

## Sheaf Theory through Examples



# Sheaf Theory through Examples

Daniel Rosiak

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To Fernando Zalamea, who guided my early foray into sheaves, and whose encouragement and example helped my love of category theory flourish.



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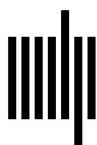
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