Predicting the Present Value Distribution of a Forest Plantation Investment

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ABSTRACT. Forestry investment decisions may be based on the probability distribution of financial return in addition to a point estimate of mean return. This study describes an approach to predicting the present value distribution of a plantation investment using actual data on timber price and yield. Changes in stumpage price are modeled with a lognormal diffusion process called geometric Brownian motion (GBM). Timber yield is modeled with a variant of GBM that includes an age-dependent growth component. Model parameters are estimated with time-series observations of loblolly pine (Pinus taeda L.) price and yield in the southeastern United States. Because GBM models have lognormally distributed errors, present value distributions are skewed with extremely long right-hand tails. The median and quartiles of the distribution provide a better measure of central tendency and spread than do the mean and standard deviation. A median-maximizing feedback cutting rule does not perform any better than a median-maximizing fixed rotation age suggesting that no economic gain can be obtained by monitoring timber price and yield under the assumptions of our models. The forecast error, measured by the distance between quartiles, is about twice the size of the median present value. System error is the primary cause, and error in the price process contributes more to the variability in present value than does error in the yield process. Parameter uncertainty increases forecast error 15 to 35%. The large forecast error raises the question of whether better predictive models can be built or whether the present value of a plantation investment is inherently uncertain. For. Sci. 42(3):378–388.

Additional Key Words. Loblolly pine, risk and uncertainty, stochastic stumpage price, stochastic yield, timber harvesting.

An important decision in forest management is whether or not to plant a site. The decision may be based in part on an estimate of financial return measured by the present value of the planting action. Recognizing that timber price and yield are stochastic processes, several authors have used numerical methods such as stochastic dynamic programming (Norström 1975, Brazee and Mendelsohn 1988, Lohmander 1988, Teeter and Caulfield 1991, Thomson 1992, Haight and Holmes 1991) or analytical methods of optimal stopping (Clarke and Reed 1989, Reed and Clarke 1990) and contingent claims (Morck et al. 1989) to determine harvest policies that maximize expected present value. Although these studies provide insight about optimal harvesting and stand valuation under stochastic price and yield, they use point estimates of mean present value as measures of financial return. In fact, present value is a random variable with a probability distribution, and properties of the distribution such as its spread and shape may influence the investment decision. The purpose of this paper is to investigate the present value distribution of a newly planted stand.

The present value distribution depends on stochastic models for the way in which timber price and yield evolve...

1 Another practical measure of financial return from a forestry plantation is the rate of return on investment. We point out later that the choice of performance measure (present value or rate of return) affects the shape and spread of the return distribution. We use present value because of its traditional use in stand valuation and suggest that a complete analysis of the rate distribution be an area of future work.
over time. We assume that stumpage price follows a lognormal diffusion process called geometric Brownian motion (GBM) in which random price changes do not depend on current or past price observations. Such a process characterizes price movements in an informationally efficient market (Fama 1970, Washburn and Binkley 1990) and has been used widely to model financial markets (see e.g., Pindyck 1991) as well as stumpage prices (Morck et al. 1989, Clarke and Reed 1989, Reed and Clarke 1990, Thomson 1992). We examine the fit of this model to a time series of stumpage price observations for loblolly pine (Pinus taeda L.) in the southeastern United States and explore the consequences of this model specification on the price distribution.

Although stochastic models for stumpage price have been widely employed, stochastic models of timber yield are much less common. Yield has been modeled using variants of GBM (Morck et al. 1989, Clarke and Reed 1989, Reed and Clarke 1990), which require the specification of relatively few parameters. Alternatively, yield can be modeled using a more complicated stochastic simulator (e.g., Taylor and Fortson 1991). To minimize the number of estimated parameters, we fit a variant of GBM that includes an age-dependent growth component to a time series of observations of loblolly pine plantation yield.

Two sources of uncertainty affect the present value distribution: system error in the price and yield models and sampling error in the parameter estimates. System error is the inherent randomness in the stochastic process (represented by the error variance), and sampling error is the uncertainty in estimates of model parameters. Although stand valuation studies analyze the effects of system error, they ignore sampling error and use ad hoc plausible parameter values or simple point estimates. We address the statistical problem of estimating parameters for GBM models for timber price and yield, and we determine the impacts of both system error and sampling error on the present value distribution.

The paper is organized as follows. The first two sections deal with statistical analysis and model fitting for timber price and yield data. The next section addresses the estimation of the present value distribution. First, we investigate the properties of present value distributions incorporating system error in the timber price and yield processes. Then, we analyze the effects of adding sampling error in the parameter estimates. In each case, we compute present value distributions for fixed age and feedback cutting rules, which allows us to estimate the gain in present value obtainable by monitoring timber price and yield. The final section summarizes the results and places them in the context of previous studies and future work.

Price Data and Model

Quarterly stumpage prices in dollars per thousand board feet (mbf) (International 1/4 in. scale) for loblolly pine in the piedmont region of North Carolina were used to fit a model of the price process. The price observations span the period 1977–1988 and represent prices at the beginning of months January, April, June and September reported in Timber Mart-South. The prices were adjusted for inflation (base age 1988) and plotted in Figure 1. There appears to be a slight downward trend over the 12 yr period.

A model for the behavior of prices widely used in the economics literature (including forestry economics—see e.g., Pindyck 1991, Morck et al. 1989, Clarke and Reed 1989, Reed and Clarke 1990, Thomson 1992) is that of geometric Brownian motion (GBM):

\[ dP = bPdt + \sigma_P dw_P(t) \]  

where \( P(t) \) is the price at time \( t \) (measured in years from the beginning of 1977) and \( w_P(t) \) is a standard Wiener process (see e.g., Karlin and Taylor 1981). This model assumes that the incremental proportional change in price at any time comprises a fixed component (\( bdt \)) and a random component (\( \sigma_P dw_P(t) \)). If this model is correct, it can easily be shown (e.g., Clarke and Reed 1989) that the first differences in the log-price series,

\[ y_i = \ln(P_i) - \ln(P_{i-1}) = \ln \left( \frac{P_i}{P_{i-1}} \right), \quad \text{for } i = 2, 3, \ldots, n \]  


Figure 1. Quarterly stumpage prices for loblolly pine sawtimber in the piedmont region of North Carolina.

2 We used stumpage price observations spanning the period 1977–1988 because the collection process changed significantly in 1988. Up until mid 1988, price observations were reported monthly; thereafter, prices were reported quarterly. Because quarterly averaging can change the structure of the underlying price model (Haigh and Holmes 1991), we limited our data to those collected on a monthly interval. We sampled prices on a quarterly interval assuming that timber sellers sample the spot market roughly every 3 months to decide whether or not to sell.
where \( t \) indexes the quarter and \( P_t \) is the observed price at the beginning of that quarter) should be independent, normally distributed (n.i.d.) random variables with mean \( b/4 \) and variance \( \sigma_p^2 / 4 \). The maximum likelihood (ML) estimates of the parameters \( b \) and \( \sigma_p^2 \) of the GBM model (1) are thus simply:

\[
\hat{b} = 4 \bar{y} = -0.01188
\]

and

\[
\hat{\sigma}_p^2 = 4\bar{s}^2 = 0.07281
\]

where

\[
\bar{s}^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})^2.
\]

To check the model’s validity, we tested the residuals

\[
e_i = y_i - \bar{y}
\]

for normality and independence. Plots of the standardized residuals showed strong resemblance to a normal form. The Shapiro-Wilk test for lack of normality (see, e.g., Bickel and Doksum 1977) produced a value of \( W = 0.986 \) with a corresponding \( P \) value of 0.93. Thus, there appears to be no evidence to contradict the normality property. Of four tests for serial correlation (Box-Pierce, Turning Point, Difference-Sign, and Rank—see, e.g., Brockwell and Davis 1991), none produced a significant result (\( P \) values 0.92, 0.46, 0.07, and 0.97, respectively). As a further check on independence, the sample autocorrelation and partial autocorrelation functions of the residuals were computed. None of the lagged autocorrelations or partial autocorrelations fell outside of the 95% confidence bounds, indicating compatibility with white noise (i.e., the residuals are i.i.d. random variables). Thus, the quarterly price data appear to be compatible with the GBM model.\(^3\)

To estimate the future distribution of stumpage prices, we used an equivalent discrete-time model with a quarterly interval to generate 10,000 random price paths over a 30 yr horizon (the starting price \( P_0 \) was $125/mbf). The price model is

\[
P_t = P_{t-1} \exp\left\{ \frac{b}{4} + \frac{\hat{\sigma}_p}{2} Z \right\}
\]

where \( Z \) is a \( N(0,1) \) random variable and \( i \) is the quarterly index. Statistics describing the price distribution in year 30 are in Table 1. Prices are lognormally distributed with the mean exceeding the median by a factor of 3. Further, the mean has increased over time while the median has decreased. These results are explained by the formulas for the ML estimates of the mean and median of \( P_t \):

\[
E[P_t] = P_0 \exp\left\{ \bar{y} + \frac{\hat{s}^2}{2} \right\}
\]

and

\[
m[P_t] = P_0 \exp\{\bar{y}i\}.
\]

Note that the ML estimate of the mean exceeds the median by a factor \( \exp[\frac{1}{2}\hat{s}^2] \). Furthermore, the median decreases because \( \bar{y} < 0 \), and the mean increases because \( \bar{y} + \frac{\hat{s}^2}{2} > 0 \). The mean price grows exponentially because the model predicts a small chance of very large price increases. For example, the model predicts with probability 0.05 that price will grow more than 7% per year and will exceed $1,000/mbf by year 30.

**Yield Data and Model**

Observations on timber volume (in board feet International 1/4 in. scale) in four test plantations of loblolly pine grown in the piedmont region of South Carolina (Buford 1991) were used to fit a model for the volume growth process. The plantations grow on sites of similar quality (site index 65 ft, base age 25) and had initial 12 x 12 ft spacing. The data were collected at 5 yr intervals for 30 yr (Figure 2).\(^4\)

A possible continuous-time model for the volume growth is given by the following stochastic differential equation

\[
dX = g(t)Xdt + \sigma_XXdw_X(t)
\]

where \( X(t) \) is the volume at age \( t \) (measured in years from the planting date). The function \( g(t) \) is an age-dependent

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\(^3\) Washburn and Binkley (1990) and Haight and Holmes (1991) found evidence of autocorrelation in the monthly series of loblolly pine stumpage prices. The latter authors also found that one test (the unit root test) gave evidence of autocorrelation in the quarterly series. Thus, there appears to be some short-order autocorrelation, which we ignore to concentrate on the effects of GBM. Further work is needed to investigate the effects of alternative price models (e.g., AR models) and the consequences of model misspecification.

\(^4\) Long-term observations of plantation yields in the piedmont region of North Carolina were not available. Assuming that land with a piedmont site index of 65 ft (base age 25 yr) has the same productive capacity in North Carolina as in South Carolina, the yield model results should be applicable.
Plantation yield (mbf/ac)

- Plantation 1
- Plantation 2
- Plantation 3
- Plantation 4
- Predicted yield

Figure 2. Observed and predicted timber yield in loblolly pine plantations in the piedmont region of South Carolina.

deterministic component of proportional volume growth, and \( \sigma_X dw_X(t) \) is the random growth component \([w_X(t) \text{ is another standard Wiener process, assumed to be independent of the } w_P(t) \text{ process}].\) If this model is correct, then the first differences in the logarithms of volumes at 5 yr intervals

\[ x_j = \ln(X_j) - \ln(X_{j-1}) \quad \text{for } j = 2, 3, \ldots \]  

should satisfy

\[ x_j = \int_{s_{(j-1)}}^{s_j} g(s) ds + \varepsilon_j \]  

where the \( \varepsilon_j \) are n.i.d. random variables with mean 0 and variance \( 5\sigma_X^2 \). The normality of the error terms, \( \varepsilon_j \), allows for the maximum likelihood estimates of the parameters of a parametrically specified growth function \( g(t) \) and of \( \sigma_X^2 \) to be obtained using nonlinear least squares.

Many possible parametric forms could be used for the function \( g(t) \). One would expect \( g(t) \) to be decreasing at least for \( t \) suitably large (reflecting diminishing proportional growth as a stand ages). The simplest form would be linear: \( g(t) = \alpha - \beta t. \) Unfortunately, the pattern of residuals from a linear model indicated model lack of fit.

An alternative 3-parameter form

\[ g(t) = \frac{Ke^{-K(t-\gamma)}}{1 + (\theta - 1)e^{-K(t-\gamma)}} \]  

was considered instead. This corresponds to the Richards Curve (see e.g., Seber and Wild 1989, p. 332) with

\[ X(t) = X(15) \exp \left\{ \int_{15}^{t} g(s) ds + \sigma_X w_X(t-15) \right\} \]

\[ = \alpha Y \frac{1 + (\theta - 1)e^{-K(t-\gamma)}}{1 + (\theta - 1)e^{-K(15-\gamma)}} \]

where \( \alpha \) is a parameter corresponding to the volume at age 15, and \( Y \) is a log-normally distributed random variable \((= \exp [\sigma_X w_X(t-15)])\). This form is quite general and includes as special cases the von Bertalanffy model \((\theta = 2/3)\), the logistic model \((\theta = 2)\), the monomolecular model \((\theta = 0)\), and the Gompertz model \((\theta \rightarrow 1)\).

To fit the model the integral in (11) needs to be evaluated. It can readily be shown to be

\[ \int_{s_{(j-1)}}^{s_j} g(s) ds = \frac{1}{1 - \theta} \ln \left\{ \frac{1 + (\theta - 1)e^{K(s_j-\gamma)}}{1 + (\theta - 1)e^{K(s_{(j-1)}-\gamma)}} \right\} \]

\[ = \phi_j(\gamma, \theta, K). \]

Thus, the successive first differences of \( \ln(\text{volume}) \) on a given plot should satisfy

\[ x_j = \phi_j(\gamma, \theta, K) + \varepsilon_j. \]

However, because the four test plots were grown over the same time period, the values of \( x_j \) between plots may be correlated. To allow correlation, let

\[ x_{ij} = \ln(X^i_j) - \ln(X^i_{j-1}) \]

denote the first difference in \( \ln(\text{volume}) \) in period \( j \) on plot \( i \) \((i = 1, \ldots, 4)\). We can fit the model

\[ x_{ij} = \phi_j(\gamma, \theta, K) + \varepsilon_{ij} \]

where the \( \varepsilon_{ij} \) are normal random variables with zero mean and covariances zero except for within a period:

\[ \text{cov}(\varepsilon_{ij}, \varepsilon_{ik}) = \rho \sigma^2 \quad i \neq k \]

\[ = \sigma^2 \quad i = k. \]

The parameter \( \tau \) represents the correlation between the random components of growth during a given 5 yr period on different plots, and \( \sigma^2 = 5\sigma_X^2. \)
To find the ML estimates of $\rho$, $\theta$, $\gamma$, $Z$, and $\sigma^2$, we transformed the $x_{ij}$ to yield n.i.d. error terms (see Draper and Smith 1981, p. 108), and used ordinary nonlinear least squares methods. We found that the ML estimate of $\theta$ was zero so that the fitted form of the Richards curve was the monomolecular form with median $m_0$ volume following

$$m_0[X(t)] = \alpha \left[ \frac{1 - e^{-K(t-\gamma)}}{1 - e^{-K(15-\gamma)}} \right]. \quad (19)$$

The ML estimate of the correlation parameter $\rho$ was $\hat{\rho} = 0.089$, which is not significantly different from zero, so that between-plot correlation can be ignored. The other ML parameter estimates were (with standard errors in brackets)

$$\hat{\gamma} = 13.471 \ (0.1131),$$

$$\hat{K} = 0.0831 \ (0.00553), \quad (20)$$

$$\hat{\sigma}^2 = 0.08376 \ (0.001861).$$

The asymptotic correlation between the estimates $\hat{\gamma}$ and $\hat{K}$ was 0.811. Residual plots showed no sign of model misspecification. An F-test for lack of fit was not significant ($P = 0.28$).

Because we use first differences in $\ln$(volume), the parameter $t$ does not appear in the estimation Equation (17). Instead of estimating the parameter $t$ in (13), we can think of the problem as one of estimating an initial condition for the stochastic differential equation (9) (Seber and Wild 1989, p. 335). The initial age at which observations of board foot volume were made was 15 yr. If we assume that the trees were growing according to (9) before that time, the volume at age 15 follows a log-normal distribution, and the four observed volumes at age 15 come from such a distribution. The parameters $\mu_0$ and $\sigma^2_0$ of this distribution can be estimated by the mean and sample variance of the logarithms of the four volumes at age 15 (i.e., $\hat{\mu}_0 = 7.910$ and $\hat{\sigma}^2_0 = 0.1382$). A single point estimate for the initial (age 15) volume would be given by

$$\hat{X}_0 = \exp(\hat{\mu}_0 + 0.5\hat{\sigma}^2_0) = 2,919$$

board feet per acre.

The expected volume versus age curve can be obtained by integrating (9) for a given initial condition $X(15) = X_0$:

$$E[X(t)] = X_0 \exp\left\{ \frac{\sigma^2}{2} \int_{15}^{t} g(s)ds + R(t-15) \right\} \left[ \frac{1 - e^{-K(t-\gamma)}}{1 - e^{-K(15-\gamma)}} \right]. \quad (22)$$

for $t \geq 15$.

Using the ML estimates of $K$, $\sigma^2$ and $\sigma^2 = \sigma^2 / 5$ along with the point estimate $\hat{X}_0 = \exp(\hat{\mu}_0 + 0.5\hat{\sigma}^2_0)$, we obtain an estimated mean-value curve for plantation volume versus age (Figure 2).

To estimate the future distribution of plantation yield, we used an equivalent discrete-time model with a quarterly interval to generate 10,000 random yield paths between ages 15 and 30. The yield model is

$$X_i = X_{i-1} \left[ \frac{1 - e^{-K(i-\gamma)}}{1 - e^{-K(15-\gamma)}} \right] \exp\left\{ \frac{\hat{\sigma}^2}{2} \right\} \left[ \frac{1 - e^{-K(t-\gamma)}}{1 - e^{-K(15-\gamma)}} \right]. \quad (23)$$

where $Z$ is a $N(0,1)$ random variable and $i$ is the quarterly index. Statistics describing the yield distribution at age 30 are in Table 1. The yield distribution is not skewed as much as the price distribution: the median yield (16.9 mbf/ac) is about 12% less than the mean yield (19.3 mbf/ac). The quartile estimates (12.2 and 23.8 mbf/ac) are slightly less than the minimum and maximum plot volumes observed at age 30 (Figure 2). Because of the lognormal distribution assumption, there is a small probability of a very high yield (e.g., less than a 5% chance of yield being greater than 40.0 mbf/ac). Except for the long right-hand tail, the predicted yield distribution appears plausible.

### Present Value Distribution

Here, we estimate the present value distribution of a plantation investment using the stochastic price and yield models from the previous two sections. Distributions are computed under two sources of uncertainty. First, properties of the distributions are investigated incorporating system error in the timber price and yield processes. Then, the effects of adding sampling error in the parameter estimates are analyzed. In each case, distributions are computed for fixed age and feedback cutting rules.

#### System Error

Estimating the distribution of financial return requires assumptions about the future evolution of timber price and yield. In the previous two sections, stochastic differential equation models were fitted to past observations. To predict future prices and yield, we assume that the same stochastic differential equation models govern their future behavior. Because predictions may be needed up to 50 yr or more into the future, this assumption is indeed heroic. Nonetheless, this assumption is better than no model at all, and the results obtained provide a baseline against which other projections can be compared.

In addition to models for timber price and yield, a rule for timber harvest is required. We evaluate the performance of two cutting rules: a fixed rotation age and a feedback cutting...
rule. A fixed rotation is a simple rule and for some managers not very realistic. Nevertheless, it is worth computing to estimate the gain in present value obtainable by monitoring timber price and yield and employing a feedback cutting rule. Furthermore, using a fixed-age harvest rule reveals an important relationship between the mean and median of the present value distribution, readily understood in this simple situation, and also present with more complex rules.

Our final assumption involves the treatment of land value. In contrast to the deterministic Faustmann formula in which an infinite series of identical plantations is assumed, we use a single-rotation formula for present value. At rotation age, bare land value is added to the revenue from harvest. To focus attention on the relative effects of system and sampling error associated with the stochastic price and yield processes, we assume that land value is fixed and known with certainty.

**Fixed-Age Cutting Rule**

We now develop formulas for the mean and median of the present value distribution assuming that the stand is harvested at age \( T \). We start with an equation for present value. If \( P(T)X(T) \) is net revenue at rotation age, \( L \) is the value of bare land, and \( \delta \) is the instantaneous discount rate, the present value (PV) of (or financial return from) a newly planted site is

\[
PV = e^{-\delta T}[P(T)X(T) + L].
\]  

(24)

Note that the initial planting cost is not included. The present value of a newly planted site can be compared with an appropriate planting cost to evaluate the profitability of the planting investment. Timber sale administration cost (expressed as a percentage of timber sale revenue) could easily be handled by multiplying \( P(T)X(T) \) by some value less than 1.0.

Now we derive formulas for the mean and median present value. These formulas are obtained by focusing on the random variable \( P(T)X(T) \). Integrating the stochastic differential equations (1) and (9), we obtain

\[
P(T)X(T) = P_0 X_0 \exp\left\{bT + \int_{15}^{T} g(s) ds + \sigma_p w_p(T) + \sigma_x w_x(T - 15)\right\}
\]  

(25)

\[
= P_0 X_0 \left[\frac{1 - e^{-\gamma(T-\gamma)}}{1 - e^{-K(T-\gamma)}}\right]
\]  

\[
\exp\left\{bT + \sigma_p w_p(T) + \sigma_x w_x(T - 15)\right\}
\]

where \( P_0 \) is the (assumed known) price of timber at the planting date, \( X_0 \) is a random variable denoting the volume of the stand at age 15, and \( w_p \) and \( w_x \) are standard Wiener processes. If the volume \( X_0 \) at age 15 has a log-normal distribution with parameters \( \mu_0 \) and \( \sigma_0^2 \), then \( P(T)X(T) \) is log-normally distributed, with median

\[
m[P(T)X(T)] = P_0 \left[\frac{1 - e^{-K(T-\gamma)}}{1 - e^{-K(T-\gamma)}}\right] \exp\{\mu_0 + bT\}
\]

(26)

and mean

\[
E[P(T)X(T)] = P_0 \left[\frac{1 - e^{-K(T-\gamma)}}{1 - e^{-K(T-\gamma)}}\right] \exp\{\mu_0 + bT + 0.5(\sigma_0^2 + \sigma_p^2 T + \sigma_x^2 (T - 15))\}
\]

(27)

Substituting (26) and (27) into (24) yields formulas for the median and mean of the present value distribution.

A fundamental property of the distribution of present values is revealed by comparing the formulas for the median and mean of \( P(T)X(T) \). From (26) and (27), we see that the mean exceeds the median by a factor depending on the sum of several variances. The reason for this difference is the high degree of skewness in the log normal distribution, which results from the GBM specification for the price process and from the similar model specification for biological growth. As a consequence of this property, the distribution of present values is also skewed with the mean greatly exceeding the median present value.

To demonstrate the skewness of the present value distribution, we use (26) and (27) to compute the median and mean present values associated with rotation ages between 15 and 40 yr. Land value is $200/ac, the real instantaneous discount rate is 0.05, and the initial stumpage price is $125/mbf. The mean present values are up to four times greater than the median present values (Figure 3). We used stochastic simulation to determine the quartiles of the present value distribution for each rotation age (not shown). In each case, the mean present value resides in the upper quartile indicating the extremely wide right-hand tail of the distribution.

Another important result revealed in Figure 3 is the effect of the decision criterion on the optimal rotation age. Using expected present value as the optimization criterion, the best rotation age is 34 yr, realizing an expected present value of $1,267. The maximum median present value, in contrast, is $439/ac achieved at a rotation age of 23 yr. The difference in optimal rotation age can be explained by a property of the price model. For values of the parameters of the GBM price process estimated from the data, the mean price grows while the median price decays (see Table 1). Thus, cutting at a later age results in a higher mean price, but a lower median price.

The skewed distribution of present values resulting from GBM models for price and biological growth is of considerable importance to forest managers. The estimated mean present value could overestimate the actual financial return by a considerable amount because it is heavily influenced by the small probability of a bonanza resulting from a dramatic (but unlikely) increase in price. Although the mean present value is not very realistic.
value for the optimal rotation age is $1,267/ac, the probability that the owner achieves this return is small (less than 0.25, since the mean PV lies above the upper quartile). A risk-neutral or risk-averse manager is probably more interested in the median return because it measures the central tendency of the skewed distribution (i.e., there is a 50% chance that observed present value is above or below the median). Consequently, we focus on the median and quartiles of the distribution of present value to measure the effects of system and parameter uncertainty.

Feedback Cutting Rule

It is well known that when timber price is modeled as an autoregressive stochastic process, feedback harvest rules produce higher expected return than do fixed rotation ages because observations of current and past prices can be used to predict future price changes and time harvests to periods when prices are above average (e.g., Brazee and Mendelsohn 1988, Haight and Holmes 1991). Less is known about the performance of feedback harvest rules when price changes evolve as GBM. In this case, the performance of a feedback policy should not be much better than a fixed rotation age because random price changes are independent of current and past prices. Consistent with this assertion, Clarke and Reed (1989) show that when fixed costs are ignored, the mean-maximizing feedback harvest rule is a fixed rotation age. However, Thomson (1992) shows that when fixed costs and revenues are included, gains in expected present value can be obtained. Gains result from price-dependent abandonment options (with a fixed reward) and harvest options (with a fixed reforestation cost).

Our analysis of feedback policies takes a different approach. Because of the skewness of the present value distribution produced by GBM models of price and yield, we seek a feedback harvest rule that maximizes the median present value rather than the mean present value. Further, we seek a feedback rule that is conditional on current price and yield. Consequently, the traditional application of stochastic dynamic programming as in the above papers does not apply.

Our approach to developing a feedback harvest policy involves the derivation of a myopic look ahead (MLA) rule, which is known to provide a good approximation to the optimal stopping rule in other forestry applications (see Reed and Clarke 1990). We use a single rotation formulation for present value in which a fixed land value is added to the harvest revenue at rotation [see Equation (24)]. Letting $V'(t) = P(t)X(t)$, the optimal rotation age in a deterministic context occurs when

$$\frac{\partial V'(t)}{\partial t} = \delta]\{V(t) + L\}.$$

(28)

Recognizing that harvest value $V(t)$ is a lognormally distributed random variable, we define the MLA rule so that the stand is harvested when the growth in the median harvest value drops below the interest charge on the current growing stock and land:

$$\frac{\partial V_m(t)}{\partial t} \leq \delta]\{V(t) + L\}.$$

(29)

Differentiating (26) with respect to time, we find that

$$\frac{\partial V_m(t)}{\partial t} = V_m(t)\{b + g(t)\}$$

where $g(t)$ is the volume growth rate [Equation (12)]. Letting $V_m(t)$ equal the current stand value $V(t)$ and substituting into (29) yields the MLA rule:

$$V(t) \leq \frac{\delta L}{b + g(t) - \delta}.$$

(30)

The rule is to harvest as soon as the current stand value $V(t)$ drops below the age-dependent reservation value defined by the right-hand side of (30). Using the estimated parameters for timber price and yield, we computed reservation values and plotted them against age in Figure 4. Reservation values increase with age so that harvest takes place before age 25.

We evaluated the MLA rule using 10,000 simulations of the timber price and yield equations. The time step was 0.25 yr, and the horizon was 25 yr. The MLA rule resulted in an average cutting age of 22.3 yr with a median present value $443/ac (Table 2A). The distribution of present values is highly skewed. The mean present value ($1,075/ac) falls in the upper quartile of the distribution. As measured by the median and quartiles of the present value distribution, the MLA rule performs about the same as the median-maximizing fixed rotation age of 23 yr (Table 2A). Although MLA rules are known to be suboptimal, it is unlikely that the MLA rule can be greatly improved. Numerical comparisons of
Reservation value ($/ac)

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>0</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Harvest</td>
<td>0</td>
<td>400</td>
<td>800</td>
<td>1200</td>
<td>1400</td>
</tr>
<tr>
<td>Wait</td>
<td>1200</td>
<td>1000</td>
<td>800</td>
<td>600</td>
<td>400</td>
</tr>
</tbody>
</table>

**Figure 4. Reservation value versus age for a loblolly pine plantation.**

MLA rules and optimal feedback rules in other harvesting problems have shown that the difference is not great (Reed and Clarke 1990).

System error in the price process contributes more to the variability of the present value distribution than does system error in the yield process. To reach this conclusion, we first computed the present value distribution assuming no uncertainty in the yield process and then repeated the experiment using a deterministic price process. We measured the variability of the distributions using distances between quartiles. The between-quartile distance obtained with price uncertainty alone ($762) is between two and three times the size of the interquartile distance obtained with volume uncertainty alone ($273). These affects are roughly consistent with the relative sizes of the standard deviations of the models for price (0.27) and yield (0.13).

**Table 2. Statistics describing the predicted distributions of present value and rotation age for median-maximizing cutting rules for a loblolly pine plantation. The statistics are computed with system error (A) and with system and sampling error (B) in the timber price and yield processes.**

<table>
<thead>
<tr>
<th>Cutting rule</th>
<th>Present value ($/ac)</th>
<th>Rotation age (yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Median (quartiles)</td>
<td>Mean (std. dev.)</td>
</tr>
<tr>
<td>Fixed rotation</td>
<td>A. System error</td>
<td>439 (210-1,022)</td>
</tr>
<tr>
<td></td>
<td>B. System and sampling error</td>
<td>443 (206-1,028)</td>
</tr>
<tr>
<td>Median feedback</td>
<td>A. System error</td>
<td>439 (210-1,022)</td>
</tr>
<tr>
<td></td>
<td>B. System and sampling error</td>
<td>443 (206-1,028)</td>
</tr>
<tr>
<td>Fixed rotation</td>
<td>A. System error</td>
<td>439 (191-1,144)</td>
</tr>
<tr>
<td></td>
<td>B. System and sampling error</td>
<td>444 (184-1,297)</td>
</tr>
<tr>
<td>Median feedback</td>
<td>A. System error</td>
<td>439 (210-1,022)</td>
</tr>
<tr>
<td></td>
<td>B. System and sampling error</td>
<td>443 (206-1,028)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**System and Sampling Error**

Up to now, we have assumed that the five statistical parameters of the timber price and yield processes ($b, \sigma_b^2, \gamma, K, \sigma_X^2$) were equal to their point estimates obtained from the statistical analyses. In fact, there is a considerable degree of uncertainty with respect to actual parameter values (Table 3). For example, the standard error of $\hat{b}$, the estimate of the mean growth rate in timber price, is 0.04068. Because the sampling distribution of $\hat{b}$ is approximately normal it follows that the range of values $\hat{b} \pm 2 \text{s.e.}(\hat{b})$, or -0.0932 to 0.0695 should all be plausible values for the parameter $b$ (it is an approximate 95% confidence interval for $b$). In other words, the time series data on timber prices are compatible with everything from an average 9% per annum downward trend to a 7% per annum upward trend. Consequently, the sampling error in the parameter estimates implies a great deal of uncertainty in predicting financial return.

**Simulation Method**

To better estimate the distributions of financial return under alternative harvest policies, we have developed a simulation procedure that incorporates the system error in price and volume growth with the sampling error in the parameter estimates. The simulator requires a method of describing the relative plausibilities of various combinations of parameter values. One approach is to use a probability distribution of possible parameter values. This requires a Bayesian, or at least a fiducial, approach to parameter estimation (Kalbfleisch 1985). Rather than try to specify a joint prior probability distribution for the parameters, which a full-blowen Bayesian analysis would require, we have chosen instead to use a fiducial approach. Although not strictly interpretable in a frequentist sense, fiducial probability does provide a measure of the degree of uncertainty concerning parameter values. With the fiducial approach, we view the parameters of the timber price and yield processes as random variables with (fiducial) normal distributions centered at their estimated values. For example, because the asymptotic distribution of the ML estimators $\hat{b}$ and $\hat{\sigma}_p^2$ of the price process (1) is bivariate normal centered at $\hat{b}$ and $\hat{\sigma}_p^2$, and with zero covariance, the fiducial distribution of $\hat{b}$ and $\hat{\sigma}_p^2$ will be bivariate normal with zero covariance and centered at the ML point estimates ($\hat{b} = -0.01188$ and $\hat{\sigma}_p^2 = 0.07281$).

The fiducial variances of $\hat{b}$ and $\hat{\sigma}_p^2$ are 0.00165 and 0.000236 obtained by squaring the standard errors of $\hat{b}$ and $\hat{\sigma}_p^2$ in Table 3. Similarly, the parameters $K, \gamma$, and $\sigma_X^2$ of the growth process can be regarded as approximately jointly normal random variables with means given by the point estimates.

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6 The notion of fiducial probability, first proposed by R.A. Fisher, has always been somewhat controversial in the discipline of statistics. An alternative way of justifying the use of the likelihood as a probability distribution of parameter values would be to adopt a Bayesian point of view with uniform prior distributions for all of the parameters.
and variances obtained by squaring their standard errors. In addition, the parameters K and γ have a correlation coefficient equal to 0.811 (the asymptotic correlation of \( \hat{K} \) and \( \hat{\gamma} \) obtained from the ML estimation).

The simulation model was used to estimate the present value distributions of fixed rotation ages between 20 and 30 yr. The time step was 0.25 yr, and the distributions were estimated using 10,000 independent simulations of the price and yield processes. The generation of parameter values from their fiducial distributions is the basis for each simulation run. A single run consists of the following steps:

1. Generate a set of parameter values from the fiducial distributions of \( b, \sigma_p, K, \gamma, \) and \( \sigma_x^2 \) using a pseudo-random number generator.
2. Generate an age 15 volume \( X_0 \) from a log normal distribution with the parameter values \( \mu_p \) and \( \sigma_p^2 \).
3. Generate a sample path of the price process [Equation (6)] with initial (time zero) price ($125/mbf).
4. Generate a sample path of the yield process [Equation (23)] with starting value \( X_0 \) (determined in step 2) at time 15 yr.
5. Determine the cutting age and resulting present value using a particular cutting rule.

The same procedure was used to estimate the present value distributions associated with feedback harvest policies. The feedback harvest rule is to cut when the expected growth in the median harvest revenue drops below the interest charge on the value of the current growing stock and land [Equation (30)]. Recognizing that reservation prices depend on parameters of the timber price and yield processes and that these parameters are stochastic, we used two different approaches to setting their values. In the first approach, we compute one set of reservation prices for all simulations using the point estimates of the parameter values obtained from the data. This harvest rule with parameters estimated (Figure 4) uses less information than would be available in practice. In the second approach, we compute a new set of reservation prices for each simulation using the actual parameter values generated from the fiducial distributions. This harvest rule with parameters known uses more information than would be available in practice. Thus, the performance of the two feedback rules with parameters estimated and known should provide lower and upper bounds of the actual performance of these rules in practice.

**Results**

The addition of sampling error does not affect the median present values associated with optimal fixed-age and feedback harvest policies (Table 2B). The median-maximizing rotation age is 23 yr with present value $439/ac. The median feedback rule with estimated parameters produces a median return is $444/ac with average cutting age of 22.1 yr. Furthermore, no gain in median present value is obtained using a feedback rule in which parameters are known.

Including sampling error increases the spread and the skewness of the distribution of present values associated with the median-maximizing cutting rules (Table 2). With a fixed rotation age, the distance between quartiles ($953) is 17% greater than the distance between quartiles computed with system error only ($812). The distance between quartiles computed with the feedback harvest rule ($1,113) is 35% greater than the corresponding distance computed with system error only ($822). The mean present values associated with fixed rotation age and feedback cutting rules occur in the interval between the 90th and 95th percentiles of the distribution. The large difference between mean and median present values results from the rare occurrence of extremely high present values associated with price growth parameters at the high end of their range.

Uncertainty in the estimate of the mean growth rate \( \hat{b} \) of the price process is by far the most important source of additional variability in the present value distribution (after system error). To reach this conclusion, we computed the present value distribution under system error and assuming error in only one parameter at a time. For all of the parameters except \( \hat{b} \), the distance between quartiles of the associated present value distribution was no greater than the distance obtained with system error alone. When \( \hat{b} \) was the only uncertain parameter, the distance between quartiles was equal to that obtained assuming error in all of the parameters.

**Summary and Discussion**

We have addressed the problem of estimating the probability distribution of present value of a forest plantation investment using actual data on timber price and yield. Estimates are based on stochastic differential equations taking the form of geometric Brownian motion, which provide plausible statistical models for loblolly pine timber price and yield evolution. Although GBM models of price have been used elsewhere to determine optimal timber cutting rules (Clarke and Reed 1989, Reed and Clarke 1990, Thomson 1992), until now the impacts of these models on the present value distribution have not been investigated.

For both fixed-age and feedback harvest rules, the distributions of present value are skewed with extremely long right-hand tails. The effect of this skewness is to make the mean present value much greater than the median present value. Because the mean present value falls in the upper quartile of the distribution, the mean is a poor descriptor of
the central location of the distribution. Mean present value could overestimate the actual present value by a considerable amount because it is heavily influenced by the small probability of a bonanza resulting from a dramatic (but unlikely) increase in price. A risk-neutral or risk-averse manager is probably more interested in the median return because it measures the central tendency of the skewed distribution (i.e., there is a 50% chance that observed present value is above or below the median). Consequently, we use the median present value to measure the performance of alternative harvest policies and at the quartiles of the present value distribution to measure the effects of system and sampling error.

Another way to deal with a skewed distribution of revenues is to develop measures of financial return using the natural logarithm of revenue. This transformation reduces the skewness of the revenue distribution causing the mean and median of the distribution to be closer together. Maximizing the expected present value of the logarithm of revenue can be thought of as maximizing expected discounted utility with a risk-averse utility function. Because the inverse transformation of this performance measure is an estimate of median discounted revenue, maximizing expected discounted utility is similar to maximizing the median discounted revenue. Another measure of financial return is the rate of return on investment. Some investors are more interested in the mean and variance of the rate of return of a plantation investment. Because the rate of return is a function of the natural logarithm of the revenue obtained at rotation age, the mean and median of the distribution of the rate of return should be closer together. The inverse transformation of the rate of return is an estimate of the median net annual return associated with a particular rotation age.

Simulation results with our models show that a median-maximizing feedback cutting rule does not perform any better than a median-maximizing fixed rotation age. We designed the feedback rule so that harvest takes place when the expected growth in median present value is less than the interest charge on the current value of timberland. The rule is implemented by cutting when current stand value is less than a reservation value that increases with stand age. Results on the performance of this policy suggest that no gain in median present value is obtained by monitoring current timber price and yield. Further, no gain in median present value is obtained by monitoring and updating the parameters of the timber price and yield processes. Although these results are consistent with the assertion that no economic gain can be obtained by monitoring timber prices in a market that is informationally efficient (Washburn and Binkley 1990), we emphasize that the results were obtained using a single-rotation present value formulation in which land value is fixed and known with certainty. Should the formulation change to include multiple rotations (e.g., Thomson 1992) or a stochastic land value process that is not completely corre-

lated with the price process, it is likely that gains in median present value can be obtained using a feedback policy based on joint monitoring of tree and land values.

Two sources of uncertainty affect the present value distribution. System error (inherent randomness in the price and yield models) is the primary source, and in our case, error in the price process contributed more to the variability in present value than did error in the yield process. With system error alone, the distance between quartiles is about twice the size of the median present value. Including parameter uncertainty (statistical uncertainty in the estimates of model parameters) increased the distance 15 to 35%. Almost all of the additional variability is caused by error in the growth parameter of the price model.

Predictions of the distribution of financial return from a forest investment provide more information than point estimates of mean return and can be compared with risks inherent in other investments. In our case, using the rotation age that maximizes median present value (23 yr), a discount rate of 5%, and a $200/ac land value, we conclude that there is a 50% probability that the present value of a newly planted loblolly pine stand will be somewhere in the range of $200 to $1200/ac with a median value around $440/ac. There is a 25% chance that present value will be between 0 and $200/ac. At the same time, there is 25% chance that present value will exceed $1200/ac with the possibility that financial return exceeds $1200/ac by a very large amount. The size of the forecast error (represented by the distance between quartiles) may be disconcerting to risk-averse investors who want a more secure estimate of income. The forecast error may be even larger if we consider additional sources of uncertainty such as model misspecification and catastrophic loss. The results raise the question of whether better price and yield prediction models can be developed to reduce forecast error or whether the present value of a planting investment is inherently uncertain.

Literature Cited


