Acquisition of a Forest Estate: A Stochastic Optimization Approach for Financing and Management

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ABSTRACT. This article addresses the financing and managing problem of a private investor who considers purchasing a forest estate. The focus is on financing the investment, timing of harvesting, and financial and nonfinancial benefits over years succeeding an acquisition. We take into account uncertainty concerning future timber prices, real rate of interest, and rate of inflation. To aid in the solution of the problem, we propose a multistage stochastic optimization model. The total amount of loan available is assumed given, while alternative interest and repayment schedules are considered for loans. The model accounts for the initial state of the forest, growth, and allowable cutting. The objective function is the expected utility. In an illustrative example, the utility for a risk averse investor is based on income for consumption. Variations of this model demonstrate the use of decision aids. These concern valuation of a forest estate, initial age distribution of forest stands, timber price volatility, initial equity, time-varying preferences for consumption, and nonfinancial benefits. For. Sci. 49(5):706–718.

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WHEN ACQUIRING A FOREST ESTATE, an important question is how to finance and manage the estate, given preferences of the investor. A key problem in the short as well as in the long run is, how should owned and borrowed capital as well as revenues from forest cutting be allocated, given the price of the estate, debt payments, expected costs of silvicultural activities, relevant aspects of uncertainty, and desired levels of consumption expenditure and nonfinancial benefits. While considering such an investment, this problem is of central importance. The complexity of the problem calls for an efficient tool for decision support for which a stochastic optimization model is proposed here.

An essential part of the purchase may be financed by cutting the forest assets at the time of purchase. However, the forest represents both a production process and the product itself. Thus, large clearcuttings not only reduce the growing stock of round wood but also lessen the cutting opportunities in the future. A desired level of cuttings depends on expected benefits of keeping tree capital in the forest and on expected costs of obtaining funding from the credit markets.

The opportunities for credit depend on more or less uncertain value of the forest estate over time. This value is related to the total volume of standing forest in combination with the current value per cubic meter. An important concern is the cost of capital invested. The equity is not free as competing investment alternatives exist, resulting in an opportunity cost for equity capital. The interest and amortization payments are connected with liquidity aspects. The cash flow into the estate should be large enough to cover...
the cash flow out; i.e., interest payments, debt repayment, needs of silvicultural activities, private consumption, etc. A reason for the complexity of problems involving liquidity is that loan payment schedules are not synchronized with the yield of long-term assets, such as forestland, particularly with simultaneous influence of inflation.

An essential factor is risk, primarily related to the uncertain future of timber prices, real interest rate, and rate of inflation. Consequently, one cannot know precisely the profitability of the forest estate at the time of purchase. What can be even more serious is that the purchaser may find oneself in a situation (e.g., a sudden decline in timber prices), where funds are totally drained in a short time. Obviously, such a situation emphasizes the liquidity aspects of the planning problem as well. To hedge against the risk of illiquidity, the investor may want to hold a liquidity reserve: i.e., the buyer may prefer to invest part of the capital in liquid assets, such as bank accounts.

For agricultural firms, there are a number of studies concerning optimal financial management under uncertainty (Barry 1994, Collings and Karp 1993, Featherstone et al. 1993, 1990, Gwinn et al. 1992). Also, the number of articles in forest economics dealing with risk and uncertainty aspects of round wood prices and the optimal harvest age has increased rapidly since the late 1980s. For example, stochastic processes, such as random walk or geometric Brownian motion, have been employed to model fluctuating stumpage prices (e.g., Clarke and Reed 1989, Thomson 1991). Others have assumed a random sample (e.g., Brazee and Mendelsohn 1988, Gong 1995). An important area for empirical research has been the performance of forest investments in comparison with nonforest investments. Some studies present risk measures for forest investments (Redmond and Cubbage 1988, Washburn and Binkley 1990, Lönnstedt and Svensson 2000a, Svensson 1997) while others address the role of forestry investments in diversified portfolios (Thomson 1991, Caulfield 1998, Heikkinen 1999). The joint problem of financing and management associated to acquisitions of NIPF estates was addressed in the doctoral thesis by Rutegård (Rutegård 1995 and 1997). Our task is to present the results with some extensions.

In this article, we develop a stochastic optimization model to support financial and managerial planning for acquisitions of forest estates. In terms of preferences, an objective is the utility of the investor’s consumption over the planning period, while other objectives may take into account nonfinancial benefits of the forest estate. Optimization results in a plan which simultaneously takes into account a number of considerations, such as the price of the forest estate, available amounts of equity capital and loans, loan payment schedules, liquidity reserves, initial state of the forest, expected future yield of the forest, and stochastic characteristics of timber price, interest rates and inflation. Finding an optimal plan calls for dynamic stochastic optimization. The advantage of using a dynamic model instead of a single-period model derives from the ability of a multi-period model to look ahead and adapt the current decisions to what might happen afterwards. This statement also applies to our financial planning problem and the associated scheduling of cutting decisions over time.

A standard method of dealing with sequential decisions under uncertainty is stochastic dynamic programming introduced by Bellman (1957). There is an extensive literature on forestry applications of this principle. For examples, see Lohmander (1987) and Mörling (1994). However, because of a large number of state variables, dynamic programming suffers from a well-known curse of dimensionality, which prevents its application here. Our approach is multistage stochastic programming proposed by Dantzig and Madansky (1961). For subsequent developments, see the state of the art review in Wets and Ziemba (1999).

We proceed with development of a forest estate planning model for financing and management. The formulation is relatively general; however, many additional features desired in practical applications are easily adapted to our formulation. Thereafter, an illustrative example of our optimization model is introduced, while some of the data of this model is discussed already in connection to the general formulation of the preceding section. Variations of this model are used to demonstrate the versatility and the power for decision aid. A formal introduction to standard convex multistage stochastic programming problems is presented in the Appendix, which sets the basis for the model development of our article.

A Financial Planning Model for Acquisitions

We now develop a financial planning model for an acquisition of a forest estate. We employ the multistage stochastic programming framework (24)–(27) in the Appendix, with a concave objective function and linear constraints. The planning horizon of such a model is subdivided into a number of stages \( t, t = 0,1,\ldots,T \), and realizations of uncertainties are represented by scenarios denoted by \( \omega \). Stochastic models for the rate of interest, inflation, and timber price are briefly discussed first. Such models are used to generate the scenario tree with realizations of the random variables for each node of the tree. Next, the objective function based on expected utility is developed. Finally, various constraints for the model are formally introduced. These include cash flow equations, liquidity buffer requirements, constraints on loans, a submodel of forest growth, thinning and harvesting, and various conditions related to the beginning and the end of the planning horizon. Without further notice, all decision variables are nonnegative. Similarly, nonanticipativity constraints (27) are not explicitly restated.

Timber Price, Rate of Interest and Inflation

In this context, risk is related to the uncertain future of timber prices, real interest rate, and rate of inflation. A number of alternative models from econometric time series analysis may be employed to describe the stochastic variables; see, e.g., Enders (1995) and Lütkepohl (1993). In the numerical illustration, we employ a vector auto regressive (VAR) model.

The scenario approach adjusted to multistage optimization is employed. In each scenario, a sequence of realizations of
random variables is defined over the planning horizon. In our model, such realizations refer to the three stochastic variables. The planning horizon is subdivided into a number of subperiods, and the sequential changes in each scenario correspond to the periods. Decisions in the model are made at the beginning of each time period. In our numerical examples below, the planning horizon comprises a 20 yr period, which is subdivided into six subperiods. The time of acquisition is at the beginning of the first subperiod. The realizations of the three stochastic variables are based on Monte Carlo simulation using the estimated VAR model.

**Objective Function**

According to the axiomatic theory of expected utility proposed by von Neumann and Morgenstern (1947), given assumptions concerning existence of a preference relation and its transitivity, continuity, and independence, a utility function exists such that utility function values for outcomes rank alternative outcomes in preference order. Furthermore, if an outcome involves uncertainty, then the utility function can be evaluated as expected value of the utility function. It is important to note that utility functions are specific to individuals. Despite some extra effort needed in assessing the utility, this is a theoretically sound approach which is applicable in practice. Utility theory is widely accepted as a basis for modeling choice behavior of agents in economic theory. However, in the presence of low probability high consequence events, for instance, such as catastrophic risks, expected utility theory has been criticized; in particular, independence assumption has been questioned. Consequently, improvements such as nonexpected utility theories have been proposed (see, e.g., Machina 1983, p. 89–123).

In this article, however, we restrict the discussion to expected utility. The utility function takes into account all relevant aspects concerning the decision maker’s preferences. For a forest owner, such aspects may concern, for instance, financial gains and losses as well as nonfinancial benefits and sacrifices. Forests often serve multiple purposes, such as timber supply, water shed, wildlife preservation, and outdoor recreation.

To operationalize utility theory for a particular application, one has to specify which aspects of outcomes matter in preferences. Hence, a utility function is defined to take into account all such aspects. However, in pragmatic applications, only some aspects may be taken into account by the utility function, while others are subject to explicit constraints.

In common applications of utility theory, the outcomes are financial streams defined over time (see, e.g., Kreps, 1988, p. 71–97). If such streams are stochastic, then expected utility accounts for individual preferences concerning the level and uncertainty in financial returns. Similarly, we begin by specifying the outcome as a cash flow for consumption over time. However, at the end of this subsection, we discuss a simple case with both financial and nonfinancial returns determining the utility of an outcome.

Consider an objective to maximize expected utility of consumption over the planning period, thereby taking into account individual risk aversion of the decision maker. The formulation is closely related to intertemporal analysis by Barry et al. (1996), for instance. Formally, our objective is stated as follows:

\[
\max \sum_{\omega} p_\omega \sum_{t} \sum_{\tau \in T_t} \beta^\tau U_t(c_{\omega t}),
\]

where \(p_\omega\) is the probability for scenario \(\omega\), \(\beta^\tau\) is an annual discounting factor for utility accounting for time preference of the investor, \(U_t\) is a utility function and \(c_{\omega t}\) is annual consumption in period \(t, t = 0, ..., T - 1\), and in scenario \(\omega, \omega = 1, ..., \Omega\). Due to our method of simulation in numerical illustrations, all scenarios are equally likely; i.e., the scenario probability is \(p_\omega = 1/\Omega\) for all \(\omega\). Hence, in the objective function, the model considers combined impact of all scenarios, with an equal weight for each one. Some of the periods \(t\) consist of several years denoted by the set \(T_t\). Hence, summation over \(\tau \in T_t\) takes into account each year separately, and \(\tau\) indicates the number of preceding years in the planning horizon. In scenario \(\omega\), annual consumption \(c_{\omega t}\) is assumed equal for each year \(\tau \in T_t\), but the annual discounted utility is defined separately for each year \(\tau\).

As a special case, objective (1) includes risk neutral individuals, who choose alternatives based on expected consumption. Different attitudes toward risk can be represented by different utility functions \(U_t(c)\), where \(c\) stands for consumable wealth. A common assumption in financial modeling is that investors prefer more wealth to less, so that \(U_t\) is a strictly monotone increasing function; i.e., \(U_t''(c) > 0\). A further common supposition is that the utility function is concave; i.e., \(U_t'(c) < 0\), which means decreasing marginal utility for additional consumption and indicates a risk-averse decision maker. In the current study, both of these assumptions are adopted; for further discussion on validity of the assumption, see, e.g., Rutegård (1995), Lönnstedt and Svensson (2000b), and Löfgren (1994). Subsequently, for numerical illustration, we choose the following negative exponential utility function:

\[
U_t(c) = -\exp[-\gamma(c - \hat{c}_t)]
\]

where \(\gamma\) is a risk aversion parameter, \(c\) is annual consumption and \(\hat{c}_t\) is a target consumption. In this notation, the Arrow-Pratt risk aversion coefficient is \(-U_t''/U_t' = \gamma\), which in our case is independent of the level of consumption. Risk aversion increases along with an increasing risk aversion coefficient, so that different individual risk attitudes can be represented by the functional form (2). In the base case of numerical illustrations we use \(\gamma = 0.1\) and \(\hat{c}_t = 50,000\) SEK[2] per year, for all \(t\). However, foreseen changes in needs can be incorporated in the model with time varying target consumption. For demonstrations below we consider planned retirement. Also random changes in needs due to illness or a natural catastrophe, for instance, can be incorporated by means of time and scenario dependent target consumption \(\hat{c}_{\omega t}\).

To illustrate the impact of nonfinancial return in the definition of the utility function, let \(S_{\omega t}\) denote the volume of growing stock in scenario \(\omega\) at the end of period \(t, t =
1,2, ..., T. We use $S_t^{\omega}$ as a measure of nonfinancial benefits, and replace the function $U_t$ of (2) in (1) by

$$U_t(c, S) = -\exp[-\gamma(c - \hat{c})] - \exp[-\lambda(S - \hat{S})],$$

where $\lambda$ is a positive parameter, $S_t$ is the growing stock and $\hat{S}$ is a target volume. Thus, the objective function in (1) becomes a separable function having components both for financial and for nonfinancial return. For estimation of parametrized utility functions, see Keeney and Raiffa 1976. A numerical illustration of the combined utility function is provided where $S_t$ is the initial level of growing stock, for all $t$.

**Cash Flow Equation and Debt Restrictions**

A source of complexity of problem relates to the fact that loans and loan payment schedules under inflation are not synchronized to the cash earned from forest land. Ideally, we might plan an increase of debt whenever a need arises. However, this rule may not be practically valid or may not be allowed by creditors. Opportunities for additional credit depend, among others, on the uncertain collateral value of the forest estate over time. Therefore, the forest owner cannot be sure that a lender is willing to issue further credits at a future time. Consequently, we do not consider the possibility of additional loans after the initial stage.

The accounting equation for cash is introduced as a model constraint. Let $E_t^{\omega}$ denote cash available in nominal currency at the beginning of time period $t$, $t = 0, 1, ..., T$, where $t = T$ refers to the end of the planning horizon. An exogenous lower bound $\hat{E}_t$ on $E_t^{\omega}$ acts as a liquidity buffer:

$$E_t^{\omega} \geq \hat{E}_t$$

for all $t$. Of course, $\hat{E}_t$ can be made dependent on $\omega$ if needed. For $t < T$, $E_{t+1}^{\omega}$ is determined as a sum of a number of factors. First, cash $E_t^{\omega}$ at the beginning of period $t$ is adjusted by total nominal interest return, which is the product of total real interest return $R_t^{\omega}$ and annual inflation multiplier $I_t^{\omega}$. Thus during the period of $N_t$ years the initial cash of period $t$ grows to $(R_t^{\omega}I_t^{\omega})^{N_t}E_t^{\omega}$. Second, the net revenues of timber selling during period $t$ in scenario $\omega$ result from an exogenous annual harvesting volume $Q_t^{\omega}$, from exogenous nominal timber price $P_t^{\omega}$ and nominal harvesting, silvicultural, and administrative costs $C_t^{\omega}$. These net increments in cash flow over $N_t$ years sum up to $N_t(P_t^{\omega} - C_t^{\omega})Q_t^{\omega}$. Third, an annual consumption $c_t^{\omega}$ over $N_t$ years results in a negative increment of $N_t c_t^{\omega}$. Fourth, amortization and interest payments on current debt result in negative cash flow increments as follows. Let $D = D_t$ denote a row vector whose components refer to different types of loans so that the value $D_k$ is the endogenous debt of type $k$ borrowed at the beginning of the planning horizon. Let $L_t^{\omega}$ be an exogenous column vector, with components $L_{kt}^{\omega}$ similarly by loan type, so that the value $L_t^{\omega}$ yields the sum of amortization and interest payments in period $t$ and scenario $\omega$ per unit of loan $k$ borrowed initially. Then the total impact of debt is the inner product $DL_t^{\omega}$. Loan types may include adjustable rate loans $k$, for which $L_{kt}^{\omega}$ is scenario dependent. Then, both interest rate and amortization may depend on the scenario. On the other hand, for fixed rate annuity loans $k$, $L_{kt}^{\omega}$ is scenario independent and proportional to the number of remaining loan years overlapping with period $t$. In summary, the state equation for cash is given by

$$E_{t+1}^{\omega} = (R_t^{\omega}I_t^{\omega})^{N_t}E_t^{\omega} + N_t(P_t^{\omega} - C_t^{\omega})Q_t^{\omega} - N_t c_t^{\omega} - DL_t^{\omega},$$

for $t = 0, ..., T - 1$, and $\omega = 1, ..., \Omega$.

The harvesting costs occur during the year of cutting. The silvicultural and administrative costs may be delayed by some years after the cutting year. However, for the sake of simplicity, such delays are not incorporated in formulation (5) above.

The interest rates generated in the scenario tree together with inflation are directly applied to calculate nominal interest payments for loans. The excess cash $E_t^{\omega}$ is assumed to be deposited in a bank account. This motivates the reduction of the interest rate by a margin, say 3%, as compared with the loan rates.

We assume a given maximum level of debt-to-equity, which restricts the amount of loans available. Let $\bar{D}$ denote an exogenous upper limit on total debt based on an acceptable debt to equity ratio. Hence, for amounts $D_k$ of different types of loan, we require

$$\sum_k D_k \leq \bar{D}$$

In addition, for each $k$, we may require that the share of loan type $k$ in total debt is at least $\mu_k$, with

$$\sum_k \mu_k \leq 1$$

i.e., we require

$$D_k \geq \mu_k \sum_i D_i.$$

**Growing Stock and Harvesting**

The optimal financial plan should also involve an optimal cutting program over the planning period. However, given the main focus of our model, we employ a simplified treatment of forest management. The possible timber volume to be harvested over the planning period is dependent on the initial standing volume, its age distribution, as well as on the assumed forest growth. Spatial detail could be included in the model in order to account for site dependent harvesting costs. For simplicity, however, we omit such considerations here.

The forest of the estate is divided into $n$ age classes; in our example in $n = 10$ age classes, with a 10 yr interval in each age class. At the beginning of period $t$ and in scenario $\omega$, let $A_{it}^{\omega}$ be the forest area (in hectares) in age class $i$, $i = 1, ..., n$. We assume a simple Markov process to describe forest growth.
For $i=1,2,...,n$, every year a share $\kappa_i$ of the area not harvested moves to next age class $i+1$. For the oldest age class $i=n$, we have $\kappa_n = 0$; i.e. the area stays in age class $n$, unless harvested. If an area is clearcut in a given year, the same area moves to age class $i = 1$ subsequently.

Let $H_{it}$ denote the annual area harvested in age class $i$, in period $t$ of $N_i$ years and in scenario $\omega$. Then, for age class $i = 1$, we have

$$A_{i+1} = (1 - \kappa_i)(A_i - N_i H_{it}) + \sum_{i=1}^{n} N_i H_{it}, \quad (8)$$

and for $i > 1$,

$$A_{i+1} = (1 - \kappa_i)(A_i - N_i H_{it}) + \kappa_{i-1}(A_{i-1} - N_i H_{i-1}), \quad (9)$$

Conforming with the Swedish Forestry Act, clearcut harvesting is allowed only in age classes exceeding the minimum final felling age. This limit is related to the stand’s main species, its site (productivity) index and geographical location. In the model, final felling is allowed only in age classes $i \geq r$, where $r$ refers to the minimum final felling age.

In addition, depending on tree saving motivation, we may further restrict the share of the area to be clearcut. Based on the above, we have exogenous parameters $\alpha_i$, so that

$$N_i H_{it} \leq \alpha_i A_{it} \quad (10)$$

Thinning is treated as a mandatory operation so that annually a fraction $\delta_{it}$ of age class $i$ is thinned during period $t$. In our example, 3% of age class $i = 5$ and $i = 7$ are thinned each year, and thereby 3% of the volume is harvested in these two age classes. The thinned volume is added to clear cutting volume, together yielding the yearly harvesting volume.

Each age class $i$ represents a certain volume $V_i$ per hectare. This relates to an average site index which is assumed to be constant within each age class. The total volume $S_i$ of growing stock is then given by

$$S_i = \sum_{i=1}^{n} V_i A_{it}, \quad (11)$$

for $t = 0,...,T$, and $\omega = 1,...,\Omega$, and the annual volume of harvesting is

$$Q_i = \sum_{i=1}^{n} V_i (H_{it} + \delta_{it} A_{it}). \quad (12)$$

**Initial, Terminal, and Intermediate Conditions**

There are a number of conditions concerning the initial age distribution of timber, growing stock, initial cash, and equity at the end of the horizon.

Let $\bar{A}_i$ denote the area of forest in age class $i$ at the time of purchasing the estate. Hence, for each $i$ and $\omega$, we have the initial condition

$$A_{i0} = \bar{A}_i. \quad (13)$$

We demonstrate below the model with both even and uneven initial age distribution.

Note that the age class areas as well as the growing stock and the harvesting volume are all scenario-dependent so that the areas of each age class will change along with the scenario specific cutting volumes. At the beginning of each time period $t$, $t = 1,...,T$, the model allows a restriction of the minimum growing stock relative to the initial growing stock

$$\bar{S}_i = \sum_{i=1}^{n} V_i \bar{A}_i.$$

Employing policy parameters $\rho_t$, we require

$$S_i \geq \rho_t \bar{S} \quad (14)$$

For $t = T$, (14) sets a terminal condition for the growing stock. With $\rho_t > 0$, the manager can force the model solution to maintain a desired level of growing stock throughout the planning horizon. If such a desire does not exist, we omit (14) or set $\rho_t = 0$, for all $t$.

At the time of acquisition, initial capital resources must be given in the model. Let $\bar{E}$ denote the amount of equity capital before the purchase and let $\bar{P}$ denote the price of the estate. The following initial condition is required concerning excess cash at the beginning of period $t = 0$, but after the loans $D_k$ and the purchase of the forest estate:

$$E_0 = \bar{E} + \sum_k D_k - \bar{P}. \quad (15)$$

Given our finite planning horizon, it is desirable to restrict the model not to consume the equity over the planning period. Hence, a further requirement is a terminal condition for the total equity. The inequality (16) below limits the terminal equity to a share $\theta$ of the initial equity $\bar{E}$ in currency of the date of purchase. If $\theta T$ is the cumulative inflation multiplier for the planning horizon in scenario $\omega$, then the lower limit to terminal equity in nominal currency is $\theta_T \bar{E}$. To maintain the real value of equity at least at the initial level, implies $\theta = 1$. Final equity consists of several components. First, at the end there is an excess cash of $(E_T - \bar{E})$. Second, the value of timber is $(P_T - C_T) S_T$, where the current timber price $P_T$ reduced by harvesting costs, silvicultural costs and administrative costs $C_T$ is used to value the growing stock $S_T$. Third, to account for the remaining loan payments of loan type $k$ at $t = T$ in scenario $\omega$, let $F_k$ be the remaining value of cash flow per unit of loan $k$. Then, the remaining value of loan $k$ is $D_k F_k$. In conclusion, for each $\omega$, we require

$$E_T + \sum_k (P_T - C_T) S_T - \sum_k D_k F_k \geq \theta_T \bar{E}. \quad (16)$$

To summarize, the problem is to find nonnegative values to the decision variables $c_t$, $e_t$, $Q_t$, $D_k$, $\bar{A}_i$, $H_{it}$ and $S_i$ (for all $t$, $\omega$, $k$, and $i$) to maximize the objective function
given by (1)–(2) subject to (4)–(16) and nonanticipativity constraints.

Illustrative Examples

In this section, a series of examples illustrates the use of our model. We begin by introducing the data for the base case of the model called Base. We use Monte Carlo simulation to generate the scenario tree based on a VAR model for timber price, interest rate, and inflation. Thereafter the optimal solution for Base is summarized. Finally, several variations of the base model are presented. These concern valuation of the forest estate based on finance theory, impact of time-varying preferences for consumption, uneven initial age distribution of forest stands, increased timber price volatility, as well as increased and decreased initial equity. At the end we also present a simple example, where both financial and nonfinancial returns are taken into account in the utility function.

For brevity, each case is presented compactly and thereby in a somewhat mechanical way. Because we always deal with optimal solutions, economic interpretations can be obtained from Karush-Kuhn-Tucker conditions, which are met by optimal solutions. However, unlike in simple economic models, where analytical solutions are available for an optimum, here economic interpretation of numerical solutions can be cumbersome because of interactions of a large number of primal and dual variables.

Consider the purchase of a forest estate of 40 ha. At the time of purchase in 1996, the price is \( P = 1.150 \text{ SEK}. \) Initial private equity is 0.700 mill SEK and, based on an acceptable debt to equity ratio, the maximum amount of loans available is 0.850 mill SEK. Let \( D \) denote an exogenous upper limit on total debt based on an acceptable debt to equity ratio.

The planning horizon of 20 yr begins in 1996 and ends in 2016. To maintain the model size manageable from the point of view of computer storage and CPU requirements, this period is subdivided into six periods of \((1 + 1 + 3 + 5 + 5 + 5)\) yr. A reason for choosing shorter initial periods is the greater interest in what happens during the first years after the acquisition.

At the end of the planning horizon, in year 2016, we require the expected real value of equity to be not less than the initial value of 0.700 mill SEK. Note that this requirement is lower limits to these loans are specified by (7) with \( (\mu_0) = (0.2, 0.2, 0.3, 0.3) \).

The forest area is subdivided into 10 age classes \( i \) with a 10 yr interval in each. Initial distribution of forest area over the age classes is even. For \( i > 10\% \) of the area in an age class \( i \) moves in 1 yr to class \( i + 1 \). Timber volume by age class is given by the vector \( (V_i) = (10, 30, 69, 99, 138, 167, 188, 214, 216, 235) \) in \( m^3/ha \). Harvesting is only allowed in the two oldest age classes. Thinning reduces the timber volume by three percent annually in age classes \( i = 5 \) and \( i = 7 \). Lower limits on growing stock are given by (14) with coefficients \( p_i = 0.6 \), for \( i < T \), and \( p_T = 0.9 \) at the end. The harvesting etc. costs \( C_{t,0} \) are 160 SEK/\( m^3 \) initially in 1996, and they decrease by one SEK/\( m^3 \) each year thereafter due to improvements in efficiency.

For the objective function (1) with \( U_t \) given by (2), the target consumption is \( \hat{c}_t = 50,000 \) SEK/year, the risk aversion coefficient is \( \gamma = 0.1 \) and the discounting factor for utility is \( \beta = 0.95 \).

For uncertainty related to future timber prices, real interest rate, and rate of inflation, we employ a VAR model as follows. Let \( \tau \) denote the time in years elapsed from the beginning of the planning horizon. Let a stochastic vector \( x_t = (x_{t,0}) \) denote the three variables so that \( x_{1,0} \) is the real price of timber, \( x_{2,0} \) is the real rate of interest, and \( x_{3,0} \) is the rate of inflation, at the beginning of year \( \tau, \tau = 0,1,2, \ldots \). The basic VAR model for \( x_t \) is defined as

\[
  x_{t+1} = \eta + Bx_t + \varepsilon
\]  

where \( \eta \) is a vector of constants, \( B \) is a coefficient matrix, and \( \varepsilon \) is a stochastic vector with a multivariate normal distribution of 0 mean and covariance matrix \( \Sigma \).

Let \( C \) denote the correlation matrix of \( \varepsilon \), let \( \sigma = (\sigma_i) \) denote the vector of standard deviations of \( \varepsilon \), and let \( \Delta = \text{diag}(\sigma) \) be the diagonal matrix of standard deviations. The covariance matrix is then given by \( \Sigma = \Delta C \Delta \).

Yearly data of Sweden from 1967–1995 constitute the basis for estimating the VAR model (Rutegård 1997). The following estimation results are used in the model:

\[
  \eta = \begin{bmatrix} 182 \\ 0.0225 \\ 0.0635 \end{bmatrix}, \quad B = \begin{bmatrix} 0.576 & -470 & 0 \\ 0 & 0.582 & 0.111 \\ 0 & -0.382 & 0.277 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 49.3 & 0.0246 \\ 0.0246 & 0.539 \end{bmatrix}.
\]

Employing Monte-Carlo simulation, the realizations of the stochastic variables are obtained according to the probability distribution specified by the VAR model (17).

As described in the Appendix, such scenarios are generated in the form of a scenario tree. In each period \( t \), a desired number of branches succeed the node of realizations prevailing at the beginning of period \( t \). In our example, for periods \( t = 0,1, \ldots, 5 \), the respective number of branches succeeding a node at time \( t \) is \( 4, 4, 2, 2, 2 \) and 2 resulting...
in \( \Omega = 4 \times 4 \times 2 \times 2 \times 2 \times 2 = 256 \) scenarios for which a summary is presented in Table 1.

The value of \( x_\tau \) at the root node with \( \tau = 0 \) is given. In our example, \( x_0 \approx (345, 0.083, 0.019) \), referring to timber price 345 SEK/m\(^3\), 8.3% interest rate, and 1.9% inflation. For other nodes in the scenario tree, the realizations are generated as follows. Let \( T_\tau \) denote the set of years \( \tau \) in period \( t \). Given that realizations already have been generated for a node at the beginning of period \( t \), subsequent realizations for a branch starting with that node are given by

\[
x_{\tau+1} = \eta + B x_\tau + e_\tau,
\]

for each \( \tau \in T_t \). Here vectors \( e_\tau \) are independently generated realizations of the stochastic vector \( \varepsilon \) in (17). These realizations are, in turn, computed as follows. Let \( y_\tau \) for all \( \tau \in T_t \), be independently generated realizations of a stochastic vector with independent normally \( N(0,1) \) distributed components. Standard algorithms for generating pseudo-random normal variables are employed to produce vectors \( y_\tau \).

Thereafter, realizations \( e_\tau \) are given by \( e_\tau = Ly_\tau \), where \( L \) is the Cholesky factor of the covariance matrix \( \Sigma = LL^T \); see Golub and van Loan (1996, p. 143).

**The Base Case**

The *Base* model, employing the data introduced above is written for GAMS/MINOS (Brooke et al., 1992, p. 3–32), which produces 8,959 constraints and 9,213 decision variables after substitutions using nonanticipativity constraints. In the optimal solution, 0.450 mill SKR is borrowed in 1996: 0.090 mill SEK in both medium- and long-term annuity loans, as well as 0.135 mill SEK in both short- and long-term adjustable rate loans. To summarize the results concerning annual consumption and harvesting, as well as cash and growing stock at the beginning of each period, Table 1 shows the average figures and standard deviations (relative to the averages) over all scenarios, for each time period.

Because loan terms require 30% of debt to be repaid in 1 yr, and because real rate of interest is favorable against forest growth yield, we observe a large harvesting volume of 1,824 m\(^3\) in 1996 resulting in a drastic reduction in growing stock. The income from 1996 is largely saved in cash and used later for loan payments and steadily increasing consumption. Annual loan interest and amortization payments are 189,000 SEK during the first year, declining thereafter to 40,000 and further to 26,000 SEK. Due to our risk averse objective formulation, standard deviation of consumption is moderate. Simple sensitivity analysis indicates that the impact of decreasing versus constant harvesting etc. costs is 4% on average optimal consumption. Large variations of cash are due to terminal condition on equity together with variations of timber price in 2016.

**Real Option Valuation**

We may view the opportunity to acquire and utilize the forest estate as a real option; see, e.g., Trigeorgis (1997, p. 1–21). Employing the option pricing method by Smith and Nau (1995), we may use our model to value the forest estate as follows. First, consider the model *Status quo*, which is the *Base* model without the forest estate. Hence the problem is to optimize savings, borrowing, and consumption. Definition of the utility function remains unchanged. Second, consider the model *Option*, which is obtained from *Base* by replacing the price \( \bar{P} \) of the estate by some parameter \( V \). We search for a value \( V = \bar{V}_* \), such that the optimal values of objective functions for *Status quo* and *Option* are equal. This implies that the person is indifferent between buying the estate at price \( V_* \) and not buying; in other words \( V_* \) represents the maximum price which the purchaser is willing to pay for the estate.

Computations yield \( V_* = 1.063 \) mill SEK, which is somewhat less than the price \( \bar{P} = 1.150 \) mill SEK. The solutions of *Status quo* and *Option* with \( V = V_* \) are summarized in Table 2. Based on results of Smith and Nau, we regard \( V_* \) as an option value in the simple case where only risk-free investment is competing with the investment in the forest estate. In a slightly more sophisticated approach, other competing financial and other investments would be included as well, and such increased competition is likely to decrease the option value.

| Table 1. Summary of scenarios for timber price, rate of interest and inflation, and summary of optimal solution for consumption, cash, growing stock and harvesting. |
|-------------------------------------------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Price of timber (SEK/m\(^3\)) | 345 | 342 | 346 | 354 | 356 | 356 | 356 |
| Rate of interest (%) | 8.3 | 7.3 | 6.9 | 6.7 | 6.8 | 6.8 | 6.8 |
| Rate of inflation (%) | 1.9 | 3.7 | 4.6 | 5.2 | 5.2 | 5.2 | 5.2 |
| Consumption (1000 SEK/yr) | 15 | 15 | 15 | 18 | 23 | 42 |
| Cash (1000 SEK) | 267 | 261 | 181 | 65 | 14 |
| Loan payment (1000 SEK/yr) | 189 | 39 | 40 | 26 | 26 |
| Growing stock (m\(^3\)) | 5464 | 3840 | 3849 | 4106 | 4375 | 4609 | 4918 |
| Harvesting (m\(^3\)/yr) | 1824 | 116 | 51 | 89 | 98 | 133 |
| Price of timber | 12 | 15 | 22 | 16 | 20 | 18 |
| Rate of interest | 29 | 37 | 37 | 30 | 36 | 36 |
| Rate of inflation | 73 | 53 | 49 | 46 | 57 | 49 |
| Consumption | 9 | 12 | 24 | 42 | 49 |
| Cash | 3 | 10 | 63 | 295 |
| Growing stock | 1 | 1 | 3 | 1 |
| Harvesting | 15 | 19 | 31 | 27 | 17 |
Impact of Initial Equity

For sensitivity analysis purposes, we alter the initial equity from $E = 0.7$ mill SEK in Base up to 1.40 mill SEK as well as down to 0.65 mill SEK resulting in models Wealthy and Poor, respectively. Solutions of both models are summarized in Table 2. In Wealthy, borrowing is no longer desired, and consumption is increased to about fivefold from the base case. Heavy cutting takes place during the first year, resulting largely in savings in cash. This is due to return in interests which are higher than return provided by forest growth. In Poor, the level of harvesting is close to the levels in the base case. However, despite the relatively small decrease in initial equity, consumption is decreased by one-third during the first 10 yr and by one-fifth thereafter.

Variations in Target Consumption

Varying needs for cash over time may be represented in our model as illustrated by the following example. Consider the purchaser is planning to retire after 10 yr. For this aim we set the target consumption $\hat{c}_t$ to 10,000 SEK for the first 10 yr and to $\hat{c}_t = 50,000$ SEK thereafter. For the resulting model Retirement, Table 2 indicates, as expected, a major increase in savings in early years and a sudden jump in the level of consumption after 10 yr. Variations both up and down in levels of harvesting are needed in comparison with the base case.

Uneven Initial Age Distribution of the Forest

Initial distribution of total forest area in Base model is assumed even over the age classes. We now redefine the age distribution by $(A_i) = (0, 0, 0, 4, 8, 8, 8, 6, 4, 2)$ which indicates hectares initially in each age class $i$. Hence, there is no forest in age classes 0–30 yr, and the majority are between 40 and 70 yr. The solution of the resulting model Uneven is summarized in Table 2, indicating changes in comparison with the Base model. Consumption is largely unaffected, while major adjustments take place in harvesting and savings.

Increased Volatility of Timber Prices

According to our VAR model, the variance of timber prices in Sweden is relatively small as compared with...
some other regions of the world. In Table 1 of the base case, the standard deviation of timber price over time relative to average ranges from 12 to 22%. We now carry out a simple and artificial test to observe the impact on increased volatility in timber prices. For this purpose, first in Monte Carlo simulation, we increase the standard deviation of timber prices by factor 4. The standard deviation now ranges from 50% to 80% relative to average price. Solution of the resulting model Volatile indicates a major increase in savings during the second half of the planning period. This is explained by a hedging strategy, where savings in early years are generated to help increase consumption later on during times when timber prices are unfavorable. In other words, if such savings would not take place, then income generated for consumption only by harvests would vary more dramatically. Due to risk aversion reflected by the utility function, such variability is undesirable.

Financial and Nonfinancial Returns

As a final illustration, we consider the base case with a modified utility function. In this case we measure nonfinancial returns of the forest by growing stock. Consequently, we employ the separable objective function (1) where functions \( U_t \) are now given by \( U_t(c,S) \) in (3), where \( c \) denotes consumption and \( S \) is the growing stock at the end of period \( t \). We use \( \bar{S} \) is the initial level of growing stock, and \( \lambda = 0.01 \). In comparison with the base case, the optimal solution of the resulting model Stock indicates a significant increase in growing stock at the expense of a decrease in consumption during the entire 20 yr period. Hence, the utility is increased while consumption is decreased. However, also in this case a major harvesting of 1,113 m²3 takes place during the first year. This is explained by loan interest and repayment schedules, which require 30% of debt to be repaid in 1 yr. Initial loans exceed the minimum needed to finance the purchase. The extra loan provides additional flexibility in forest management and thereby a higher utility during coming years.

Summary and Conclusions

The goal of this article is to develop a computable model for aiding an investor in financial and managerial planning of acquisitions of forest estates, taking into account uncertainty concerning future timber prices, real rate of interest, and rate of inflation. Each possible strategy of actions in financing and forest management results in outcomes (see Kreps 1988, p. 1–6) such as financial and nonfinancial returns. Financial return, for instance, may be given by consumable wealth over time. In general, such outcomes are defined over time, and they involve uncertainty. To begin, the investor has to define which aspects of an outcome must be considered while judging preference ranking of outcomes. Such aspects may simultaneously refer to the level and risk of financial return and of nonfinancial benefits or sacrifices.

Given individual preferences, some of the key questions to be answered are as follows: (1) what is the maximum price under consideration; (2) given the price for the forest estate and an acceptable maximal debt to equity ratio, how to allocate initial equity and debt to finance the acquisition; (3) to what extent should alternative debt instruments be employed; (4) how to schedule cutting and silvicultural activities over time, given timber market uncertainties.

To answer such interdependent questions, a multistage stochastic optimization model is proposed. The formulation is relatively general; however, many additional features desired in practical applications can be easily adapted to our formulation. Due to page limitations, we only employ a small sample of examples for numerical illustrations. In this model, the investor’s preferences are taken into account employing the expected utility approach. Hence a utility function is used, such that preference ranking for outcomes can be based on utility function values. In this article, two utility functions are illustrated: first, an additive, negative exponential utility function of consumption stream over time, and second, a modification of the former such that nonfinancial returns are measured by the volume of growing stock over time. In the former case, time dependent needs for net income from forestry can be taken into account, and in addition to the level of income, individual risk and time preferences determine the utility. In the latter case, while a similar interpretation applies to nonforest income as well, the entire utility function can be interpreted as a weighted sum of financial and nonfinancial utilities.

The model accounts for financial considerations as well as characteristics of the forest and its management over the planning period. For finances, this includes an upper limit to initial debt-to-equity ratio, a liquidity buffer requirement, repayment schedules for loans, as well as cash balance dynamics. A simple Markov process describing forest growth, restrictions on cuttings, and requirements on thinning comprise the forestry model. Given a finite planning horizon, we restrict the model from consuming the equity over the planning period. Hence, a further requirement is a terminal condition for the total equity.

For financial valuation of a forest estate, a common procedure is to calculate the expected net present value using risk adjusted interest rate for discounting. However, recent literature in real options question such method (see, e.g., Smith and Nau 1995). The opportunity to acquire and utilize the forest estate is a real option. Therefore, we answer question (1) employing methods of real option valuation. In particular, following Smith and Nau (1995), we ask at which price is the person indifferent between buying the estate and not buying. In the case where only risk-free investment is competing with investment in the forest estate, such value can be found as illustrated above. In general, other competing financial and other investments could be included in the model as well, and, consequently, it can be shown that the indifference value based on financial return is consistent with arbitrage pricing theory in financial economics. If nonfinancial returns are included in the utility function as well, then the valuation based on indifference applies, but, of course, the result deviates from the real option value of financial returns.
Given a price for the forest estate, questions (2)–(5) can be answered simultaneously by an optimal solution of our stochastic programming model. The optimal plan suggests a hedging strategy against uncertainty concerning timber prices, rate of interest and inflation. In theory, this is a well established and known fact. However, it is a nontrivial task in practical situations to determine a hedging strategy which is optimal under a chosen utility function. A common but simplified goal for hedging is minimization of variance of financial return, for instance (see, e.g., Luenberger 1998, p. 283). Even if nonfinancial returns are omitted, such strategy is in general inconsistent with our optimal hedging based on expected utility. Numerical examples are presented with a base case and some variations concerning time-varying preferences on consumable wealth, age distribution of forest stands, timber price processes, and initial equity of the investor. Furthermore, an illustration is provided with both financial and nonfinancial returns determining the utility function.

One of the most critical issues of the approach is a proper specification of the utility function to account for true preferences of the investor adequately. Estimation of parametrized functional forms of utility functions, such as ours, is a well established area of research (see, e.g., Keeney and Raiffa 1976).

Another critical issue in stochastic programming applications is the model’s ability to represent future uncertainty in an appropriate way. In this respect, the stochastic model of timber price, rate of interest and inflation, as well as the scenario generation process to obtain representative realizations of the three stochastic variables, are of utmost importance.

Obviously, there are sources of errors that might have an adverse impact on the results. For example, considering timber prices for which historical data is available on an annual basis only, a very long period is needed to obtain a reasonably large set of data. In turn, this may cause further problems because of insufficiently reliable data, or because data have been compiled in different ways over time. However, these problems should be put in perspective with the general difficulty in taking into account future uncertainty, whatever model is used. The main task of a model for decision aid, such as ours, is to get prepared for an unforeseen future taking into account true preferences of the investor.

Endnotes

[3] For smaller values, such as for $F = 0.35$ mill SEK, for instance, our model has no feasible solution because forestry income is not sufficient to repay loans.

Literature Cited

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APPENDIX—Multistage Stochastic Optimization

In this review, we first introduce the general convex optimization problem with reference to its solution techniques. Thereafter, it is specialized to dynamic problems under uncertainty. The resulting model is the multistage stochastic optimization problem. The presentation follows Ainassari, Kallio, and Ranne (1998).

Consider the following convex programming problem:

$$\max_{x \in X} f(x)$$

subject to

$$g_i(x) \geq 0 \quad \text{for } i = 1, \ldots, k \quad (22)$$

$$h_j(x) = 0 \quad \text{for } j = 1, \ldots, l \quad (23)$$

where $X$ is a convex and closed set of $n$ vectors, $f$ and $g_i$ are concave, and $h_j$ affine functions defined on $X$. The function $f$ is the objective function. The vector $x$ is an $n$ vector of decision variables. The problem is solved for the values of the variables $x_1, \ldots, x_n$ that satisfy the constraints (22)–(23) and maximize the function $f$.

A vector $x \in X$ satisfying all the constraints (22)–(23) is a feasible solution to the problem and the union of all feasible solutions is the feasible region. Under our assumptions, the feasible region is a convex set. Hence, the problem is to find a feasible point $x$ such that $f(x)$ is a feasible solution to the problem and the union of all optimal solutions is the feasible region. As such it does not take the uncertainty explicitly into account. In order to model the uncertainty more accurately, we follow the line of thought introduced by Dantzig and Madansky (1961). In the multistage stochastic programming approach, the uncertainty is modeled as a scenario tree. For an example, see Figure A-1. The tree displays the timing and sequence of uncertain events. Each scenario $\omega, \omega = 1, \ldots, \Omega$, begins from the same node, the root (on the left of Figure 1), representing the situation at the initial stage. The branches emanating from the node represent the realizations of uncertain events at that stage. At the next stage, each branch diverges further. The process continues until one has reached the end of the time horizon. Thus, each scenario is a path through the tree representing a possible sequence of realizations of stochastic variables. Associated with each scenario $\omega$ is a probability of occurrence denoted by $P_\omega$.
The nodes, i.e. the branching points, of the scenario tree divide our finite time horizon into several stages $t$, $t = 0, 1, 2, \ldots, T$. Thereby we subdivide the time horizon into a number of periods in each scenario. Such a subdivision of the time horizon may be scenario dependent. However, for the sake of simplicity, we assume that the time periods $t$ ($t = 0, 1, 2, \ldots, T$) are the same in each scenario. The beginning of each time period represents an opportunity to make a decision and a decision vector will be associated with such points in time. It is assumed that subsequent branching is not observed while making such a decision.

At the beginning, the decision maker does not know which scenario will occur. Therefore, the decisions in the root have to be identical for all scenarios. That is, the information at hand is the same for all scenarios, and there is no reason for different decisions. The constraints enforcing this principle are nonanticipativity constraints. Likewise, at stage two, the decisions for scenarios 9 and 10 must be the same, as well as the decisions for scenarios 11 and 12.

To generalize this idea, we define for all $t$ and $\omega$, an information set $S_t^\omega$ as the set of scenarios, for which the path in the scenario tree from the beginning of the time horizon to the beginning of period $t$ matches such a path associated with scenario $\omega$. Therefore, if two scenarios $\omega$ and $\nu$ belong to $S_t^\omega$, then the information available for decision at the beginning of period $t$ is the same for both scenarios, and hence, there is no reason for different decisions at this stage in the two scenarios.

Let $x_t^\omega$ denote the vector of decision variables for period $t$ in scenario $\omega$, and let $x \in R^n$ be the overall decision vector consisting of decision variables in $x_t^\omega$ taking into account all $t$ and $\omega$. As above, we may define an objective function $f(x)$ for evaluating the performance of alternative decisions $x$. For practical applications again, concavity of $f$ is essential. For many applications, two additional assumptions are justifiable. First, the objective function is an expected value of scenario-wise objective functions. Second, scenario-wise objectives are separable in time steps; in other words, if $f_t(x_t^\omega)$ denotes the objective function for period $t$, then the scenario-wise objective is

$$\sum_t f_t(x_t^\omega).$$

Such objective function is often chosen to represent the expected utility, while the scenario-wise objectives represent discounted sums of utility over time. With these assumptions and with scenario probabilities $p_{\omega}$, the overall multistage stochastic optimization problem is to

$$\max\sum_\omega \sum_t p_{\omega} f_t(x_t^\omega)$$

subject to

$$A_t x_{t-1}^\omega + B_t x_t^\omega = b_t^\omega \text{ for all } \omega \text{ and } t$$

$$l_t^\omega \leq x_t^\omega \leq u_t^\omega \text{ for all } \omega \text{ and } t$$

$$x_t^\omega = x_t^\nu \text{ for all } \omega \text{ and } \nu \in S_t^\omega.$$
where $A_t^{\omega}$ and $B_t^{\omega}$ are data matrices (of suitable dimensions), and $b_t^{\omega}$, $l_t^{\omega}$, and $u_t^{\omega}$ are vectors of problem data, for all $t$ and $\omega$. For $t = 0$, we define $x_{-1}^{\omega} = 0$ in (25).

The stochastic optimization problem (24)–(27) is a special case of the convex optimization problem (21)–(23). It is often called the “deterministic equivalent.” The dynamic constraints (25) represent affine constraints of (23), while the convex set $X$ is defined by simple bounds (26) and by nonanticipativity constraints (27). Here, the deterministic equivalent does not include concave inequality constraints (22). If desired, such constraints may be included without too much extra computational difficulty in solving the problem; see, e.g., Kallio and Rosa (1999). We compute an optimal solution for the problem (24)–(27) employing only a single decision vector per information set; i.e., using (27), for all $\omega$, we substitute $x_t^{\omega}$ for $x_t^v$, for all $v \in S_t^{\omega}$, and thereby we reduce dimensions of the problem.