On the Optimality of a Normal Forest with Multiple Land Classes

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ABSTRACT. We analyze a discrete time forestry model with nonlinear utility and any number of forest age and land classes. It is shown that under discounting, this model possesses a cyclical stationary state instead of an even flow of timber. The cycle length in timber flow equals the least common multiple of the rotation lengths of the various land classes. The cycle structure is dependent on the initial age-class allocation. As the discount factor approaches unity, the cyclicality in total timber harvest vanishes, but the forest age-class structure may not approach a normal forest in any land class. An aggregate even timber flow is associated with a normal forest in each land class only if there does not exist any pair of land classes with rotation periods having a common divisor greater than one. For. Sci. 48(3):530–542.

Key Words: Forest rotation, even flow of timber, regulated forest, endogenous timber price, model II.

Economic models on harvesting trees at the forest level involve numerous open issues such as whether the age-class structure has a tendency to converge toward some optimal long run stationary state. Optimal harvesting should select age classes with the highest financial maturity. An age-class-specific harvesting activity determines how the amount of financially mature timber evolves in time. Under “diminishing marginal returns,” the problem is to find a simultaneous optimum for the rotation length and for smoothing the annual fluctuation in mature age classes.

In the forestry literature, an even flow of timber, or more traditionally a specific structure called a “normal,” “regulated,” or “synchronized” forest, represents a desirable long run target for forest age classes. It is implicitly or explicitly included in numerous classical area and volume control formulas as well as in many linear programming applications (see, e.g., Davis and Johnson 1986). Reed (1986) writes: “The ideal of the normal forest is thus linked to that of sustained yield, and it has, it seems, occupied a central place in forestry thinking, for as long as a scientific discipline called ‘Forestry’ can be said to have existed.” However, the forestry literature has not offered detailed analysis on the conditions under which some form of the normal forest or even flow of timber will endogenously became an optimal long run stationary equilibrium. This article presents several new results on whether an even flow of timber, together with some form of normal forest age class structure, constitutes an optimal long run stationary state in a model with any number of both age and land classes and where the timber price is determined endogenously.

The problem of choosing the optimal forest rotation age for a single stand, with linear utility or exogenously determined timber price has already been formulated by Faustmann (1849). Under the linearity assumptions, the extension to a system with multiple age classes is straightforward since the harvesting activities of all the stands can be optimized separately. However, if the periodic utility is strictly concave or the timber price endogenous, separability between the stands no longer holds. The Faustmann formula applied to an age-class structure that deviates from the normal forest implies wide annual fluctuations in endogenous timber price or marginal utility. In forestry this has been intuitively understood for centuries and is reflected in the concept of the normal forest and more recently in preference for an even flow of timber. In a normal forest (with homogeneous land quality), timber harvesting is even over time since the total...
land area is evenly allocated between the existing age classes, and in each period the oldest age-class is clearcut and replanted. In forest management and planning, one established line of research applies a linear objective function and linear programming (e.g., Hoganson and McDill 1993). Often these forest management scheduling applications include woodflow smoothness or nondeclining yield constraints or they may include the normal forest as an exogenous endpoint condition. These applications may have high practical relevance. However, from a theoretical point of view it is more fundamental to ask whether an even timber flow or some form of normal forest structure will follow endogenously if the objective function is nonlinear and reflects diminishing marginal utility.”

An age-class forestry model with a nonlinear objective function or endogenous timber price is applied by Lyon and Sedjo (1983), Sedjo and Lyon (1990) and Adams et al. (1996). In many respects their work is excellent. However, the computations rely, without explicit analysis, on the assumption that some form of normal forest structure is the optimal long run steady state in the model. The fact that these central issues are open in forest and resource economics is emphasized by several authors such as Clark (1976, p. 256; 1990, p. 267), Hyde (1980, p. 83), Binkley (1987), Getz and Haight (1989), Montgomery and Adams (1995), and Adams et al. (1996).

Kemp and Moore (1979) present a model formulation and a conjecture that the normal forest must be the optimal long run stationary state. However, no analytical or numerical results are available. Next Heaps (1984) presents a continuous time and continuous forestland model with strictly convex harvesting costs. His specification leads to a complex system with delayed differential equations and, as regards the question of the normal forest, “complete proof remains to be found” (see also Wan 1985). Next Mitra and Wan (1986) study the issue using a model based on discrete time, continuous forestland, and no discounting. They are able to prove that an optimal solution always exists and that with strictly concave periodic utility, the optimal solution converges toward the normal forest from any initial age-class structure. Their other study (Mitra and Wan 1985) includes discounting and shows that a normal forest with Faustmann rotation is an optimal stationary state. However, no analytical or numerical results are available.

Mitra and Wan (1985) discrete time model and apply the fruitful two age-class simplification by Wan (1994). The analyses was based on elementary Karush-Kuhn-Tucker theory instead of price supports and the von Neumann Facets used by Mitra and Wan (1986) and Wan (1994). The existence of optimal stationary cycles was proved, and, using the Implicit Function Theorem, it was shown that, given discounting, a strictly concave utility function, and any initial age-class structure, the optimal solutions reach stationary cycles in finite time. In Salo and Tahvonen (2002b), the Mitra and Wan (1985) model is studied with any number of age classes, under discounting and strictly concave utility. The results by Mitra and Wan (1985) are generalized, and it is proved analytically that around the normal forest stationary state there always exists a set of age-class structures that will lead to periodic solutions without convergence to the normal forest. This model is specified in discrete time, but when the period length was shortened toward zero the cycles vanish.

Although continuous time may work against stationary cycles, the original discrete time age-class model deserves attention. It is computationally highly efficient, in line with numerous formulations in forest management and planning (cf. “model two” in Johnson and Scheurman 1977) and is used frequently in forest policy analysis (e.g., Sedjo and Lyon 1990, Adams et al. 1996). In addition, due to seasonality in harvesting and reproductive dynamics, discrete time is often found to be the most natural framework for specifying renewable resource models (Getz and Haight 1989).

This study extends the discrete time model by Mitra and Wan (1985) to contain any number of age and land classes (cf. Sedjo and Lyon 1990). This is an extension of the previous theoretical analyses that has thus far included only a single land class. We develop the necessary optimality conditions using the Karush-Kuhn-Tucker theorem. The existence of optimal stationary cycles is proved analytically. Thus, under discounting, there exists a continuum of stationary solutions where the optimal total timber harvesting follows a cyclical pattern without converging toward any constant level. We show that the cycle length for total harvesting equals the least common multiple of the optimal rotation lengths for all the land classes. A numerical example suggests that adding land classes decreases the radius of the harvesting cycle but increases the radius of the harvesting cycle in each land class. With zero rate of discount, the stationary cycles in total harvesting disappear, as in the one-land-class analysis by Mitra and Wan (1986). However, we show that in contrast to their results, a zero rate of discount is not typically associated with the normal forest structure in each land class. The normal forest structure for each land class follows only if the

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rotation period lengths for each land class have only one as a common divisor. If some land classes have rotation periods with a common divisor greater than one, the constant stationary consumption level is associated with a continuum of forest age-class structures. These analytical results are supported by numerically computed results, and together these help to interpret the volatility found in empirical forestry models.

The article is organized as follows. The first section develops the optimization problem and derives the necessary optimality conditions. The second section proves the existence of stationary cycles for total harvesting and shows the cyclicity of forest age-class evolution with no discounting and constant consumption. The third section presents an approach for computing the solutions numerically and gives several numerical results that characterize the analytical findings. The last section summarizes the main results and presents some directions for future research.

A Model for Any Number of Forest Age and Land Classes

Let the forestland area consist of land classes \( i = 1, \ldots, h \). Let \( s \) denote the age of a tree age-class, with \( s = 1, \ldots, n \), where \( n \) is a period after which trees lose their commercial value. It is reasonable to assume that \( n \) must be finite. For simplicity of notation, \( n \) is assumed the same for all land classes. The land area of type \( i \) allocated for age-class \( s \) in period \( t \) is \( x_{ist} \).

The harvest for a land area of type \( i \) with trees of \( s \) at the end of period \( t \) is given by \( z_{ist} \) (in land area units). The timber content per land unit of type \( i \) in age-class \( s \) is given by \( f_{ist} \), and is assumed to satisfy \( 0 \leq f_{ist} \leq \ldots \leq f_{ist}^{in}, i = 1, \ldots, h \). If the discount factor is denoted by \( b \), the Faustmann rotation age for land class \( i \) is \( m_i \), and it satisfies \( 0 \leq m_i \leq n \) and

\[
b^{m_i} f_{ist} \leq b^{m_i} f_{ist} (1 - b^{m_i}), \text{ for } s = 1, \ldots, n. \tag{1}\]

The land area constraints for site class \( i \) can be given as

\[
\sum_{s=1}^{n} x_{ist} = 1
\]

Note that size differences in land areas between land classes can be taken into account in the timber content coefficients \( f_{ist} \), \( i = 1, \ldots, h \). Assume that the instantaneous utility of harvesting timber is given by \( U(c_t) \), where \( U \) is a continuous, twice differentiable, increasing, and strictly concave utility function and \( c_t \) the level of total timber harvest (or “consumption”) in period \( t \). In line with Lyon and Sedjo (1990) and others, we write

\[
U(c_t) = \int_0^c D(y)dy
\]

where \( D(c_t) \) is the inverse demand function. Thus \( U \) equals the price of timber.

To keep the notation simple where possible, let \( S \in \mathbb{R}^n \) be the unit simplex, i.e.

\[
S = \left\{ x \in \mathbb{R}^n \mid x \geq 0, \sum_{s=1}^{n} x_s = 1 \right\}
\]

Let \( x_{it} = (x_{i1}, \ldots, x_{in}) \) denote the age-class structure of land class \( i, i = 1, \ldots, h \) and \( x_i = (x_{1i}, \ldots, x_{ni}) \) the vector of age-class structures of all forest types. Respectively, let \( z_{it} = (z_{i1}, \ldots, z_{in}), z_i = (z_{1i}, \ldots, z_{ni}) \) and \( z = (z_{i1}, z_{i2}, \ldots) \).

The planning problem is to maximize the present value of utility from harvesting different types of forests with initial land allocations \( x_{0it} = 1, \ldots, h \):

\[
v(x_0) = \max_z \sum_{t=0}^{\infty} b^t U(c_t) \tag{2}
\]

subject to

\[
c_t = \sum_{i=1}^{h} \sum_{s=1}^{n} f_{ist} z_{ist} \tag{3}
\]

\[
x_{it+1} - z_{ist}, s = 1, \ldots, n, i = 1, \ldots, h
\]

\[
z_i \geq 0 \tag{4}
\]

\[
x_{ist} \in S, i = 1, \ldots, h \tag{5}
\]

\[
z_{ist} = x_{ist} + s = 1, \ldots, n, i = 1, \ldots, h \tag{6}
\]

for all \( t = 0, 1, \ldots \), and where the initial allocations satisfy \( x_{0it} \in S, i = 1, \ldots, h \). If \( h = 1 \), problem (2–7) is mathematically equivalent to that in Mitra and Wan (1985).

We can eliminate \( c_t \) by (3), \( x_{ist} \) by (6), and \( z_{ist} \) by (4) and (7). The constraints \( z_{ist} \geq 0 \) take the form \( x_{ist+1} - x_{ist} \leq x_{ist}, s = 1, \ldots, n-1, i = 1, \ldots, h \). We obtain the Lagrangian

\[
L = \sum_{t=0}^{\infty} b^t U(c_t) + \sum_{i=1}^{h} \sum_{s=1}^{n-1} p_{ist+1} (x_{ist} - x_{ist+1}),
\]

and the necessary optimality conditions for all \( i = 1, \ldots, h, t = 0, 1, \ldots \) are

\[
\begin{align*}
b^{-t} \partial L / \partial x_{ist+1} &= -f_{ist} U'(c_t) + b(f_{ist} - f_{ist}) U'(c_{ist+1}) \\
&\quad + b(p_{ist+2} - p_{ist+1}) - p_{ist+1} \leq 0, s = 1, \ldots, n-2
\end{align*}
\]

\[
\begin{align*}
b^{-t} \partial L / \partial x_{ist} &= -f_{ist} U'(c_t) + b(f_{ist} - f_{ist}) U'(c_{ist+1}) \\
&\quad + b(p_{ist+2} - p_{ist+1}) - p_{ist+1} \leq 0
\end{align*}
\]

\[
x_{ist+1} \partial L / \partial x_{ist+1} = 0, \quad s = 2, \ldots, n \tag{10}
\]
where the \( p_{ist} \) are the Lagrangian multipliers for \( i = 1, \ldots, h \), \( s = 1, \ldots, n \), \( t = 0,1, \ldots \). The existence of an optimal harvesting program in the case of bounded utility and \( b < 1 \) follows from theorem 4.6, in Stokey and Lucas (1989, p. 79). For \( b = 1 \), one land class, and bounded utility, the existence is shown by Mitra and Wan (1986). For using the Lagrange method in dynamic optimization, see, e.g., Chow (1997).

### The Existence of Stationary Cycles

We next study the existence of the cyclical solutions with any number of age classes, any number of land classes, and with the discount factor equal to or less than unity. For this purpose, define Faustmann Harvesting as a program where the oldest age classes satisfying (1) are clearcut in each land class \( i = 1, \ldots, h \), while the other age classes are left untouched. A forest is called an Optimal Faustmann Forest (OFF), if the Faustmann harvesting is periodic with period length \( m_i \) for all \( i = 1, \ldots, m_i \) and periods \( t + k \), \( k = 0, \ldots, m_i - 1 \), a system of \( m \times (m_i - 1) \) equations separately for each land class \( i \):

\[
b(p_{i,s+2+t+k} - p_{i,s+t+k}) - p_{i,s+t+k} = f_i(U'(c_{i+k}) - b(f_i - f_{i+1})U'(c_{i+t+k})), \quad \text{for } s = 1, \ldots, m_i - 2
\]

\[
b(p_{i,s+t+k+1} - p_{i,s+t+k}) = f_iU'(c_{i+t+k+1}) - f_iU'(c_{i+t+k})
\]

\[
- b p_{i,s+2+t+k} - p_{l,m_i,t+k} = f_i m_i - U'(c_{i+t+k})
\]

Let us assume that \( m_i = 1, \ldots, h \) is unique for each \( i \). We shall show that besides the normal forest \( x = (1/m_1, \ldots, 1/m_p, 0, \ldots, 0) \), \( i = 1, \ldots, h \), there are other OFFs with uneven land allocation structures which lead to periodic but not to optimal harvesting. It turns out that any forest is an OFF if each land class \( i = 1, \ldots, h \) holds that \( x_i = (1/m_1 + \phi_1, \ldots, 1/m_i + \phi_1, \ldots, x_i) \) is small enough, \( s = 1, \ldots, m_i \) and \( x_i = 0 \) for \( s = m_i + 1, \ldots, n \). If the Faustmann harvesting is optimal for the problem (2)–(7) when \( x_{i0} = x_i, t = 1, \ldots, h \). Define \( K = \{ x = (x_1, \ldots, x_n) | x \text{ is OFF} \} \). We call an OFF an interior Optimal Faustmann Forest if \( x_{iy} > 0 \) for \( s = 1, \ldots, m_i \), \( t = 1, \ldots, h \).

We next study the existence of the cyclical solutions with any number of age classes, any number of land classes, and with the discount factor equal to or less than unity. For this purpose, define Faustmann Harvesting as a program where the oldest age classes satisfying (1) are clearcut in each land class \( i = 1, \ldots, h \), while the other age classes are left untouched. A forest is called an Optimal Faustmann Forest (OFF). Under Faustmann harvesting, the age-class structure \( x_i = (x_{i1}, \ldots, x_{in}) \) for every land class type \( i = 1, \ldots, h \) has the property \( x_i \) is in \( S_i \) if the Faustmann harvesting is the optimal solution for the problem (2)–(7), if there are multipliers \( x_i \) satisfying \( x_i \) is in \( S_i \) for all \( i \), \( s = 1, \ldots, m_i \) and \( x_i = 0 \) for \( s = m_i + 1, \ldots, n \). Define \( \phi^K \) as the largest number \( \phi \) which satisfies \( x_i = (1/m_1 + \phi_1, \ldots, 1/m_i + \phi_1, \ldots, x_i) \) is in \( S_i \), for all \( i \), \( s = 1, \ldots, m_i \) and \( x_i = 0 \) for \( s = m_i + 1, \ldots, n \). Note that \( \phi^K > 0 \) indicates the existence of OFFs with uneven land allocation structures.

**Proposition 1.** There exists a set of interior Optimal Faustmann Forests with \( \phi^K > 0 \).

**Proof:** Let \( m \) denote the least common multiple of integers \( m_1, \ldots, m_p \). Under Faustmann harvesting, the age-class structure of a forest of land class \( i \) is periodic with period length \( m_i \) but the harvesting is periodic with period length \( m \). By convexity of problems (2)–(7), if there are multipliers \( p_{ist} \) satisfying conditions (8)–(11) under Faustmann harvesting, \( x_i \) is an OFF. Under Faustmann harvesting, conditions (8) and (11), for interior point OFFs imply that, for all \( i = 1, \ldots, h \), \( t \geq 0 \),

\[
x_{i,s+t+1} = x_{i,s} \Rightarrow p_{i,s+t+1} \geq 0, s = 1, \ldots, m_i - 1
\]

\[
x_{i,0} = x_{i,m_i} = 1 - x_{i,1}, \ldots, - x_{i,m_i,1}
\]

\[
c_i = \sum_{i=1}^{m_i} f_{im_i} / m_i
\]

\[
x_{i,s+t+k} = 0, s = 1, \ldots, m_i, t \geq 0
\]

where \( m_i - k \) is to be understood as modulo \( m_i \) and thus to obtain the values \( 1, \ldots, m_i \) cyclically with cycle length \( m_i \).

The cycle length for the whole forest equals \( m \) periods and thus \( p_{ist} \) for \( s = 1, \ldots, n \) for each land class \( i \). Condition (8) (and condition (9), if \( m_i = n \)) yields, for \( s = 1, \ldots, m_i - 1 \) and periods \( t + k \), \( k = 0, \ldots, m_i - 1 \), a system of \( m \times (m_i - 1) \) equations separately for each land class \( i \):

\[
x_{i,s+t} \geq 0, s = 1, \ldots, m_i, t = 1, \ldots, m_i - 1 \Rightarrow p_{im_i+t+1} = 0
\]

\[
x_{i,t+1} = x_{i,t+m_i}, s = 1, \ldots, m_i
\]
Interpretations

In the cyclical solution, each land class follows a cycle with length equal to the land class-specific rotation period. This then leads very naturally to the fact that the length of the harvesting cycle must equal the least common multiple of the land class-specific rotation periods.

Let us analyze and interpret condition (19) in the simplest possible case, i.e., when \( h = 1, m = n = 2 \). The conditions reduce to the following single equation:

\[
p_{2\tau} = \frac{f_{\tau} b^2}{1 - b^2} \left[ U'(c_{\tau+1})/b - U'(c_{\tau+2}) \right] - f_{\tau} U'(c_{\tau})
\]

With \( h = 1 \) and \( m = n = 2 \), the cycle length must be two periods, implying \( c_{\tau+2} = c_{\tau} \). Using condition (11) and \( p_{2\tau} \ge 0 \), we can write the condition in the form

\[
U'(c_{\tau}) f_{\tau} b^2 - U'(c_{\tau+1}) f_{\tau} b + U'(c_{\tau+1}) f_{\tau} b^2 - U'(c_{\tau+2}) f_{\tau} b^2 \ge 0 \quad (19')
\]

or equivalently

\[
\frac{U'(c_{\tau})}{U'(c_{\tau+1})} \le \frac{b f_{\tau}}{f_{\tau+1} b^2 - f_{\tau-1} b^2} \equiv \eta \quad \forall \tau \ge 0 \quad (19'')
\]

Equation (19'') is equivalent to condition (10) in Wan (1994), who studies the case \( h = 1, m = n = 2 \). (However, note the typing error on the LHS of Wan’s condition (10) and that his

\[
\eta = \frac{(1 + \sigma)}{(1 - \sigma)}
\]

reduces to \( \eta = b/[a + (1 - a)b^2] \), which is equivalent to \( \eta \) given in (19''), since Wan (1994) assumes \( f_1 = a \) and \( f_2 = 1 \).

Any optimal stationary cycle must satisfy (19''). We obtain \( \eta > 1 \) by (1) and \( m = 2 \). Thus (19'') is clearly satisfied for the normal forest with \( c_1 \) constant. However, by the strict concavity of \( U \), (19'') is trivially satisfied when \( c_{\tau} > c_{\tau+1} \) but also for same values where \( c_{\tau} < c_{\tau+1} \), which then makes the existence of optimal cycles evident. Condition (19') states that a cyclical solution is optimal and that smoothing the cycle cannot be optimal when the sum of the marginal utility of increasing this period’s “low” harvest, \( U'(c_{\tau}) f_1 \), and the related increase in the present value utility from periods \( t + 2, t + 4, \ldots \) harvests does not exceed the present value marginal utility from periods \( t + 1, t + 3, \ldots \) harvests. If we let \( c_{\tau} \rightarrow c_{\tau+1} \), condition (19') reduces to \( U'(c_{\tau}) f_2 b^2(1 - b^2) + f_1 - f_2 b(1 - b^2) \le 0 \) or \( f_2 b^2(1 - b^2) - f_1 b(1 - b) \ge 0 \), which holds as a strict inequality by the Faustmann optimal rotation condition for \( m = 2 \). Thus, smoothing the cycle up to the normal forest would violate the Faustmann condition. This has the following interpretation: Adjusting the age-class structure closer to the normal forest requires cutting trees one period before their Faustmann financial maturity. This is optimal as long as the differences in periodic marginal utilities are large enough. However, due to discrete time, the marginal adjustment costs always remain strictly positive while marginal benefits from the smoother periodic harvest approach zero when the land allocation is arbitrarily close to the normal forest. Adding any number of land and age classes does not change these facts although, as will be shown next, a smooth time path for total harvesting can be obtained with much more freedom in the age-class structure.

An Example

To demonstrate the optimal stationary cycles let us study the following example:

\[
U(c) = 4c - \frac{1}{2}c^2, \quad n = 2, \quad b = \frac{3}{4},
\]

\[
f_{11} = \frac{1}{2}, \quad f_{21} = \frac{4}{3}, \quad f_{22} = \frac{2}{3}, \quad f_{22} = 2.
\]

Because \( f_{11} < b(f_{12} - f_{11}) \) and \( f_{21} < b(f_{22} - f_{21}) \) we obtain \( m_1 = m_2 = m = n = 2 \). To simplify we eliminate the younger age classes and let \( x_{it} \) denote the land area under the older age class in land class \( i, i = 1,2 \). By (13) and (14), we obtain, for one complete cycle under Faustmann harvesting,

\[
X_{1,t+1} = 1 - x_{1t},
\]

\[
X_{2,t+1} = 1 - x_{2t},
\]

\[
c_{\tau} = f_{12} x_{1t} + f_{22} x_{2t}, \quad c_{\tau+1} = f_{12} x_{1,t+1} + f_{22} x_{2,t+1} = f_{12} + f_{22} - f_{12} x_{1t} - f_{22} x_{2t}
\]

\[
= f_{12} + f_{22} - c_{\tau}
\]

This periodic solution is optimal if (11) and (19) hold, i.e., if

\[
p_{\tau} = (-f_{12} + (f_{12} - f_{11}) b^2) U'(c_{\tau}) + b f_{12} U'(c_{\tau+1}) \ge 0
\]

for all \( \tau \ge 0 \), \( I = 1,2 \). This yields

\[
\frac{U'(c_{\tau})}{U'(c_{\tau+1})} \le \frac{b f_{12}}{f_{11} + (f_{12} - f_{11}) b^2} = \frac{32}{31} \tag{20a}
\]

\[
\frac{U'(c_{\tau+1})}{U'(c_{\tau})} \le \frac{b f_{22}}{f_{21} + (f_{22} - f_{21}) b^2} = \frac{32}{31} \tag{20b}
\]

and

\[
\frac{U'(c_{\tau})}{U'(c_{\tau+1})} \le \frac{b f_{22}}{f_{21} + (f_{22} - f_{21}) b^2} = \frac{18}{17} \tag{21a}
\]

\[
\frac{U'(c_{\tau+1})}{U'(c_{\tau})} \le \frac{b f_{22}}{f_{21} + (f_{22} - f_{21}) b^2} = \frac{18}{17} \tag{21b}
\]
Computing shows that only \((20,a,b)\) are binding and we obtain

\[
\frac{44}{27} \leq c_t \leq \frac{46}{27}
\]

corresponding to the continuum of age-class structures

\[
\frac{44}{27} \leq \frac{4}{3} x_t + 2 x_{2t} \leq \frac{46}{27},
\]

\[0 \leq x_t \leq 1, 0 \leq x_{2t} \leq 1
\]

Note that if we restrict ourselves to solutions giving the constant total harvest \(c = (f_{12} + f_{22})/2 = 5/3\), we still obtain a continuum of OFF age-class structures given by the conditions

\[
\frac{5}{3} \leq \frac{4}{3} x_t + 2 x_{2t},
\]

\[0 \leq x_t \leq 1, 0 \leq x_{2t} \leq 1
\]

To get a more general picture, define \(\phi_y\) as the maximal cycle radius for variable \(y\) in the sense that \(y \in [y_\infty - \phi_y, y_\infty + \phi_y]\) in the two period cycle and where \(y_\infty\) is the steady state level of \(y\). Thus, in the above example, the maximal cycle radius for \(c_t\) is 1/27 and, for \(x_{1t}\) and \(x_{2t}\), 1/2 and 19/54, respectively, since

\[
\frac{44}{27} = 4 \cdot \frac{1}{3} + 2 \left( \frac{1}{2} - \frac{19}{54} \right)
\]

and

\[
\frac{46}{27} = 0 + 2 \left( \frac{1}{2} - \frac{19}{54} \right)
\]

The maximal cycle radius depends on the discount factor. This is depicted in Figures 1a,b. In Figure 1a, the maximal radius for the harvest level is \(\phi_c^2\) if only land class 1 exists and \(\phi_c^2\) if only land class 2 exists. With both land classes, the maximal cycle radius is \(\phi_c\). Figure 1b shows the maximum cycle radius for the land areas. If the model consists only of land class 1 or 2, the maximal radius is \(\phi'_{x1}\) or \(\phi'_{x2}\), respectively. With both land classes allowed, the maximal radii are \(\phi'_{x1}\) and \(\phi'_{x2}\). We can observe that adding another land class decreases the maximal harvesting cycle but increases the maximal cycle for the land areas. In addition, cycles for the land areas may occur even with \(b = 1\) and constant harvesting.

The example studied shows that cycles or continua of age-class structures may exist even with constant harvesting flow. We next study the existence of stationary continua for the case of constant harvesting.

### Optimal Age Class Structures with Constant Harvesting Levels

**Lemma 1.** Let \(x\) be an OFF with \(c_t\) constant under optimal harvesting. Then

\[
c_t = \sum_{i=1}^{h} f_{im_i} / m_i \quad \text{for all } i \geq 0.
\]

**Proof:** Under Faustmann harvesting, a complete harvesting cycle is given by (14). Summing over the cycle, i.e., over \(k = 0, \ldots, m - 1\) in (14), observing the land area constraints and noting that \(x_{i,m_i-k,i}\) is cyclic with cycle length \(m_i\), we obtain

\[
mc_i = \sum_{i=1}^{h} (m_i / m_i) f_{im_i}, \text{ i.e.,}
\]

\[
c_t = \sum_{i=1}^{h} (f_{im_i} / m_i)
\]

which is the total harvest given by the normal forest. ■

**Proposition 2.** There is a continuum of Optimal Faustmann Forests with constant optimal harvest if and only if there are \(i\) and \(j\), \(i \neq j\), such that \(m_i\) and \(m_j\) have a common divisor \(m' > 1\). In any OFF with constant optimal harvest, land class \(i\) forms a normal forest if \(m_i\) and \(m_j\) have no common divisors greater than one for all \(j \neq i\).
Proof: Let \( m' > 1 \) be a divisor of \( m_i \) and \( m_j \). Let land class \( k \) be a normal forest, \( i \neq k \neq j \). Let the forest structure in land classes \( i \) and \( j \) be cyclic with cycle length \( m' \) under Faustmann harvesting. Then, e.g., land class \( i \) is subdivided into \( m_i / m' \) groups of age classes with identical forest structures. Under Faustmann harvesting, the cycle length of the whole forest is \( m' \), and the harvest from land class \( k, i \neq k \neq j \), is \( f_{s, m_i} / m_k \), i.e., constant. Then, by Lemma 1, we have an OFF if the harvest from land classes \( i \) and \( j \) together is the constant \( f_{s, m_i} / m_i \) over the whole cycle, i.e., if

\[
f_{s, m_i} x_{ik} + f_{s, m_j} x_{jk} = f_{s, m_i} / m_i + f_{s, m_j} / m_j, k = 1, ..., m'.
\]

Observing the land area constraints

\[
\sum_{k=1}^{m'} x_{ik} = m' / m_i, l = i, j
\]

we note that this is a system of linear equations with \( 2m' - 2 \) variables and \( m' \) equations. The last equation is linearly dependent on the other equations, as can be seen by summing the equations over \( k = 1, ..., m' \). Because the normal forest provides one solution for this underdetermined system of equations, it has a continuum of solutions. This proves the “if” part of the first claim. The “only if” part follows from the second claim.

To prove the second claim, assume \( m_i \) and \( m_j \) have no common divisors greater than one for all \( j \neq i \), and consider an OFF with constant optimal harvest. Let \( m = m_l m_j \). Then for all \( t \geq 0 \), the forest structure and the harvest from site class \( j \) will be the same in periods \( t + km', k = 0, ..., m_i - 1 \), separately for all \( j \neq i \). However, because \( m' \) and \( m_i \) have no common divisors, the harvest in site class \( i \) must go through all the values of one complete cycle of length \( m_i \). By assumption, the total harvest is constant and thus the harvest from land class \( i \) must be constant also, which proves the claim.

In their study, Mitra and Wan (1986) show that, with discount factor equal to unity, no cycles occur, and the optimal solution converges toward the normal forest from any initial state. We can use Proposition 2 for studying whether a similar result holds with multiple land classes. It can be shown that with \( b = 1 \), a solution with constant harvesting satisfies conditions (8)–(11), given that the rotation periods satisfy the maximum sustainable yield condition [cf.(1)]

\[
f_{s, m_i} / m_i \geq f_{s, m_j} / m_j, \text{ for } s = 1, ..., n, i = 1, ..., h.
\]

Next we observe from Proposition 2 that a normal forest is the unique stationary age-class structure only if there exist no \( m_i \) and \( m_j \), \( i \neq j \), with a common divisor \( m' > 1 \). If such a common divisor exists for any two land classes, there always exists a continuum of stationary age-class structures, in addition to the normal forest. Thus the Mitra and Wan (1986) result that, with zero rate of discount, the forest age-class structure convergences toward the normal forest does not hold with multiple land classes.

It is, however, possible to relax the definition of the normal forest. One definition for the normal forest may require that there exists the normal forest structure for each land class separately. Under this definition the Mitra and Wan (1986) result that zero rate of discount implies the convergence of the optimal solution to the normal forest does not carry over to the case with multiple land classes. Another possibility is to label a forest a normal forest if harvesting the oldest age-class per each land class in each period yields constant total timber flow over time.

Our Proposition 1 shows that under positive discounting a stationary cycle with cyclically evolving harvest is likely to form the long run stationary solution instead of such generalized normal forest structures. A zero rate of discount rules out cycles in total harvesting and is thus consistent with this generalized definition of the normal forest. This, however, relies on the assumption that timber flows from different land classes are perfect substitutes. In an extended model that includes timber quality aspects, it should be required that in a normal forest both the total harvest level and the quality of harvest be constant over time (see, e.g., Davis and Johnson 1987, p. 540). This would in turn favor the narrow definition of the normal forest.

The world timber supply model by Sedjo and Lyon (1990) has a structure with many features similar to those of the model analyzed here. Their model includes 23 land classes distributed over several timber supply regions, and one demand function for the total timber harvested. They solve the model by computing a time path that reaches a steady state where “all periods are alike” and the volume of timber harvested in each land class is equal to its steady state level (Sedjo and Lyon 1990, p. 109). At the global level, this steady state is characterized as the foresters’ ideal, i.e., as the normal forest, although the forest age class structure in each region is not required to represent the normal forest. Sedjo and Lyon (1990, p. 130) observe from their results that, although the entire global forest system approaches the foresters’ ideal, there is typically high volatility in regional harvests.

These findings can be explained by our Propositions 1 and 2. By Proposition 1, we note that with a positive rate of discount such a stationary state (i.e., the “foresters’ ideal”) may be restrictive, since typically the optimal solution approaches a stationary cycle with fluctuating total harvest levels. The fact that at their stationary state there is high volatility between the regions is strictly in line with Proposition 2. Forests in different regions in any given land class, each with its region specific initial age class structure, can be interpreted as land classes with exactly the same growth function. Assume that in spite of the possible differences in various cost parameters in their model, the rotation periods are the same in different regions for a given land class. In such a case, there are clearly rotation periods with a common divisor greater than one. This then explains the high volatility between the regions in their model.

In classical forestry literature (Davis and Johnson 1987, p. 550), the normal forest concept is typically defined under the
Computation of Optimal Harvesting Programs

We still have the open question of the optimal evolution of the age-class structure if $x_0 \notin K$. Salo and Tahvonen (2002a), using the Implicit Function Theorem, prove that, given $h=1$, $n=m=2$, the optimal solution converges toward stationary cycles, given any initial age-class structure. In numerical calculations, this finite convergence can be easily tested, and, moreover, proven optimal for a model that includes any number of land and age classes. For this purpose we employ a formulation, in which optimization is done only over finitely many periods and substitute a scrap value function for the rest of the periods. If the optimal solution reaches the set of OFFs in a finite number of periods, the optimal solution for the rest of the periods will apply Faustmann Harvesting, and thus the utility after reaching the OFF can be calculated as a function of the initial age-class structure in the OFF. Hence, consider maximizing

$$v(x_0, T) = \max_{\{x_t, t=0, \ldots, T-1\}} \left\{ \sum_{t=0}^{T-1} b^t U(c_t) + b^T v_K(x_T) \right\} \tag{20}$$

s.t. (3)–(7) for all $t=0, \ldots, T-1$, where the initial allocations satisfy the land area constraints and

$$v_K(x_T) = \sum_{k=1}^{m} b^{m-k} U(c_{T+m-k}) \tag{21}$$

where

$$c_{T+m-k} = \sum_{i=1}^{h} f_{im} x_{[k]T+k} = 1, \ldots, m, k = k \mod m_i.$$

The scrap value $v_K(x_T)$ is the present value utility if the Faustmann harvesting policy is applied to the initial allocation $x_T$. Salo and Tahvonen (2002b) prove that, given $T$ is large enough and $h=1$, the necessary conditions for maximizing (20) subject to (3)–(7) are equivalent to the necessary conditions of problems (2)–(7). Thus computing the optimal solution using the scrap value formulation yields the optimal solution, given that $T$ is large enough. Following Salo and Tahvonen (2002b), we may prove analogously the following Proposition.

Proposition 3. If $x^*_t, t=0, \ldots, T$, is the optimal sequence for problem (20) s.t. (3)–(7) from $x_0$ satisfying $x_0 \in \mathcal{S}, i = 1, \ldots, h$ and $x_{T+k} \in K$, where $k = \max\{1, n-m, f = i, \ldots, h\}$, then the optimal sequence for problem (2) s.t. (3)–(7) from $x_0$ is $x^*_t, t=0, \ldots, T^*$, continued by the Faustmann periodic solution from $x_T$.

The proof of Proposition 3 is a straightforward extension of the proof for the case $h=1$ given in Salo and Tahvonen (2000b) and is not repeated here. We next use specification (20)–(21) and present numerically computed solutions for demonstrating Propositions 1 and 2 and the cases where the initial age-class structure is outside the cyclical stationary state. For these computations, we use the Matlab programming framework and a solution algorithm by Andersen and Ye (1999).

In Figures 2a,b it is assumed that $n=13$, $h=6$, $T=50$, $b=0.9999$, $U(c) = kc - 1/2c^2$, $k=5000$, $f_1=0, 51, 123, \ldots, 123, f_2=0, 43, 87, 132, \ldots, 132, f_3=0, 47, 104, 120, 280, \ldots, 280, f_4=0, 39, 90, 120, 170, 200, 250, \ldots, 250, f_5=0, 40, 60, 80, 110, 140, 160, 200, 270, 300, 350, 420, 420, 420$. The initial state satisfies $x_0 \in \mathcal{S}, i = 1$ and $x_{1,3,0} = 1$. Hence, consider maximizing (20) subject to (3)–(7) for all $t=0, \ldots, T-1$, where the initial allocations satisfy the land area constraints and

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discount and \( b = 0.98 \). Although \( m_1 = 8 \) and \( m_2 = 10 \), the effect of discounting is clear: both age classes converge toward the stationary cycle with greater radius than in Figure 4a, and the total timber harvesting converges toward a cycle instead of the even flow of timber found in Figure 4b.

**Discussion**

The question of what is the optimal long run stationary state for a forest age-class system in models with discounting and nonlinear utility has remained open in forest economics. This study extends the previous analytical studies by including not only any number of age classes but also any number of land classes, each with its own forest growth function. We show that under discounting there exists a set of stationary cyclical solutions around a solution where timber harvest is constant over time. This shows that the “foresters’ ideal” age-class structure will appear as the long run optimal stationary state only accidentally. More typically, the optimal timber harvesting level will follow a cycle with a length that equals the least common multiple of the land class-specific rotation periods. Previous results by Mitra and Wan (1986) show that with one land class, the optimal solution converges to the normal forest from any initial age-class structure if there is no discounting. With multiple land classes, a zero rate of discount implies only that in the long run, timber harvest will be constant. However, this is typically associated with a collection of land-class-specific age class structures that deviate from the normal forest.

The model in this study applies discontinuous time, as do the most age-class models in renewable resource economics and population ecology (see Getz and Haight 1989). In forestry models, the amount of land or space must be specified. We measure it by continuous units represented by vectors of land areas for each age-class, i.e., \( \mathbf{x}_t = (x_{1t}, ..., x_{nt}), t = 1, ..., h \). In this specification, the optimal cycles occur since, due to discrete time, the marginal costs of smoothing the age-class structure toward the normal forest are always strictly positive whereas since space is continuous, the marginal benefits from additional smoothing approaches zero as the age-class structure ap-
proaches the normal forest. In Salo and Tahvonen (2002b),
the period length is decreased toward zero, with the impli-
cation that the maximum cycle radius converges toward
zero. However, this does not change the fact that, given
any strictly positive period length (e.g., 1 yr or 1 hr), the cycle
radius will increase toward some physically determined
maximum as the nonlinearity of the utility function is de-
creased toward zero. On the other hand, this result suggests
that using a “long” period length like 10 yr may yield
excessively high volatility for timber harvesting and price.

Heaps (1984) applies continuous time and continuous
space and finds support for the hypothesis that the optimal
solution may approach the normal forest although, due to
mathematical difficulties, the final proof remains open. How-
ever, it must be noted that if both time and space are continu-
ous, the model describes a forest where the age-class struc-
ture forms a continuum. The same outcome follows in the
discrete time/continuous space model as the period length is
decreased toward zero. Continua of age-class structures are
difficult to interpret, at least in boreal forests, where 1 yr may
represent a natural period length.

To avoid the mathematical difficulties in Heaps (1984)
and to recognize the fact that forests cannot be harvested in
units smaller than one tree, Tahvonen and Salo (1999, 2000)
and Tahvonen et al. (2001) develop a model where time is
continuous and space is discrete. In this formulation, forest-
land is measured in units that each contain at least one tree.
In a model version comparable to the models referred to here,
the authors prove the convergence of the optimal solution
toward the normal forest given that the interest rate is not “too
high,” and the initial age-class structure not “too far” away
from the normal forest. Numerical computation has not
revealed any exceptions to this result, with any interest rate
level and with any initial age class structure.

Figure 3a,b. No discounting without the normal forest.
Figure 4a,b. Optimal harvesting of two land classes with no discounting.
The fourth possibility would be to apply discrete time and discrete space. However, we are unaware of whether this specification has been applied with a nonlinear objective function and infinite time horizon. The question of which of these four specifications is the correct one may be misplaced. Instead, different specifications may be useful in different contexts with none being free of some problems. It may be highly useful if theoretical forest economic research could explain the features of all these models.

The normal forest issue is further complicated if additional aspects are included in the basic model versions that we have discussed so far. Tahvonen and Salo (2000) show that optimal cycles around the normal forest may exist in the specification with continuous time and discrete space if the marginal utility remains finite as the flow of timber goes to zero. Salo and Tahvonen (2002a) prove that cycles disappear in a two-age class Mitra and Wan (1985) model if it is optimal to allocate part of the land area to an alternative use like agriculture. One interesting topic for future study would be the existence of cycles and the effects of alternative land uses in forestry models that contain multiple land and age classes. As regards high regional volatility in harvest rates, one useful extension might be to add region-specific strictly convex harvesting costs or region-specific strictly concave wood processing capacity and global markets for processed wood products instead of global markets for raw timber. Also other products of forests, like wildlife, have effects on the long run behavior of these models, and in the forestry literature these have frequently been mentioned as one reason behind the “foresters’ ideal.” Finally, it should be noted that very similar issues arise in the context of stand-level uneven-aged forestry models.

**Figure 5a,b.** Optimal harvesting of two land classes with discounting.

The fourth possibility would be to apply discrete time and discrete space. However, we are unaware of whether this specification has been applied with a nonlinear objective function and infinite time horizon. The question of which of these four specifications is the correct one may be misplaced. Instead, different specifications may be useful in different contexts with none being free of some problems. It may be highly useful if theoretical forest economic research could explain the features of all these models.

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**Literature Cited**


