly optimal archive entries will hence never be replaced by other better solutions, and thus, the elements in the archive stop improving when using Algorithm 4 at a certain stage, depending on the value of  $\epsilon$ . This leads directly to the next archiver.

#### 4.2 Archiver *ArchiveUpdateTight2*

The second archiving strategy we consider here, *ArchiveUpdateTight2*, which is shown in Algorithm 5, overcomes the problem stated above by changing the criterion to accept a candidate solution (which is in fact the only difference between Algorithms 4 and 5). In turn, by using Algorithm 5, the uniformity of the archive solutions cannot be guaranteed any more, and the (theoretical and pessimistic) upper bounds on such archives increases by one order of magnitude compared to the first archiving strategy (see Section 4.3). The difference of the two archivers *ArchiveUpdateTight1* and *ArchiveUpdateTight2* is the strategy to accept a candidate solution  $p \in P$ . Given an archive  $A_0$ , *ArchiveUpdateTight2* accepts  $p$  (i) if either term  $\mathcal{E}_1$  or  $\mathcal{E}_2$  of Equation (4.4) is true (line 3 of Algorithm 5) or (ii) if there exists an element  $a \in A_0$  that is dominated by  $p$  (line 8 of Algorithm 5). In case (i) *ArchiveUpdateTight1* also accepts the candidate solution, the difference of both archivers is case (ii), which is not considered in *ArchiveUpdateTight1*.

Under the same assumptions as made above for Algorithm 4, the following theorems show that the same monotonicity result on the approximation quality can be obtained, and the distance  $dist(A_l, P_Q)$  of the archives  $A_l$  to the Pareto set  $P_Q$  vanishes for  $l \rightarrow \infty$  (the elements of the archive converge to the Pareto set). Thus, if the limit archive exists (the sequence  $|A_l|$  of the magnitudes of the archives is not necessarily converging), this set forms a  $\Delta$ -tight  $\epsilon$ -Pareto set. Though we have to assume in Theorem 6 that  $F$  has to be injective to guarantee the convergence, this property is in fact not relevant in practice (see, e.g., Section 6.2).

**THEOREM 5:** *Let  $l \in \mathbb{N}, \epsilon \in \mathbb{R}_+^k, 0 < \Theta < 1$ , and  $0 < \tilde{\Delta} < \Delta$ . Further, let  $A_i, P_i, i = 0, \dots, l$ , be as defined in Algorithm 1, where *ArchiveUpdateTight2* $_{\epsilon, \tilde{\Delta}}$  is used to update the archive. Then*

$$\forall x \in \bigcup_{i=1}^l P_i : \exists a \in A_l : a \prec_{\Theta \epsilon} x. \tag{4.11}$$

**PROOF:** Analogous to proof of Theorem 3.  $\square$

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**Algorithm 5**  $A := \text{ArchiveUpdateTight}_{2\epsilon, \Delta} (P, A_0)$

---

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1:  $A := A_0$ 
2: for all  $p \in P$  do
3:   if  $\mathcal{E}_1$  is true or  $\mathcal{E}_2$  is true then ▷ see Equation (4.4)
4:      $A := A \cup \{p\}$ 
5:   end if
6:   for all  $a \in A$  do
7:     if  $p \prec a$  then
8:        $A := A \cup \{p\} \setminus \{a\}$ 
9:     end if
10:  end for
11: end for

```

---

**THEOREM 6:** Let (MOP) be given and  $Q \subset \mathbb{R}^n$  be compact, and let there be no weak Pareto points in  $Q \setminus \mathcal{P}_Q$ . Further, let  $F$  be injective and let the assumption in Equation (4.2) be fulfilled. Then an application of Algorithm 1, where  $\text{ArchiveUpdateTight}_{2\epsilon, \Delta}$  is used to update the archive, leads to a sequence of archives  $A_l, l \in \mathbb{N}$ , where the following holds:

(a) There exists with probability 1 an  $l_1 \in \mathbb{N}$  such that  $A_l$  is a  $\Delta$ -tight  $\epsilon$ -approximate Pareto set w.r.t.  $F|_Q$  for all  $l \geq l_1$ .

(b)

$$\lim_{l \rightarrow \infty} \text{dist}(A_l, P_Q) = 0, \quad \text{with probability 1.}$$

**PROOF:** All parts of the proof are analogous to parts in proofs of Theorem 2 and Theorem 4. □

### 4.3 Bounds on the Limit Archive Sizes

Since we are aiming for finite size representations of the Pareto front, the bounds of the magnitudes of the archives obtained by the new archiving strategies are of particular interest, which we address in this section.

#### 4.3.1 Upper Bounds

The upper bounds on the archive sizes that result by the novel archiving strategies can be derived by analogy to the bounds for the archivers presented in Schütze et al. (2008c):

The maximal archive size maintained by  $\text{ArchiveUpdateTight1}$  is the same as for  $\text{ArchiveUpdateEps1}$ , and is given by

$$|A_l| \leq \sum_{i=1}^k \prod_{\substack{j=1 \\ j \neq i}}^k \left\lceil \frac{M_j - m_j}{\Theta \epsilon_j} \right\rceil, \quad (4.12)$$

where  $m_i = \min_{x \in Q} f_i(x)$ ,  $M_i = \max_{x \in Q} f_i(x)$ ,  $1 \leq i \leq k$ , and  $|A_0| = 1$ . Note that the magnitude can only be influenced by the value of  $\epsilon$ . The existence of this bound is due to the exclusion strategy of Equation (4.4), which makes it possible that the sequence of

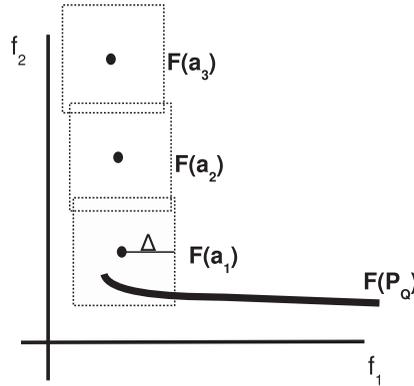


Figure 3: Possible example of a set that was generated by *ArchiveUpdateTight1* with  $dist(F(A), F(\mathcal{P}_Q)) > \Delta$ .

archives converges after finitely many steps. On the other hand, this exact feature prevents us from being able to guarantee that  $dist(F(A), F(\mathcal{P}_Q))$  and thus  $d_H(F(A), F(\mathcal{P}_Q))$  are small (say  $\leq \Delta$ ), as the following example shows (compare to Figure 3): assume that the elements  $a_3, a_2, a_1$  are inserted into the archive in this order. By construction of *ArchiveUpdateTight1*, these points will not be removed in the subsequent steps, since there exists no point  $p$  with  $F(p) \in F(A) \setminus C_{A, \Delta}$  that dominates  $a_i, i \in \{1, 2, 3\}$ . In such a manner, an example can be constructed with  $dist(F(A), F(\mathcal{P}_Q)) = \max_{i=1, \dots, k} (M_i - m_i)$ . However, this (bad) theoretical value has never been observed in computations.

The maximal archive size obtained by *ArchiveUpdateTight2* is equal to the one obtained by *ArchiveUpdateEps2*

$$|A_I| \leq \prod_{i=1}^k \left\lceil \frac{M_i - m_i}{\Theta \epsilon_i} \right\rceil,$$

where  $m_i, M_i$  are as defined above and  $|A_0| = 1$ .

### 4.3.2 Lower Bounds

The lower bound of  $|A_\infty|$  for both new archiving strategies is obviously given by 1. For this, consider, for example,  $f_1 = f_2 = \dots = f_k$  to be a convex function which takes its (unique) minimum inside  $Q$ , which leads to  $|P_Q| = 1$ . Though desired, it is hardly possible to provide meaningful lower bounds for general MOPs, since (a) the archive size is in practice mainly determined by the value of  $\epsilon$ , in particular when the entries of  $\epsilon$  are much smaller than  $\Delta$ , and since (b) the Pareto front can fall into different connected components. However, if this is not the case, one can obtain the following result in the bi-objective case, which we state without the (obvious) proof.

PROPOSITION 1: Let  $\tilde{m}_i = \min_{x \in P_Q} f_i(x)$  and  $\tilde{M}_i = \max_{x \in P_Q} f_i(x), i = 1, 2$ , and let  $F(P_Q)$  be connected. Then, when using *ArchiveUpdateTight1* $_{\epsilon, \Delta}$  or *ArchiveUpdateTight2* $_{\epsilon, \Delta}$ , the archive

size maintained in Algorithm 1 for the limit archive is bounded by

$$|A_\infty| \geq \max_{i=1,\dots,k} \left\lceil \frac{\tilde{M}_i - \tilde{m}_i}{2\Delta} \right\rceil \quad (4.13)$$

An analogous statement for  $k > 2$ , for example, by estimating the Pareto front by a  $k$ -simplex in objective space where the vertices are the minima of the objectives, however, does not hold since the  $(k - 1)$ -dimensional volume of the Pareto front can be arbitrarily small.

## 5 Integration into Iterative Search Methods

Since this work deals with the design of archiving strategies, the question that naturally arises is how these methods can be integrated *effectively* into an iterative stochastic search process such as an MOEA, which we address here.

One obvious benefit of an MOEA that is equipped with an (external) archive compared to a classical MOEA with a fixed population size is certainly given by the convergence properties of the archiver (see Theorems 4 and 6 of this paper, or the results of the works discussed in Section 1). Another possible advantage is that the globality of the search will be increased: in case the external archive is considered for the mating pool, the number of well-converged and well-distributed parent solutions is increased leading to a potentially more thorough search around the Pareto set. In fact, in Deb et al. (2005), where a MOEA with an external archive has been studied, it has been observed that this algorithm is

successful in finding well-converged and well-distributed solutions with a much smaller computational effort than a number of state-of-the-art MOEAs.

This algorithm,  $\epsilon$ -MOEA, can be viewed as a possible prototype for the integration of an archiving strategy into the evolutionary search process. The procedure is as follows: the algorithm contains a population ( $P_l$  in iteration step  $l$ ) and an archive ( $A_l$  in step  $l$ ). A new offspring  $o$  is created by crossover of an archive solution  $a_0 \in A_l$  and an individual  $p \in P_l$ . Both sets  $A_l$  and  $P_l$  are then updated by  $o$  (following different strategies). This process is repeated until a prescribed termination criterion is fulfilled. Apparently, *any* archiving strategy for the update of the external archive  $A_l$  can be used, including, for instance, the ones proposed in this paper.

However, apart from this universally applicable prototype for the integration of the archivers into an MOEA, there is a particular property of the two strategies proposed above that needs special attention: variations of existing archive entries that are too small will be discarded from the archiver for the subsequent archive. Due to the exclusion strategy in Equation (4.4) used in Algorithms 4 and 5, there exists for every solution  $a$  from a given archive  $A$  a neighborhood  $U_a$  of  $a$  such that *every* point  $u \in U_a$  will be rejected from the archive by  $a$ . Regarding this, it has to be noted that not every crossover (as well as mutation) strategy is suitable for a coupling with *ArchiveUpdateTight1* or *ArchiveUpdateTight2*. For instance, there is a potential conflict in terms of efficiency when using, for example, the simulated binary crossover operator (SBX) and the polynomial mutation, probably the most commonly used operators for crossover and mutation (Deb, 2001). When  $a_0 \in A_l$  and  $p \in P_l$  are used, for example, for

crossover,<sup>2</sup> the probability is relatively high that the offspring  $o$  is contained in  $U_{a_0}$  (or in  $U_p$ , which leads to the same problem), since the probability density has a peak at  $a_0$  (and a second peak at  $p$ ). As has been demonstrated on numerous benchmark and real world problems, SBX and polynomial mutation are highly valuable for the evolution of  $P_l$ . Thus, the operators will also be beneficial for the evolution of  $A_l$ , but probably only in the long run. In order to obtain a greedy strategy for the evolution of  $A_l$ , which is desired for fast convergence, both SBX and polynomial mutation do not seem to be well suited, since in case fast convergence is sought, a minimal distance of the offspring  $o$  to  $a_0$  is required.

In case gradient information is available (which is, e.g., assumed in many recent studies dealing with memetic strategies, e.g., Brown and Smith, 2005; Bosman and de Jong, 2005; Sharma et al., 2007; Harada et al., 2007; Schütze et al., 2008a), the minimal distance of  $a_0$  and  $o$  such that  $o \notin U_{a_0}$ , i.e., that  $o$  is possibly accepted by the archiver—can be estimated as follows: since  $\tilde{\Delta} < \Delta$ , we can assume that we are (ideally) interested in a point  $o$  in the neighborhood of  $a_0$  such that

$$\|F(a_0) - F(o)\|_\infty = \Delta, \tag{5.1}$$

since such a point will at least not be discarded due to the exclusion strategies of the archers proposed above. In the case where  $F$  is Lipschitz continuous, there exists an  $L \geq 0$  such that

$$\|F(x) - F(y)\|_\infty \leq L\|x - y\|_\infty, \quad \forall x, y \in Q. \tag{5.2}$$

Since we are heading in this article for gap free approximations, we can assume that  $\Delta$  is small, and thus, that  $a_0$  and  $o$  are close to each other. Hence, it is sufficient to estimate the Lipschitz constant  $L$  locally around  $a_0$ , which can be done by

$$L_{a_0} := \|DF(a_0)\|_\infty = \max_{i=1, \dots, k} \|\nabla f_i(a_0)\|_1, \tag{5.3}$$

in case the gradients are available, where  $DF(a_0) \in \mathbb{R}^{m \times n}$  denotes the Jacobian and  $\nabla f_i(a_0) \in \mathbb{R}^n, i = 1, \dots, m$ , the objectives' gradients. An alternative way to approximate  $L_{a_0}$  without using gradient information is, for example,

$$\tilde{L}_{a_0} := \frac{\|F(a_0) - F(\tilde{a}_0)\|}{\|a_0 - \tilde{a}_0\|}, \tag{5.4}$$

where  $\tilde{a}_0$  is close to  $a_0$ . The approximation in Equation (5.4) is certainly less accurate than in Equation (5.3), but we think that accuracy does not play an important role in this context.

Using such a local approximation  $L_{a_0}$  [e.g., Equation (5.3) or (5.4)], the distance of  $a_0$  and an offspring  $o$  that satisfies Equation (5.1) can be estimated by

$$\|a_0 - o\|_\infty \approx \frac{\Delta}{L_{a_0}} \tag{5.5}$$

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<sup>2</sup>An analogous statement holds for the polynomial mutation.

Using this, a neighborhood search (or mutation) can, for example, be realized as follows: assume that the difference of  $F(a_0)$  and  $F(o)$  should be in the range  $[\Delta, \lambda\Delta]$ , where  $\lambda > 1$ , then it follows, analogous to Equation (5.5), that  $\|a_0 - o\| \in [\Delta/L_{a_0}, \lambda\Delta/L_{a_0}]$ , and thus, the offspring can, for example, be taken uniformly at random from the pierced sphere  $B_{\frac{\lambda\Delta}{L_{a_0}}}^\infty(a_0) \setminus B_{\frac{\Delta}{L_{a_0}}}^\infty(a_0)$ :

- (1) compute  $L_{a_0}$
- (2) choose  $d \in B_1(0)$  and  $h \in [\Delta/L_{a_0}, \lambda\Delta/L_{a_0}]$  uniformly at random
- (3) set  $o := a_0 + h \frac{d}{\|d\|_\infty}$

We do not, however, investigate the effectiveness of this mutation strategy here, since this would go beyond the scope of this article. Instead, we give evidence that the step size control in Equation (5.5) can be used in a particular local search strategy leading to a possible efficient hybridization of this method with an MOEA that is equipped with the archivers presented in Algorithms 4 and 5 (which was one motivation for the need of gap free approximations).

To be more precise, we want to demonstrate that in the underlying context, a hybridization with multi-objective continuation methods (e.g., Hillermeier, 2001; Schütze et al., 2005) could be advantageous, since both the concept of  $\epsilon$ -dominance as well as the tightness can be directly integrated into it (see also Schütze, 2004, for a similar study).

The basic idea of multi-objective continuation methods is, roughly speaking, to move *along* the set of (local) Pareto points. To be more precise, over the course of the algorithm, one is faced with the following setting: given a (locally optimal) solution  $x_0 \in Q$  and a search direction  $v \in \mathbb{R}^n$  with  $\|v\| = 1$  (obtained via linearization of the solution set at  $x_0$ ), the task is to find a suitable step size  $h_0 \in \mathbb{R}_+$  for the next guess  $y_0 = x_0 + h_0v$ . Motivated by previous considerations, one can ask, for example, for a step size  $h_0$  such that

$$\|F(x_0) - F(y_0)\|_\infty = \Delta, \tag{5.6}$$

where  $\Delta$  is the tightness value taken in Algorithms 4 and 5 [alternatively,  $\Delta$  in Equation (5.6) can be replaced, e.g., by  $\|\epsilon\|_\infty$  for Algorithms 2 and 3]. Following the discussion made above and using a local approximation of  $L_{x_0}$  as in Equation (5.3) or (5.4), one can use the following step size control:

$$h_0 = \frac{\Delta}{L_{x_0}} \tag{5.7}$$

Note that this estimation only holds for small values of  $\Delta$ , since the  $L_{x_0}$  is a local approximation. If  $\Delta$  is too large,  $L_{x_0}$  cannot serve as a suitable Lipschitz estimation, and the value of  $h_0$  may not be suitable. The following two examples show, however, that the control in Equation (5.7) can be beneficial for small values of  $\Delta$ . Such a step size control would be interesting for hybrids of continuation methods with EMO strategies (e.g., Harada et al., 2007; Schütze et al., 2008a) since in this case the archivers presented above could efficiently be integrated into the entire algorithm as external archives, which we want to demonstrate in the following two examples.

### 5.1 Example A

In order to understand the possible impact of the discussion made above on the continuation methods, we first apply the step size control for an academic example:

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$F(x) = \begin{pmatrix} (x_1 - 1)^4 + (x_2 - 1)^4 \\ (x_1 + 1)^2 + (x_2 + 1)^2 \end{pmatrix} \tag{5.8}$$

The Pareto set of the MOP in Equation (5.8) is given by

$$\mathcal{P} = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in [-1, 1] \right\}.$$

Figure 4 shows two different discretizations of  $\mathcal{P}$  and  $F(\mathcal{P})$ . In Figure 4(a), the Pareto set is approximated by points  $x_i, i = 1, \dots, N$ , which are placed equidistant in parameter space:

$$x_i = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{2i}{N} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

Next, the Pareto set was discretized using the adaptive step size control proposed above:

$$x_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad x_{i+1} = x_i + h_i \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},$$

where  $h_i$  is taken from Equation (5.7) and  $v_i = (1/\sqrt{2}, 1/\sqrt{2})^T$  was chosen as the search direction. Figure 4(b) shows the discretization points  $x_i$  for  $\epsilon = (1, 1)$ ,  $\Delta = 1$ , and  $\Theta = 0.99$ , yielding a satisfying distribution of the solutions on the Pareto front.

### 5.2 Example B

Next we consider the following MOP:

$$f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$f_i(x) = \sum_{\substack{j=1 \\ j \neq i}}^n (x_j - a_j^i)^2 + (x_i - a_i^i)^4, \tag{5.9}$$

where

$$a^1 = (1, 1, 1, 1, \dots) \in \mathbb{R}^n$$

$$a^2 = (-1, -1, -1, -1, \dots) \in \mathbb{R}^n,$$

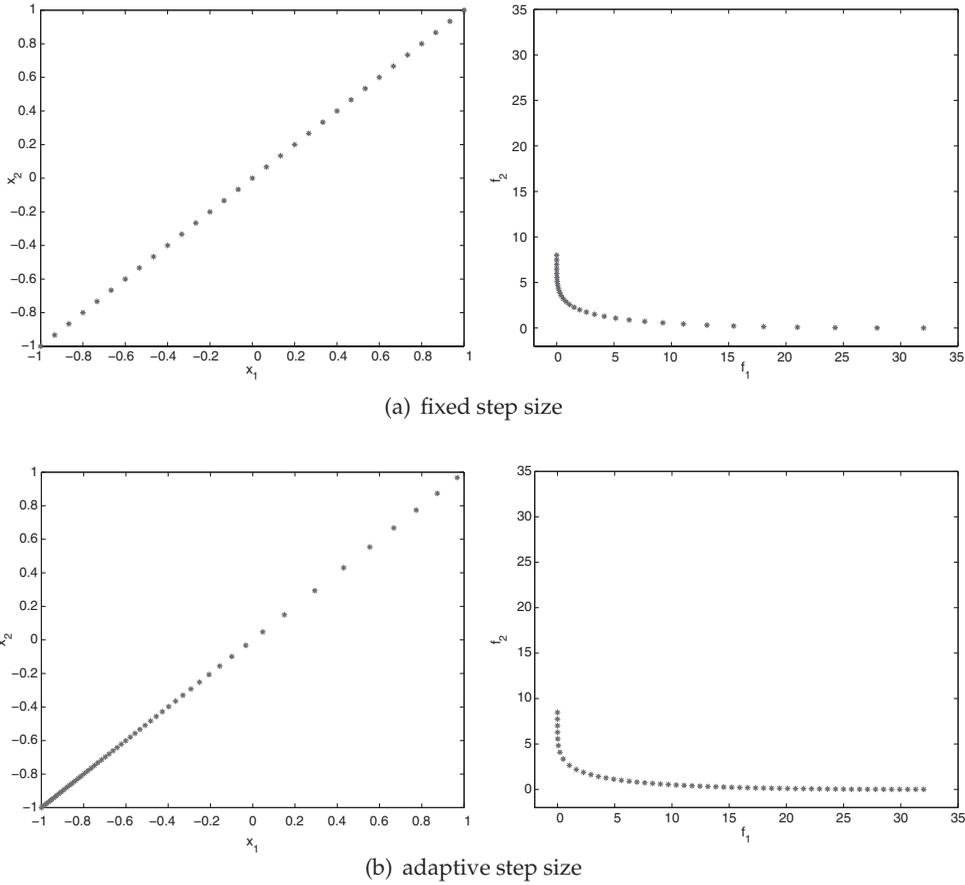


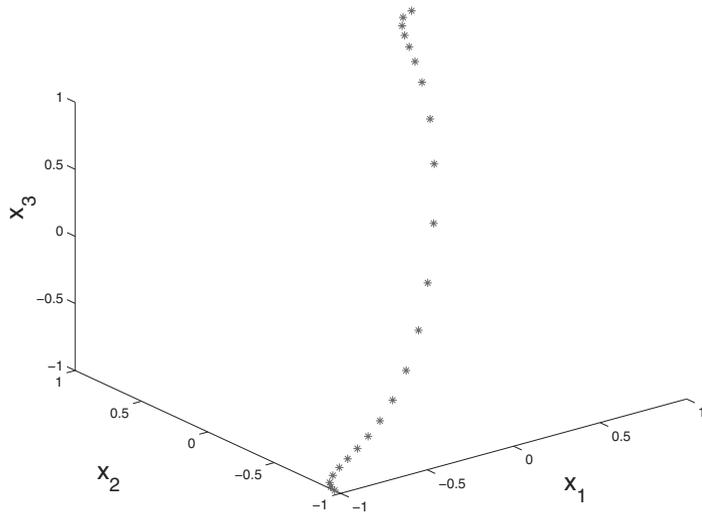
Figure 4: Discretizations of the Pareto set of the MOP in Equation (5.8) with (a) fixed step size and (b) adaptive step size control.

In Figures 5 and 6, some numerical results are presented, where we have used the continuation method proposed in Schütze et al. (2005). To be more precise, we have applied the step size control on the distance between the current solution and the predictor, since this point mainly determines the distance of two solutions.

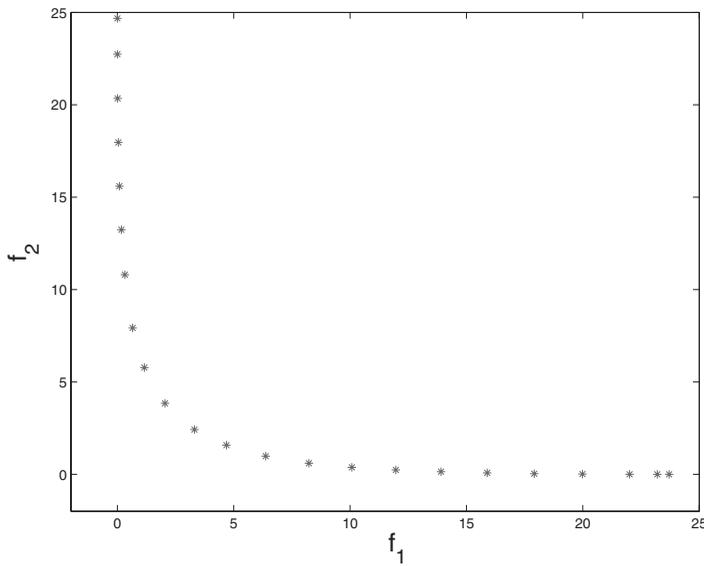
Figure 5 shows the result for  $n = 3$ ,  $\epsilon = (2, 2)$ , and  $\Delta = 2$ . In total, 23 solutions were obtained. This fits quite well with the bound given in Section 4.3 when choosing  $\Theta = 1$  and when replacing  $M_i$  by the maximal value of  $f_i$  on the Pareto set,  $\tilde{M}_i := \max_{x \in P_Q} f_i(x) = 24$ ,  $i = 1, 2$ , and  $m_i = 0$ . In that case we obtain:

$$|A_i| \leq \left\lceil \frac{24 - 0}{2} \right\rceil + \left\lceil \frac{24 - 0}{2} \right\rceil = 24. \tag{5.10}$$

This example shows the significant difference between the archivers *ArchiveUpdate-Epsi*,  $i = 1, 2$  which are merely based on  $\epsilon$ -dominance and the novel strategies that aim for a tightness of the approximation: if points with images near to the middle of the Pareto front in Figure 5(b) are inserted into the archive, for example, the points  $m_1$  with



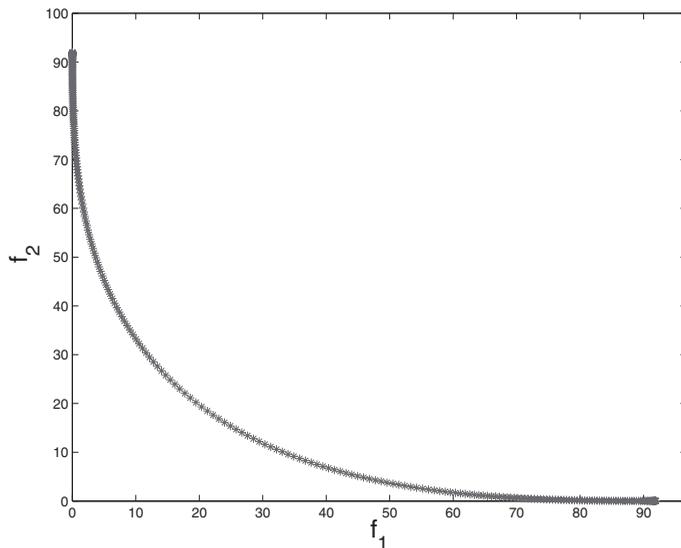
(a) Parameter space



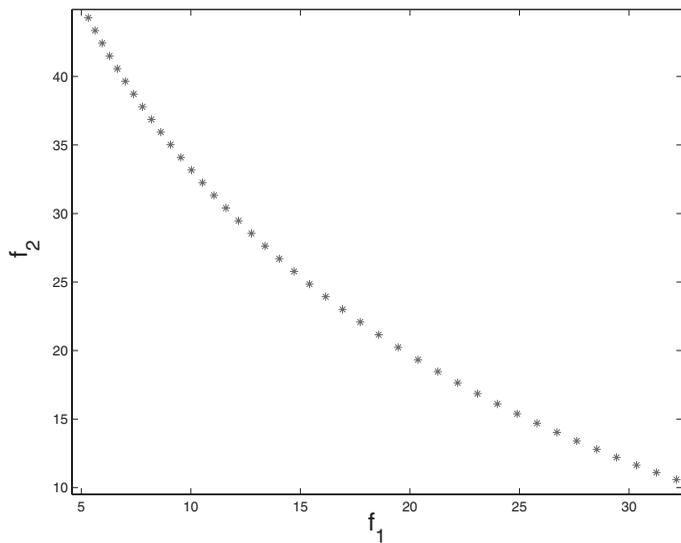
(b) Image space

Figure 5: Result of the continuation method with step size control on the MOP in Equation (5.9) for  $n = 3$  in (a) parameter space and (b) image space.

$F(m_1) = (5, 2)$  and  $m_2$  with  $F(m_1) = (2, 5)$ , then no more points with images at the ends of the Pareto curve will be added further on, since these are all  $\epsilon$ -dominated either by  $m_1$  or  $m_2$ . The resulting approximation would form an  $\epsilon$ -approximate Pareto set, but would apparently not describe the appropriate Pareto set graphically. This would change, however, for one of the novel archiving strategies, which we investigate in more detail in the next section.



(a) All solutions in objective space

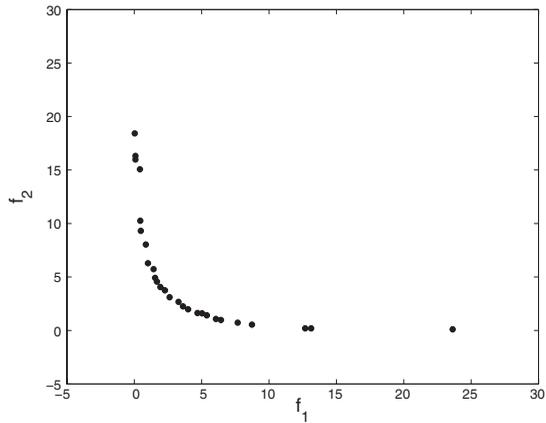


(b) Zoom of (a)

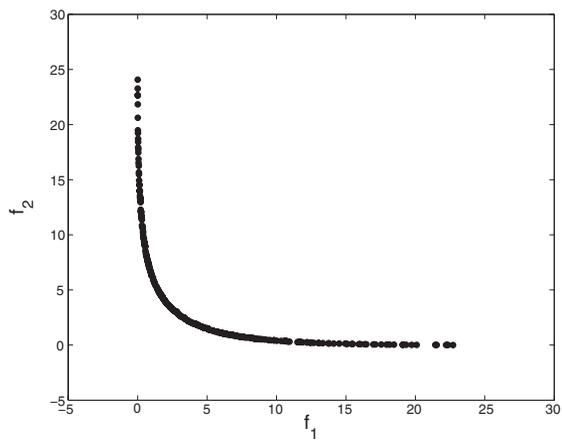
Figure 6: Result of the continuation method with step size control on the MOP in Equation (5.9) for  $n = 20$  in image space: (a) all solutions and (b) zoom.

## 6 Numerical Results

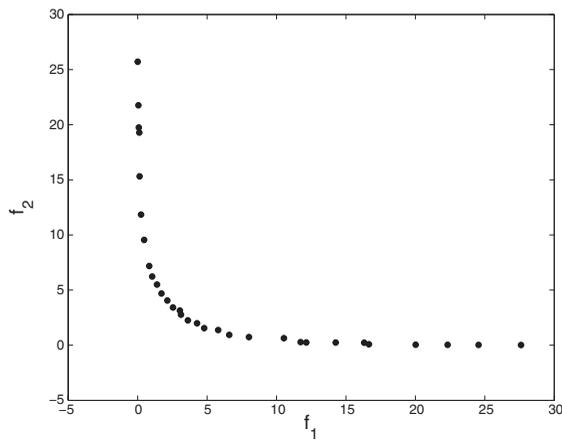
Here we make a comparative study on three test problems in order to illustrate the effect of the different archiving strategies. For the subsequent comparisons, we have used the following archiving strategies:



(a) *ArchiveUpdateEps1*



(b) nondominated solutions



(c) *ArchiveUpdateTight1*

Figure 7: Three limit archives for the MOP in Equation (5.9) obtained by different archiving strategies.

Table 1: Comparison of the magnitudes of the final archive ( $|A_N|$ , rounded) and the corresponding update times ( $T$ , in seconds) for different archiving strategies and for the MOP in Equation (5.9). We have taken the average result of 100 test runs.

	ND	Eps <sub>1</sub>	Tight <sub>1</sub>
$ A_N $	267	27	34
$T$	36.46	0.29	0.36

- (ND) *ArchiveUpdateND*,
- (Eps1) *ArchiveUpdateEps1*,
- (Tight1) *ArchiveUpdateTight1*, and
- (Tight2) *ArchiveUpdateTight2*,

where *ArchiveUpdateND* is the archiver that stores all nondominated solutions, that is,

$$ArchiveUpdateND(P, A_0) := \{x \in P \cup A_0 : y \not\prec x \ \forall y \in P \cup A_0\}.$$

For an investigation of the convergence properties of *ArchiveUpdateND*, we refer to Schütze et al. (2008a).

In order to obtain a fair comparison of the different archivers, we have decided to take a random search operator for the generation process (the same sequence of points for all settings). The computations have been done on an Intel Xeon 3.2 GHz processor. An implementation of all the archiving strategies discussed in this work including these examples can be found in <http://paradiseo.gforge.inria.fr/index.php?n=Paradiseo.Features>.

### 6.1 Example 1

First, we compare the first three different archiving strategies on the MOP in Equation (5.9) from the previous example. We have taken  $N = 200,000$  randomly chosen points in  $Q = [-1.5, 1.5]^3$ , and the values  $\epsilon = (1, 1)$  and  $\Delta = 2$ . The set obtained by *ArchiveUpdateEps1* forms probably (or is near to) an  $\epsilon$ -approximate Pareto set, but reveals gaps, which is not the case in Figure 7(c), where *ArchiveUpdateTight1* has been used, and where the solutions are much more regularly distributed. The tightest approximation in this case study is certainly obtained when all nondominated points are kept in the archive [see Figure 7(b)]. However, in that case, the time that had to be spent to update the archive<sup>3</sup> was huge compared to the two other strategies (see Table 1).

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<sup>3</sup>The elements of all the archives were stored using a linear list.

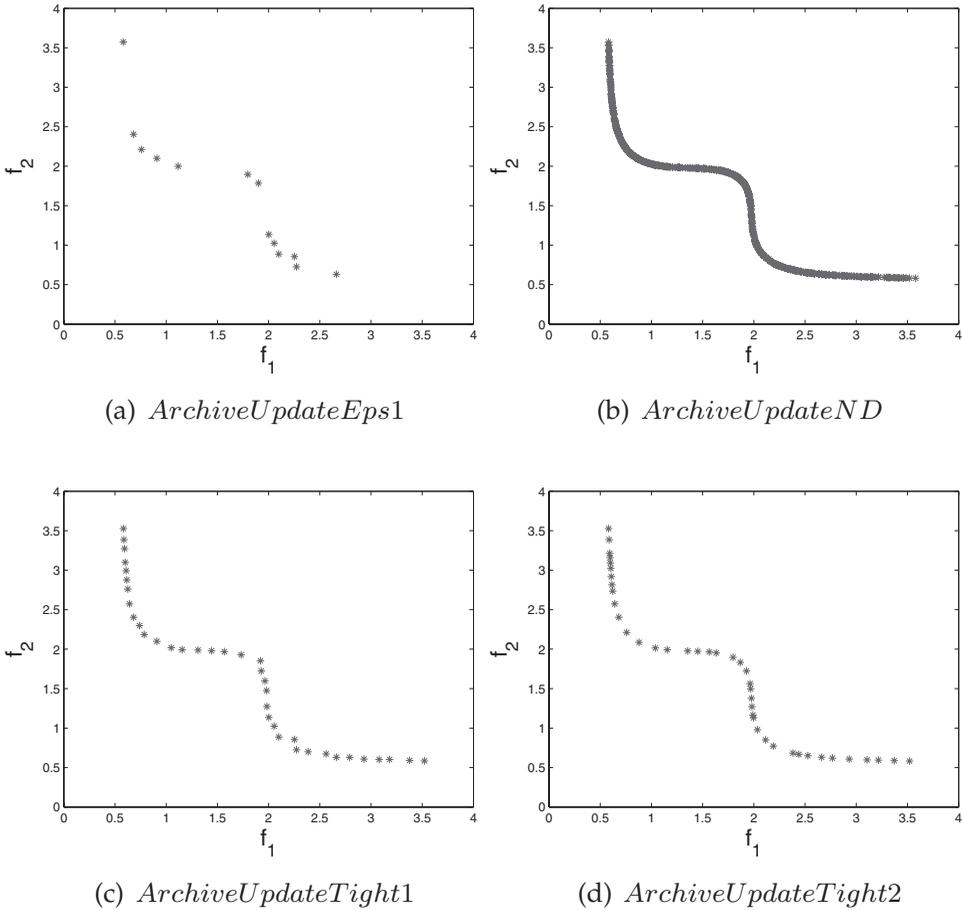


Figure 8: Four limit archives for the MOP in Equation (6.1) obtained by different archiving strategies. The magnitudes are  $|A_{\text{eps1}}| = 13$ ,  $|A_{\text{ND}}| = 511$ ,  $|A_{\text{tight1}}| = 37$ ,  $|A_{\text{tight2}}| = 41$ . See Table 2 for their approximation qualities.

### 6.2 Example 2

Next, we consider the following parameter dependent MOP (Witting and von Molo, 2006):

$$\begin{aligned}
 f_1, f_2 &: \mathbb{R}^2 \rightarrow \mathbb{R} \\
 f_1(x, y) &= \frac{1}{2}(\sqrt{1 + (x + y)^2} + \sqrt{1 + (x - y)^2} + x - y) + \lambda \cdot e^{-(x-y)^2} \\
 f_2(x, y) &= \frac{1}{2}(\sqrt{1 + (x + y)^2} + \sqrt{1 + (x - y)^2} - x + y) + \lambda \cdot e^{-(x-y)^2}
 \end{aligned} \tag{6.1}$$

Figure 8 shows examples for the resulting limit sets. Hereby, we have taken  $N = 10,000$  randomly chosen points in  $Q = [-1.5, 1.5]^2$  and  $\lambda = 0.85$  for the value of the additional

Table 2: Distances of the images of the final archives to the approximation  $F(A_{\mathcal{P}})$  of the Pareto front of the MOP in Equation (6.1) averaged over 100 test runs. The Hausdorff distances of the solutions obtained by *ArchiveUpdateTight1* and *ArchiveUpdateTight2* are nearly to the optimum which is given here by  $\Delta = 0.1$ .

	Eps1 <sub>1</sub>	Tight1	Tight2	ND
$dist(F(A_{final}), F(A_{\mathcal{P}}))$	0.1029	0.1029	0.0079	0.0135
$dist(F(A_{\mathcal{P}}), F(A_{final}))$	0.9209	0.1092	0.1042	0.0290
$d_H(F(A_{final}), F(A_{\mathcal{P}}))$	0.9209	0.1092	0.1042	0.0290

parameter, as well as the values  $\epsilon = (0.1, 0.1)$ , and  $\Delta = 0.1$  for the archiving strategies (using  $\tilde{\Delta} = \Delta$  and  $\Theta = 1$ ). Also, in this case, the result of *ArchiveUpdateEps1* reveals gaps in the approximation, which does not occur when using *ArchiveUpdateTight1* or *ArchiveUpdateTight2*. Note that the differences of the latter two solutions are subtle as expected: both approximations are similar, but the final solutions obtained by *ArchiveUpdateTight2* have converged better. The difference in the magnitudes of the archive sizes in this example (and in all other examples examined by the authors) do not differ significantly, though the theoretical upper bounds for both strategies do. Also in this example, *ArchiveUpdateND* delivers the tightest approximation, containing by far the most elements, which results in a longer running time for the update process. Since the Pareto set of this MOP is given analytically by

$$\mathcal{P} = \left\{ \begin{pmatrix} x \\ -x \end{pmatrix} : x \in [-1.5, 1.5] \right\} \tag{6.2}$$

this allows us to have a closer look at the approximation qualities of the obtained solutions. Figure 8 shows the distances between the final archives and the Pareto front. For the latter we have used the following discretization of Equation (6.2):

$$A_{\mathcal{P}} := \left\{ \begin{pmatrix} x_i \\ -x_i \end{pmatrix} : x_i = -1.5 + 3i/500, i = 0, \dots, 500 \right\} \tag{6.3}$$

The Hausdorff distances of the two final archives of the novel archivers are close to 0.1, which is the optimum in this case, since this is the chosen value of  $\Delta$ . The main difference between the two solutions is the value of  $dist(F(A_{final}), F(A_{\mathcal{P}}))$ , which means that the elements obtained by *ArchiveUpdateTight2* are nearer to the Pareto front. The gaps that can be observed when using *ArchiveUpdateEps1* [Figure 8(a)] are reflected by its relatively large value of  $dist(F(A_{final}), F(A_{\mathcal{P}}))$  in Table 2. Best values are obtained by *ArchiveUpdateND*, but this has to be paid by the much larger amount of elements in the archive. Further, it can be argued that approximations with Hausdorff distances less than 0.1 (the values of  $\Delta$  and  $\epsilon_i, i = 1, 2$ ) to the Pareto front are not needed.

### 6.3 Example 3

Finally, we consider a three-objective model. For this, we extend the MOP in Equation (5.9) by adding a third objective which is analogous to  $f_1$  and  $f_2$ , and choose  $a_3 = (1, -1, 1, -1, \dots) \in \mathbb{R}^n$ . In order to demonstrate one possible benefit of  $\epsilon$ -dominance

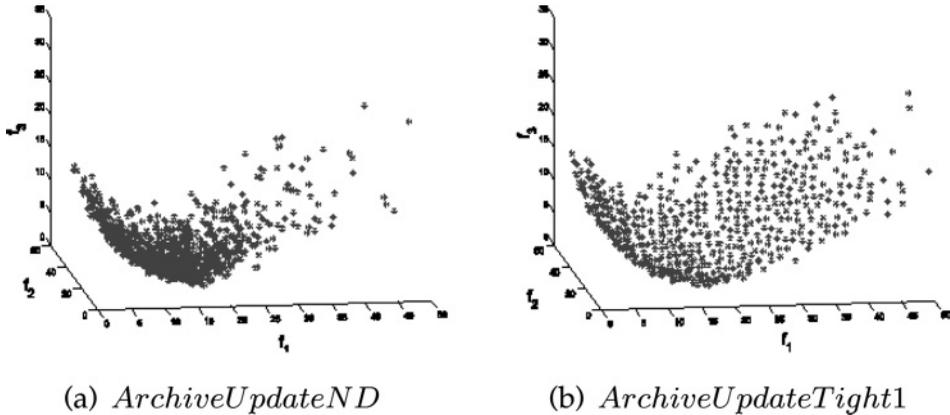


Figure 9: Two different solution sets for the MOP in Example 3. We have chosen  $\epsilon = (1, 1, 1)$ ,  $\Theta = 1$ , and  $\Delta = 5$ .

based archivers against the classical archiver *ArchiveUpdateND*, which stores all non-dominated solutions, we fix in this example the running time of the different algorithms (i.e., in our case, the stochastic search algorithm coupled with the different archiver). Figure 9 shows one comparative result for  $n = 10$  and where the running time was fixed to 5 min. Denote by  $A$  the final archive when *ArchiveUpdateND* was used, and by  $B$  the resulting archive coming from *ArchiveUpdateTight1*. The magnitudes are  $|A| = 914$  and  $|B| = 529$ . It can be observed that compared to  $A$ , the spread of the solutions of  $B$  is much better, while a larger region of the image space is covered although its magnitude is less. In order to measure the approximation quality, we use the epsilon indicator (Zitzler et al., 2003), where  $I_\epsilon(A, B)$  gives the smallest value of  $\bar{\epsilon} \in \mathbb{R}$  such that  $A$  is an  $\epsilon$ -approximate Pareto set of  $B$  where  $\epsilon = (\bar{\epsilon}, \dots, \bar{\epsilon})$ , that is,

$$I_\epsilon(A, B) := \min\{\bar{\epsilon} \in \mathbb{R} \mid \forall b \in B \exists a \in A : a \prec_\epsilon b\}. \tag{6.4}$$

In our case, we obtain for  $A$  and  $B$

$$I_\epsilon(A, B) = 0.9624 \quad \text{and} \quad I_\epsilon(B, A) = 0.85775, \tag{6.5}$$

indicating that  $B$  is a (slightly) better approximation of the Pareto front than  $A$ . If the relative  $\epsilon$ -dominance

$$x \prec_\epsilon^{\text{rel}} y \Leftrightarrow F(x) \leq_p (1 + \epsilon)F(y) \tag{6.6}$$

is used for the epsilon indicator, the difference of the approximation qualities becomes more significant:

$$I_\epsilon^{\text{rel}}(A, B) = 1.082 \quad \text{and} \quad I_\epsilon^{\text{rel}}(B, A) = 0.183. \tag{6.7}$$

These values can be interpreted as follows: if the objective values of  $A$  are scaled up to 20%, then  $A$  is entirely dominated by  $B$ . On the other hand,  $B$  has to be scaled up by more than 100% (i.e., the values have to be doubled) such that it is dominated by  $A$ .

The main reason for the difference of the approximations is that an archiver based on  $\epsilon$ -dominance accepts in general less solutions than all nondominated ones, which makes the update process much faster, and thus, more points can be evaluated by the generator within the given time budget. In this case, the algorithm using *ArchiveUpdateND* evaluated  $6.6 \times 10^5$  different test points while  $3.5 \times 10^7$  points were evaluated using *ArchiveUpdateTight1* within the same time.

## 7 Conclusions and Future Work

We have proposed two archiving strategies for obtaining finite size and gap free (or tight) Pareto front approximations by stochastic search algorithms and have proven the convergence of the resulting archives. The limit set using the first archiver forms with probability 1 a  $(\Delta, \Theta\epsilon_m)$ -tight  $\epsilon$ -approximate Pareto set, that is, a gap free Pareto front approximation, measured by the value of  $\Delta$ , that provides the guaranteed uniformity level  $\Theta\epsilon_m$ . The limit set of the second strategy forms a  $\Delta$ -tight  $\epsilon$ -Pareto set, which offers a better approximation quality measured in the Hausdorff-sense, but in turn lacks uniformity.

For future work, the development of an archiving strategy that produces a sequence of archives leading to a  $(\Delta, \epsilon_m)$ -tight  $\epsilon$ -Pareto set would be of particular interest. It could also be interesting to integrate the archiving strategies directly into the stochastic search process (as done, e.g., in Deb et al., 2005, for an evolutionary algorithm) in order to obtain a fast and reliable multi-objective optimization algorithm. Finally, the analysis of the archiving strategies could be advanced. The main focus in this article was on the limit behavior of the sequence of archives, but there are also further interesting topics worth investigating. One question that naturally arises is the speed of the convergence. Related works for single-objective optimization problems (e.g., Voigt et al., 1995, Rudolph, 1997a, 1997b) show that this is not straightforward.

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