The Bivariate Gompertz Diffusion Model for Tree Diameter and Height Distribution

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Abstract: With this research we present a new method for describing the bivariate diameter and height distribution of trees growing in a pure, uneven-aged forest. We use a stochastic differential equation framework to derive a bivariate age-dependent probability density function of tree diameter and height when the tree diameter and height follow a bivariate stochastic Gompertz shape growth process. We also adopt the two-dimensional transition probability function methodology for growth modeling of forest stands. The bivariate stochastic Gompertz model is fit to diameter and height observations for 1,575 pine trees in the Dubrava district of Lithuania. A considerable advantage of the bivariate stochastic Gompertz growth model is that the model parameters are easily interpretable. All results are implemented in the symbolic algebra system MAPLE. For. Sci. 56(3):271–280.

Keywords: bivariate Gompertz process, bivariate lognormal distribution, correlation function, pseudoresiduals

FOREST GROWTH MODELING is an intrinsic part of forest management. Forest growth has usually been modeled either with nonlinear logistic shape models or with empirical models (e.g., polynomial model). However, empirical models are devoid of any biological interpretation. For determining individual tree biomass and volume, the age of the tree takes priority among the growing stock properties. It is evident that forest stands grow under diverse environmental conditions and undergo changes in trend. Thus, detailed modeling of the size of a tree cannot be properly described by a single value but needs an age-dependent distribution for determining the relationship between the distribution of the size of a tree and the age of a tree. The traditional modeling methods do not quantify the distribution of the size of a tree subject to the age. Therefore, a novel approach for modeling the age-dependent distribution of the size of a tree is to use the Feller diffusion process, which has been widely used in the financial and actuarial fields.

Any forest stand consists of trees with different heights and diameters. Those differences depend on many concealed, underlying genetic and environmental factors, leading to an assumption that the diameter and height of trees are random variables. The joint bivariate distribution of tree diameter and height gives us a detailed, complex view of diameter and height, whereas the marginal distribution of tree diameter (height) only provides an independent view of diameter (height). The joint distribution function of diameter and height is very important for assessing timber value. Within forestry applications, the fitting of the bivariate probability distribution follows two methodologies. The first is the elliptical methodology, in which the bivariate density function is a function of quadratic forms of the marginal variables (Schreuder and Hafley 1977, Zucchini et al. 2001, Wang and Rennolls 2007, and references therein). The second is the copula approach, in which the copula density function characterizes the dependency structure separately from marginal variables (Wang et al. 2008). However, with the exception of a normal copula, the copula approach does not correspond well to the diameter and height structure arising from uneven-age stands (Wang et al. 2008).

In this work, we present the use of system stochastic differential equations (SDEs) in forestry. The methodology considers a bivariate distribution as arising from a bivariate (diameter and height) stochastic dynamical system. Stochastic dynamical system models have played an important role in scientific fields such as physics, biology, and economics. The system fluctuations, generally infiltrated from outside, are defined by a two-dimensional standard Brownian motion. The presence of correlation between two noises in the system of SDEs affects the dynamics of tree diameter and height. We show that a two-variable system of SDEs is the right model in continuous time to account for tree diameter and height distribution and that it can be obtained explicitly by solving the Fokker-Planck equation (also known as the Chapman-Kolmogorov equation) (see Risken 1989). The Gompertz growth model has been used in many scientific fields as a basic model of bactericidal kinetics (Ferrante et al. 2005), CO2 emissions (Gutiérrez et al. 2008), stand growth (Rupšys et al. 2007), and others. Next we consider a bivariate stochastic Gompertz shape model of diameter and height. In our coupled system of SDEs, the noise affects the dynamics of tree diameter and height. The main objective of our investigation was to study the transition probability density function of tree diameter and height. We provide the following new results. First, in the case of...
bivariate stochastic Gompertz diffusion process we show an age-dependent bivariate transition probability density function (BDF) of tree diameter and height. Finally, numerical illustrations indicate that our derived BDF and marginal transition probability density functions of tree diameter and age fit well.

### Bivariate Stochastic Gompertz Process

Biology literature commonly uses allometric laws to describe the relationship between various parameters of growth processes in living organisms. In this article, we will focus on the dynamics of tree diameter and height distribution of forest stands subject to the age changes. We assume that the dynamics of tree diameter and height can be expressed in terms of the bivariate Gompertz shape stochastic differential equation with multiplicative noise. Let us consider a bivariate diameter and height stochastic growth process facing stochastic fluctuations in the following (Itô 1942) system of SDEs:

\[
dX(t) = A(X(t))dt + Q(X(t))^{1/2}dW(t),
\]

where \(X(t) = (D(t), H(t))^T\), \(T\) denotes transposition, \(t \in [t_0; T]\), \(t_0 \geq 0\), \(D(t)\) is a breast height diameter (denoted as diameter in the following) at age \(t\), \(H(t)\) is a height at the age \(t\), \(X(t_0) = x_0 = (d_0, h_0)^T\) is a fixed vector \(d_0 \equiv 0\), \(h_0 \equiv 0\), the vector \(A(x)\) and the matrix \(Q(x) (x = (d, h)^T)\) are defined by

\[
A(x) = (\alpha d - \beta d \ln(d), \alpha h - \beta h \ln(h))^T,
\]

\[
Q(x) = (C(x)B^{1/2})(C(x)B^{1/2})^T = C(x)BC(x) = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}dh \\ \sigma_{12}dh & \sigma_{22}h^2 \end{pmatrix},
\]

\[
C(x) = \begin{pmatrix} d & 0 \\ 0 & h \end{pmatrix}, \quad B = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix},
\]

and \(\{W(t); t \in [t_0; T]\}\) is a bivariate standard Brownian motion. The bivariate Brownian motion can be described by a vector \(W(t) = (W_1(t), W_2(t))^T\), where \(W_1(t)\) and \(W_2(t)\) are univariate standard Brownian motions. The bivariate standard Brownian motions have the following three properties: no uncertainty at time \(t = 0\), \(W(0) = (W_1(0), W_2(0))^T = (0, 0)^T\); the process \(\{W(t) = (W_1(t), W_2(t))^T, t \in [t_0; T]\}\) has independent increments (the next increment does not depend on the present or past location); and \(W(t) - W(s)\) follows a bivariate normal distribution with zero mean and with variance equal to the length of the time interval over which the increment takes place, \(t - s\). The model parameters \(\alpha_1, \alpha_2, \beta, \sigma_{ij}, 1 \leq i, j \leq 2\) are unknown real numbers to be estimated, \(\beta\) is the intrinsic growth rate of both state variables diameter and height, the quantities \(e^{\alpha_2/\beta}\) and \(e^{\alpha_1/\beta}\) outline the saturation level of diameter and height, \(\sigma_{11}, \sigma_{22} \geq 0\) are the intensities of the Brownian motions \(W_1(t)\) and \(W_2(t)\), respectively, \(\sigma_{12} = \sigma_{21} = \rho \sqrt{\sigma_{11}\sigma_{22}}\), and \(\rho\) is the parameter characterizing the cross-correlation of the Brownian motions, \(|\rho| \leq 1\). Thus, the matrix \(B\) controls how fast the transition probability density converges to a stationary density. Notice that this formulation has the theoretical advantage of never becoming negative in any of the tree diameter and height coordinates.

The questions of the existence and the uniqueness of the strong solutions have been studied in Chung and Williams (1990) and Gutiérrez et al. (2008). Let us denote by \(p(d, h, t)\) the BDF of the bivariate stochastic process \((D(t), H(t))\) defined by Equation 1. This density obeys the following Fokker-Planck equation:

\[
\frac{\partial p(d, h, t)}{\partial t} = -\frac{\partial((\alpha_1 d - \beta d \ln(d))p(d, h, t))}{\partial d} - \frac{\partial((\alpha_2 h - \beta h \ln(h))p(d, h, t))}{\partial h} + \frac{1}{2} \left( \frac{\partial^2(\sigma_{11}d^2p(d, h, t))}{\partial d^2} + 2 \frac{\partial(\sigma_{12}dp(d, h, t))}{\partial dh} + \frac{\partial(\sigma_{22}h^2p(d, h, t))}{\partial h^2} \right).
\]

Satisfying Equation 2, the BDF \(p(d, h, t)\) of the bivariate stochastic process \(X(t) = (D(t), H(t))^T\) has a bivariate lognormal distribution \(\Lambda_\mathbf{2}(\mu(t), \Sigma(t))\) (see Gutiérrez et al. 2008) with the mean vector

\[
\mu_1(t) = \ln d_0 e^{-\beta(t-t_0)} + \frac{1 - e^{-\beta(t-t_0)}}{\beta} \left( \alpha_1 - \frac{\sigma_{11}}{2} \right), \quad \mu_2(t) = \ln h_0 e^{-\beta(t-t_0)} + \frac{1 - e^{-\beta(t-t_0)}}{\beta} \left( \alpha_2 - \frac{\sigma_{22}}{2} \right).
\]

and the variance–covariance matrix

\[
\Sigma(t) = \frac{1 - e^{-2\beta(t-t_0)}}{2\beta} B.
\]

Hence, the BDF of tree diameter and height has the form,

\[
p(d, h, t) = \frac{1}{2\pi dh|\Sigma(t)|^{1/2}} \exp \left( -\frac{1}{2} \Omega(d, h, t) \right),
\]
where $|\Sigma(t)|$ denotes the determinant of the matrix $\Sigma(t)$ and the quadratic form $\Omega(d, h, t)$ is defined by

$$\Omega(d, h, t) = (d - \mu_1(t), h - \mu_2(t))[\Sigma(t)]^{-1}(d - \mu_1(t)),$$

with the mean vector defined by Equation 3 and variance–covariance matrix defined by Equation 4, where $[\Sigma(t)]^{-1}$ denotes the inverse matrix of the matrix $\Sigma(t)$. The marginal distributions for the diameter and the height have also univariate lognormal distributions $\Lambda_1(\mu_i(t), \nu_i(t))$ with means $\mu_i(t)$, and $\nu_i(t)$ and variances $\sigma_{i1}^2$ and $\sigma_{i2}^2$.

Hence, the marginal distributions of tree diameter and height take the forms,

$$p(d, t) = \frac{1}{d\sqrt{2\pi
\sigma_{i1}^2}} \exp\left( -\frac{1}{2\nu_i(t)}(\ln d - \mu_i(t))^2\right), \quad (6)$$

$$p(h, t) = \frac{1}{h\sqrt{2\pi
\sigma_{i2}^2}} \exp\left( -\frac{1}{2\nu_i(t)}(\ln h - \mu_i(t))^2\right). \quad (7)$$

By retaining the initial conditions $P(D(t_0) = d_0) = 1, P(H(t_0) = h_0) = 1$, the marginal mean trend functions $d_m, h_m$, of the stochastic bivariate Gompertz process 1 of tree diameter and height are defined by

$$d_m(t) = \exp\left(\mu_i(t) + \frac{\sigma_{i1}^2}{4\beta} (1 - e^{-2\beta(t-t_0)})\right), \quad (8)$$

$$h_m(t) = \exp\left(\mu_i(t) + \frac{\sigma_{i2}^2}{4\beta} (1 - e^{-2\beta(t-t_0)})\right), \quad (9)$$

respectively. The marginal variance trend functions $d_v, h_v$ of tree diameter and height are defined by

$$d_v(t) = \exp\left(2\mu_i(t) + \frac{\sigma_{i1}^2}{2\beta} (1 - e^{-2\beta(t-t_0)})\right)\left(\exp\left(\frac{\sigma_{i1}^2}{2\beta} (1 - e^{-2\beta(t-t_0)})\right) - 1\right), \quad (10)$$

$$h_v(t) = \exp\left(2\mu_i(t) + \frac{\sigma_{i2}^2}{2\beta} (1 - e^{-2\beta(t-t_0)})\right)\left(\exp\left(\frac{\sigma_{i2}^2}{2\beta} (1 - e^{-2\beta(t-t_0)})\right) - 1\right). \quad (11)$$

The covariance function is defined by

$$\lambda(t) = \exp\left(\mu_i(t) + \mu_i(t) + \frac{\sigma_{i1}^2 + \sigma_{i2}^2}{4\beta} (1 - e^{-2\beta(t-t_0)})\right)\left(\exp\left(\frac{\sigma_{i1}^2}{2\beta} (1 - e^{-2\beta(t-t_0)})\right) - 1\right). \quad (12)$$

The correlation function $\rho(t) = (d(t), h(t))$ of the tree diameter and height process 1 is defined by

$$\rho(t) = \left(\exp\left(\frac{\sigma_{i2}^2}{2\beta} (1 - e^{-2\beta(t-t_0)})\right) - 1\right)\left(\exp\left(\frac{\sigma_{i1}^2}{2\beta} (1 - e^{-2\beta(t-t_0)})\right) - 1\right)^{1/2} \left(\exp\left(\frac{\sigma_{i2}^2}{2\beta} (1 - e^{-2\beta(t-t_0)})\right) - 1\right)^{-1/2}. \quad (13)$$

**Bivariate Stochastic Model Estimation**

Recently, using discrete sampling of the observed diffusion process, a variety of methods for statistical inference have been developed (Sørensen 2004, Jimenez et al. 2006, Gutiérrez et al. 2008, and references therein). The most common efficient method of estimation is based on the maximum likelihood procedure. From a theoretical point of view, a maximum likelihood method requires complete information about the distribution of the diameter and height process. Because of the Markov property of diffusion processes, the maximum likelihood function associated with the observed data set $(D(t_i), H(t_i)) = (d_i, h_i), t_0 < t_1 < \cdots < t_n$, $\Delta_i = t_i - t_{i-1}, i = 2, 3, \ldots, n$ takes the form

$$L(a_1, a_2, \beta, B) = \prod_{i=2}^n p(d_i, h_i, t_i, d_{i-1}, h_{i-1}, t_{i-1}) \quad (14)$$

$$= (2\pi \nu \beta)^{-\frac{n-1}{2}} |B|^\frac{n-1}{2} \prod_{i=2}^n (d_i, h_i)^{-1} \exp\left[ -\frac{1}{2} \left[ \ln x_i - e^{\beta t} \ln x_{i-1} - (1 - e^{-\beta t}) \frac{\gamma}{\beta} \right]^T \right]$$

$$\times v_i^{-2} B^{-1} \left[ \ln x_i - e^{-\beta t} \ln x_{i-1} - (1 - e^{-\beta t}) \frac{\gamma}{\beta} \right].$$
where \( p(d_i, h_i, t_i | d_{i-1}, h_{i-1}, t_{i-1}) \) is the conditional density of \((d_i, h_i)\) given \((d_{i-1}, h_{i-1})\), \( x_i = (d_{i-1}, h_{i-1})^T \),

\[
\frac{\partial^2}{\partial d_i \partial d_j} \ln p(d, h, t) = 0, \quad i, j = 1, 2, \ldots, n.
\]

In the following we assume that \( \Delta_i = 1, i = 2, 3, \ldots, n \). Taking the logarithm of the maximum likelihood Equation 14, calculating the differential of this function, and then equating them to zero, we obtain the maximum likelihood estimator of the parameter \( \beta \) in the form (Gutiérrez et al. 2008)

\[
\hat{\beta} = \ln \left( \frac{\sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} y_i \hat{\gamma}}{\sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} y_i \hat{\gamma}} \right),
\]

where \( y_i = (\ln d_i, \ln h_i)^T \). Finally, the estimators of the parameters \( \alpha_1, \alpha_2, \beta, \sigma_{ij}, 1 \leq i, j \leq 2 \) take the form (Gutiérrez et al. 2008)

\[
\hat{\alpha}_1 = \hat{\gamma}_1 + \frac{\hat{\sigma}_{11}}{2},
\]

\[
\hat{\alpha}_2 = \hat{\gamma}_2 + \frac{\hat{\sigma}_{22}}{2},
\]

\[
\hat{\beta} = \frac{1}{n-1} \sum_{i=2}^{n} (\hat{\psi}_i - \hat{\xi}_i \hat{\gamma})(\hat{\psi}_i - \hat{\xi}_i \hat{\gamma})^T,
\]

where

\[
\hat{\gamma} = \frac{1}{n-1} \sum_{i=2}^{n} \hat{\psi}_i, \quad \hat{\psi}_1 = (d_1, h_1)^T, \quad \hat{\psi}_i = \hat{\psi}_{i-1} (\ln d_i - e^{-\hat{\beta}_i} \ln d_{i-1}, \ln h_i - e^{-\hat{\beta}_i} \ln h_{i-1})^T,
\]

\[
\hat{\psi}_i^2 = \frac{1 - e^{-\hat{\beta}_i}}{2\hat{\beta}}, \quad \text{and} \quad \hat{\xi}_i = \hat{\psi}_i^2 - \frac{1 - e^{-\hat{\beta}_i}}{\hat{\beta}}.
\]

**Goodness of Fit of a Bivariate Lognormal Distribution**

Goodness-of-fit tests play an important role in applied statistics. They allow us to verify the correspondence between the estimated theoretical models and real data. The quantitative analysis of tree diameter distribution is usually based on tests, such as \( \chi^2 \), Kolmogorov-Smirnov, Anderson-Darling, and Cramer-von Mises (Thode 2002). Most of these tests are very sensitive to the presence of outliers in the observed data. If any test rejects the null hypothesis, it just means that the estimated distribution model is not a perfect representation of the observed data set. In addition, there are various measures for the deviation of an actual (empirical) distribution from its estimated distribution, such as the Reynolds error index, the absolute discrepancy, the stand stability index, the bias and standard error of the estimate, and many others (Reynolds et al. 1988, Alvarez et al. 2002, Cao 2004). These measures of goodness of fit can be used for comparisons between observed data sets and distribution models.

Statistical testing models are often based on a distributional assumption of normality. A useful technique for evaluating the normality of small and moderate size samples is the Shapiro-Wilk test statistic \( W \) (Shapiro and Wilk 1965). In this article, we test the normality of the pseudoresiduals defined by Zucchini and MacDonald (1999). The pseudoresiduals, \( r_i \), corresponding to the observation \((d_i, h_i, t_i)\) are defined in the form

\[
r_i = \Phi^{-1} \left( \int_0^{d_i} \int_0^{h_i} p(x, y, t) \, dx \, dy \right), \quad i = 1, 2, \ldots, n,
\]

where \( \Phi \) denotes the distribution function of the standard normal distribution, \((d_i, h_i, t_i)\) is the \( i \)th observation of diameter, height, and age. Let \((r = r_1, r_2, \ldots, r_n)\) denote an \( n \) dimensional vector of ordered pseudoresiduals. Thus, given an assumption that the transition probability density function \( p(d, h, t) \) of tree diameter and height is indeed the correct function for the observed data set \((d_i, h_i, t_i), 1, 2, \ldots, n\), the pseudo residuals \((r = r_1, r_2, \ldots, r_n)\) follow the standard normal distribution. The analysis of variance test for normality presented by Shapiro and Wilk (1965) is defined by statistic \( W \) as

\[
W = \frac{\left( \sum_{i=1}^{n} m_ir_i \right)^2}{\sum_{i=1}^{n} (r_i - \bar{r})^2},
\]

where \( m_i, i = 1, 2, \ldots, n \) are constants generated from the means, variances, and covariances of the order statistics of a sample
of size n from a standard normal distribution. That is if \((Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)})\) is an ordered sample of size n from a standard normal distribution, then \(a_i = \text{E}(Z_{(1)}), 1, 2, \ldots, n, s_{i,j} = \text{Cov}(Z_{(i)}, Z_{(j)}), i = 1, 2, \ldots, n, j = 1, 2, \ldots, n, a = (a_1, a_2, \ldots, a_n), V = [s_{i,j}],\)

\[
(m_1, m_2, \ldots, m_n) = \frac{a^2V^{-1}}{\sqrt{a^2V^{-1}V^{-1}}}.
\]

The coefficients \(m_i (i = 1, 2, \ldots, n)\) are tabulated in Shapiro and Wilk (1965). For \(n > 50\), the coefficients are approximated by the methods given in Shapiro and Wilk (1965) and Royston (1982). For residuals following a standard normal distribution the value of the statistic \(W\) tends to be close to 1 and, on the contrary, tends to be small if residuals are from a non-normal distribution.

A normal probability plot of pseudoresiduals is constructed by plotting \(r_i(a)\) against \(a_i\). The normal probability plot of pseudoresiduals enables us to visually evaluate the fit of the estimated diameter and height distribution to the observations.

**Methods**

**Data**

The tree diameter and height analysis is based on measurements in pine stands in the Dubrava district of Lithuania. The data were provided by the Lithuanian Forest Survey Service, which consists of a systematic sample of plots distributed on a square grid of 5 km. Over a 20-year period (1976–1996) in the uncut stands, sample plots were remeasured five times. In the rest of the sample plots the number of remeasurements ranged from one to four times depending on the time of the carried cuttings. The sample method used 500 m² circular plots and rectangular plots (from 0.25 to 0.4 ha). The following variables were measured: age (\(t\)), number of trees per ha, dbh (\(d\)), tree position (coordinates \(x, y\)), height (\(h\)), and descriptive variables, such as alive or dead, were also recorded. Approximately 30% of the sample trees on each plot were randomly selected for the height measurement. Height was measured to the nearest 0.1 m with a digital height meter. These observations are represented in Figure 1.

Altogether 119 pine trees were felled on inventory plots to obtain tree volume components. Minimum and maximum values, means and standard deviations of the central sample tree measurement characteristics (\(d, h,\) and \(t\)), and volumes (\(V, m^3\)) are presented in Table 1.

We shall now illustrate the observed data set by characterizing the underlying diameter and height dynamics as a stochastic bivariate Gompertz process with the multiplicative noise. For model estimation observations on 1,575 pines were used.

**Parameter Estimation**

In the forestry literature bivariate distribution models are usually estimated separately on diameter and height by the maximum likelihood procedure or other methods (see Wang et al. 2008). The SDF method enables us to write the maximum likelihood estimators in a closed form for the bivariate lognormal transition probability density function 5. All of the parameters \(\alpha_1, \alpha_2, \beta, \sigma_{i,j}, 1 \leq i, j \leq 2\) of stochastic differential equations (Equation 1) were estimated simultaneously. A MAPLE program was used to perform calculations. With use of the observed data set presented in Figure 1 and the estimators defined by Equations 15–18, the parameters of model 1 were estimated.

To assess the standard errors of the maximum likelihood estimators, a study of the Fisher information matrix (Fisher 1922) was performed. The asymptotic variance of the maximum likelihood estimator is given by the inverse of the Fisher information matrix, which is the lowest possible

![Figure 1. Plot of the observed data set from pine forests in Lithuania.](https://academic.oup.com/forestscience/article/56/3/271/4604155)
achievable variance among the competing estimators. Define $\rho(\theta) = (L(\theta))$, where $\theta = (\alpha_1, \alpha_2, \beta, \sigma_{11}, \sigma_{12}, \sigma_{22})$, and $L(\theta)$ is defined by Equation 14, the $6 \times 1$ vector $\rho(\theta) = \frac{\partial p(\theta)\partial \theta}{\partial \theta}$, and the $6 \times 6$ matrix $\rho(\theta)'' = \left[\frac{\partial^2 p(\theta)}{\partial \theta \partial \theta}\right]^T$. We have that $n^{1/2}(\hat{\theta}_n - \theta) \rightarrow^d N(0, [i(\theta)^{-1}]^{-1})$, where Fisher’s information matrix is

$$i(\theta) = E(p'(\theta)p'(\theta)^T) = -E(p''(\theta)).$$

The standard errors of the maximum likelihood estimators are defined by diagonal elements of the matrix $[i(\theta)^{-1}]$.

## Results and Discussion

The values of the maximum likelihood estimators (standard errors) are

$$\hat{\beta} = 0.06106(0.00257),$$

$$\begin{pmatrix}
\hat{d}_1 \\
\hat{d}_2
\end{pmatrix} = \begin{pmatrix}
0.21706(0.00928) \\
0.20369(0.00877)
\end{pmatrix},$$

$$\begin{pmatrix}
\hat{B} \\
\hat{B}
\end{pmatrix} = \begin{pmatrix}
0.02045(0.00036) & 0.0942(0.00026) \\
0.0942(0.00026) & 0.01378(0.00024)
\end{pmatrix}.$$  

$T$ values of all parameters are very high (between 23.2 and 57.6).

Figure 2 illustrates the estimated marginal mean trend function (EMTF) and the estimated standard deviation trend function (ESDTF) of diameter and height, defined by Equations 8–11. In each case, the EMTFs appears to fit as well as any asymmetric sigmoid curve could be expected to fit. The Gompertz shape deterministic growth model has two parameters that are easily interpretable. The parameter $\beta$ outlines the intrinsic growth rate of both state variables diameter and height. If, for convenience, we define the new parameters $K_1 = e^{\alpha_1/\beta}$ and $K_2 = e^{\alpha_2/\beta}$, then these new parameters outline the diameter and height-carrying capacities and form a numerical upper bound on the diameter and height size. In both cases the EMTFs of diameter and height monotonically evolve to carrying capacity, 34.98 cm per diameter and 28.12 m per height, and the ESDTFs of diameter and height monotonically evolve to the equilibrium value of standard deviation of diameter and height.

The estimated correlation function (ECF) is presented in Figure 3. The ECF plays a major role in modeling the dependence among diameter and height. We notice that the ECF is of a particular form. The ECF decreases as the age $t$ increases, indicating that the denser stands with smaller trees show a higher correlation between diameter and height.

### Table 1. Characteristics of the destructively sampled trees

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
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<td>$d$ (cm)</td>
<td>6.2</td>
<td>50.2</td>
<td>22.1</td>
<td>9.2</td>
</tr>
<tr>
<td>$h$ (m)</td>
<td>7.9</td>
<td>27.3</td>
<td>21.4</td>
<td>7.0</td>
</tr>
<tr>
<td>$t$ (years)</td>
<td>20</td>
<td>90</td>
<td>59</td>
<td>27</td>
</tr>
<tr>
<td>$V$ (m$^3$)</td>
<td>0.0155</td>
<td>2.5165</td>
<td>0.5103</td>
<td>0.4789</td>
</tr>
</tbody>
</table>

Figure 2. Estimated marginal trend functions: mean (continuous curve); mean ± SD (noncontinuous curves).

Figure 3. Estimated correlation function among diameter and height.
Figure 4 shows the estimated bivariate transition probability density function (EBDF) of tree diameter and height, defined by Equation 5. These surfaces indicate that the EBDF of tree diameter and height is steeper for the young stands and less steep for the mature stands.

The BVD (Equation 5) estimated to the observed data set \( \{(D(t_i), H(t_i)) = (d_i, h_i), t_0 < t_1 < \cdots < t_n\} \), \( \Delta = t_i - t_{i-1} = 1, i = 2, 3, \ldots, n \) implicitly estimates the marginal transition probability density functions (EMDFs) of the tree diameter and the height. The sections of EMDFs are represented in Figure 5. Figure 5 shows that the localization of the EMDFs decreases as the age increases.

![Density function at the age of 100 years](image1)

![Density function at the age of 60 years](image2)

![Density function at the age of 30 years](image3)

**Figure 4.** Estimated bivariate transition probability density function.

![Marginal density function of diameter](image4)

![Marginal density function of height](image5)

**Figure 5.** Estimated marginal transition probability density functions.

For the evaluation of goodness of fit of our presented lognormal shape bivariate and univariate age-dependent transition probability density functions 5–7, we use the Shapiro-Wilk statistic (Equation 20) and normal probability plots. The normal probability plots of the pseudo-residuals, using the estimates of parameters calculated for the bivariate stochastic Gompertz diffusion model by the maximum likelihood procedure, are represented in Figure 6. Figure 6 shows that both EMDFs and EBDF fit well. For the EBDF the computed Shapiro-Wilk statistic \( W \) yielded a value of 0.9839 \( (P = 0.0067) \). The EMDF of the diameter \( W \) yielded a value 0.9833 \( (P = 0.0024) \). The EMDF of the height \( W \) yielded a value 0.9827 \( (P = 0.0011) \). These results lead us to the conclusion that the observed data set is compatible with the densities 5–7. It is worth remarking that the Shapiro-Wilk statistic provides a generally superior overall measure of non-normality. The Shapiro-Wilk test may reject the null
hypothesis in the case of a single extreme observation. Moreover, the test may fail if the sample is too large.

**Application**

First, the bivariate Gompertz diffusion process statistically fitted to the dynamics of the tree diameter and height, on the basis of observations, enables us to describe the dynamics of correlation between the stand-level characteristics and to evaluate the analytical relationship between diameter and height using conditional distributions in the form

\[
    d = \int_{d>0} d \frac{p(d, h, t)}{p(h, t)} \, dd, \\
    h = \int_{h>0} h \frac{p(d, h, t)}{p(d, t)} \, dh, 
\]

(22)

respectively, where transitional probability density functions \( p(d, h, t), p(d, t), \) and \( p(h, t) \) are defined by Equations 5–7. The relationships (22) \((h = h(d, t), \ d = d(h, t))\) between the height (diameter) and the diameter (height) subject to the age are shown in Figure 7.

Second, the bivariate distribution of diameters and heights allows us to revise estimated mean tree volume in the form

\[
    \tilde{V}(t) = \int_{d>0} \int_{h>0} V(d, h, t) p(d, h, t) \, dh \, dd, 
\]

(23)

where \( V(d, h, t) \) is the tree volume on diameter, height, and age regression of power form

\[
    V = \exp(\delta_0 h^{\delta_1} d^{\delta_2} t^{\delta_3}),
\]

(24)

and parameters \( \delta_0, \delta_1, \delta_2, \delta_3 \) to be estimated. The selection of tree volume model was basically motivated by the available measured tree level characteristics. The regression approach used the weighted least-squares technique and linearization of the volume Equation 24. A MAPLE program was implemented to perform the calculations of the weighted least-squares estimators of the parameters \( \delta_0, \delta_1, \delta_2, \delta_3 \). The estimators and their standard deviations (in parentheses) are \( \hat{\delta}_0 = -9.5282(0.0127), \hat{\delta}_1 = 1.9183(0.0072), \hat{\delta}_2 = 0.8907(0.0104), \) and \( \hat{\delta}_3 = -0.0268(0.0042) \). The adjusted coefficient of determination \((R^2)\) is 0.992. The relationship between the mean tree volume and the age of the tree is shown in Figure 8.

Equations 5, 21, and 22 presented here should make a consistent and unbiased basis for evaluating stand volume subject to age. Using Equation 5 we can determine various quantities, such as the probability that a tree has a diameter from 25 to 35 cm and height from 15 to 25 m. Figure 9 shows this probability.
Conclusions

The purpose of this article was to introduce a new method for the fitting of a bivariate diameter and height distribution and to show how this novel method can be implemented. For a realistic representation of diameter and height growth, we used a bivariate Gompertz shape diffusion process. The results obtained here have shown that it is possible to relate a bivariate nonlinear diameter and height growth law and a bivariate diameter and height distribution law.

The heterogeneity of stand structure increases due to selective cutting, which maintains the uneven-aged structure of stands. This will also emphasize the need for the bivariate age-dependent distribution of tree diameter and height.

A theoretical prerequisite of our approach was the bivariate stochastic Gompertz diameter and height growth law driven by the two-dimensional standard Wiener process. Thus, the proposed method could be continued in terms of properly modifying the drift and diffusion functions of the stochastic diameter and height growth process. Last, it is our hope that this modeling effort will be a springboard to practical applications that further improve the prediction accuracy of the bivariate diameter and height distribution model.

Literature Cited


