Non-linear interaction of elastic waves in rocks

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Accepted 2013 May 21. Received 2013 March 11; in original form 2012 May 4

SUMMARY
We study theoretically the interaction of elastic waves caused by non-linearities of rock elastic moduli, and assess the possibility to use this phenomenon in hydrocarbon exploration and in the analysis of rock samples. In our calculations we use the five-constant model by Goldberg. It is shown that the interaction of plane waves in isotropic solids is completely described by five coupling coefficients, which have the same order of magnitude. By considering scattering of compressional waves generated by controlled sources at the Earth surface from a non-linear layer at the subsurface, we conclude that non-linear signals from deep formations are unlikely to be measured with the current level of technology. Our analysis of field tests where non-linear signals were measured, suggests that these signals are generated either in the shallow subsurface or in the vicinity of sources. Non-linear wave interaction might be observable in lab tests with focused ultrasonic beams. In this case, the non-linear response is generated in the secondary parametric array formed by linear beams scattered from inclusions. Although the strength of this response is controlled by non-linearity of the surrounding medium rather than by non-linearity of inclusions, its measurement can help to obtain better images of rock samples.

Key words: Numerical approximations and analysis; Body waves; Wave propagation; Acoustic properties.

1 INTRODUCTION
It has been long suggested that by studying non-linear seismic responses one might obtain additional information from underground formations, which is not available from standard seismic methods employed in hydrocarbon exploration. Non-linear techniques can provide better media characterization because the contrast in non-linear media parameters is usually large when compared to the contrast in the linear parameters.

Non-linear interaction of sound waves was originally studied in acoustic media. It results in particular in the so-called parametric array, which was theoretically described by Westervelt (1963) and by Berklay (1963). The essence of the parametric array is the generation of low frequency signals by two collinear ultrasound waves. The parametric array was observed in water (Bellin & Beyer 1962; Muir & Willette 1972) and then in air (Bennett & Blackstock 1975). This effect has been also applied to generate highly directive audible sounds (Pompei 1999).

Non-linear ultrasound techniques are used in medical applications, where a number of non-linear imaging methods have been introduced (see e.g. Ichida et al. 1983; Sarvazyan et al. 1998). Fatemi & Greenleaf (1998) have demonstrated that two ultrasound waves generate strong difference-frequency responses when they are focused on hard inclusions in soft tissues. By measuring such responses Fatemi & Greenleaf (1998) obtained images with better resolution than in standard ultrasound scanning. This observation poses the question of whether a similar non-linear imaging technique can be applied to investigate elastic media in laboratory or field settings.

Non-linear properties of solids have been extensively investigated in laboratory conditions (Rollins et al. 1964; Zarembo & Krasil’nikov 1971; Iwasaki et al. 1978; Bakulin & Protosenya 1982; Johnson et al. 1987; Johnson & Shankland 1989; Belyaeva & Timinian 1991; Meegan et al. 1993; Winkler & Liu 1996). Measurements by Bakulin & Protosenya (1982), Meegan et al. (1993) and Payan et al. (2009) showed that compound materials, like rocks and concretes, are two to three orders of magnitude more non-linear than mono crystalline solids. An extensive review of studies of non-linear properties of earth materials is given by Ostrovsky & Johnson (2001). Table 1 adopted from the paper by Donskoy et al. (1997), summarizes the results of the mentioned studies. It shows typical values of the non-linear parameter $\Gamma$ (see eq. 23) that characterizes the ratio of the third order (non-linear) elastic moduli to the second order (linear) elastic moduli. The value of $\Gamma$ ranges from 1 in gases and fluids to $10^5$ in porous media. Belyaeva & Zaitsev (1998) have predicted that the maximum value of $\Gamma$ in micro-inhomogeneous media is about $10^7$. Zaitsev et al. (2009) presented an example of a material with negative $\Gamma$, whose absolute value is in the range of $10^6$.

Although the results of laboratory tests cannot be straightforwardly up-scaled to field size, they suggest that propagation of seismic waves in underground formations can cause measurable non-linear elastic effects. Indeed, \textit{in situ} non-linear behaviour of soils during earthquakes have been reported by Beresnev & Wen...
Table 1. Values of non-linear parameter $\Gamma$ in different media (Donskoy et al. 1997).

<table>
<thead>
<tr>
<th>Fluids and gases</th>
<th>Non-porous solids</th>
<th>Foam plastic</th>
<th>Marine sediments</th>
<th>Marble</th>
<th>Sandstone and soils</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma &lt; 10$</td>
<td>$\Gamma &lt; 15$</td>
<td>$\Gamma \approx 200$</td>
<td>$\Gamma \approx 10^2\text{–}10^3$</td>
<td>$\Gamma \approx 800$</td>
<td>$\Gamma \approx 10^3\text{–}10^4$</td>
</tr>
</tbody>
</table>

(1996), Field et al. (1994), Kokusho & Matsumoto (1998), Aguirre & Irikura (2002) and Tsuda et al. (2006). Beresnev (1993) analysed theoretically non-linear reflection from deep targets of waves generated by vibrators at the Earth surface. He concluded that non-linearly scattered elastic waves can be realistically recorded for typical rock parameters. This conclusion could be of great practical importance and one of our goals is to check it.

The measurements of vibrator-induced non-linear effects in underground formations started in the former USSR in the 1980s (Aleshin et al. 1983). Beresnev & Nikolaev (1988) performed tests with sources emitting long monochromatic signals. They observed a transfer of energy from the fundamental wave to its second harmonic, which was attributed to rock non-linearity. Solov’ev (1990) however argued that higher harmonics in the data of Beresnev & Nikolaev (1988) are generated at the source site and not during the wave propagation. Similar tests with vibrators by Dimitriu (1990) were inconclusive. Further improvements of measurement techniques allowed to quantify non-linear responses from seismic waves by measuring changes in elastic moduli (Iinazaki 2004; Lawrence et al. 2008). According to Lawrence et al. (2008), non-linear responses are induced by Rayleigh surface waves.

Examples of the registration of non-linear signals from deep formations in tests with controlled sources have been reported by Khan & Khan-McGuire (2005), Khan & McGuire (2006), Shulakova (2007) and Zhukov et al. (2007). Khan & Khan-McGuire (2005) and Khan & McGuire (2006) described cross-well measurements, where non-linear seismic data was correlated with rock-pore fluids. The possibility of using non-linear seismic effects in hydrocarbon exploration was addressed by Shulakova (2007) and Zhukov et al. (2007), who used combined frequencies to reconstruct formations across-sections. According to Shulakova (2007) and Zhukov et al. (2007), non-linear components of the signal are associated with hydrocarbon bearing structures. However, Khan & Khan-McGuire (2005), Khan & McGuire (2006), Shulakova (2007) and Zhukov et al. (2007) gave neither details of their signal processing nor a theoretical explanation of their observations. For these reasons, it is difficult to check whether these correlations are structural or occasional.

Non-linearity of rocks can be of various physical origins and it exhibits itself in many different forms, such as hysteretic material behaviour (dependence of elastic moduli on the stress–strain path), change of resonant frequencies of soil layers, distorted waveforms, enhanced and strain-dependent wave damping, amplification of the free surface and high-frequency spikes in the ground acceleration. In geotechnical engineering and seismonological applications one usually analyses the above non-linear phenomena using rheological models that describe hysteretic, and damping effects (see e.g. Hartzell et al. 2004; Delépine et al. 2009). We are mainly interested in seismic prospecting. In our study we consider an ideal case, where the damping and stress–strain path dependence are neglected. What is taken into account is dependence of elastic moduli on the wave strain. Among various possible physical effects we concentrate on generation of combined (or double) harmonics.

Our approach is motivated by practical concerns. In seismic exploration one measures rock strains that are several orders of magnitude smaller than strains induced during earthquakes. For this reason the non-linear phenomena in seismic exploration are very weak and most of them are not observable. One the other hand, in seismic exploration one uses controlled sources whose frequency spectra are known with a high accuracy. In particular, one can use almost monochromatic sources. The spectrum of the response from a linear formation should contain only source frequencies. Any type of rock linearity will cause generation of combined (sum- and difference-frequency) harmonics. If such harmonics are not present in the spectrum of the sources, their appearance in rock responses would be an indication of non-linearity. Detection of combined harmonics seems to be the most straightforward way to analyse rock non-linearity in seismic exploration. Since generation of combined harmonics is a generic non-linear effect, it can be described within the framework of a non-dissipative non-linear elasticity. On the contrary, combined harmonics generated by rocks during earthquakes cannot be detected directly, because the source spectrum is not known. One needs to compare signal spectra at different locations, which reduces the sensitivity of such measurements. For a more reliable analysis one needs to consider other non-linear effects using non-linear visco-elasticity and plasticity.

In our study, we analyse the effect of generation of combined harmonics by rocks in hydrocarbon exploration and in laboratory tests. This is done by developing the theory of non-linear interaction of waves in rocks and comparing predictions of this theory with observations. In Section 2, we introduce dynamic and constitutive equations, and describe the procedure to calculate non-linear elastic responses. This procedure is based on a Green’s function formalism and the method of successive approximations. In Section 3, we study symmetry properties of the elastic non-linear force. The results from Section 3 are used in Section 4 where we explain that interactions of plane elastic waves in uniform and isotropic media are completely described by five functions. By investigating the evolution of resonantly coupled plane waves in Section 5, we establish the maximum possible amplitudes of waves that can be generated by non-linear effects in rocks. The magnitude of non-linear signals that can be generated by deep targets in field tests with controlled sources is evaluated in Section 6. In Section 7, we discuss field data. In Section 8, we consider the possible physical mechanisms that might be responsible for the generation of non-linear harmonics in the vibro-acoustography method. We conclude that the secondary parametric array dominates all other non-linear mechanism. The secondary parametric array is analysed in more detail in Section 9. Section 10 contains discussion of results. Some details of the calculations are given in Appendices A–C.

2 FORMALISM

2.1 Dynamic and constitutive equations

We use the five constant model by Goldberg (1960). This model is rather generic and it represents a number of real features of a
\[ \frac{\partial^2 \vec{u}}{\partial t^2} = \nabla \cdot \vec{\dot{a}} + \vec{f}. \]  

(1)

Here, \( \rho \) is the density of the undeformed medium, \( \vec{u} \) is the solid displacement, \( \vec{f} \) is the external force and the stress tensor \( \vec{\sigma} \) has the components

\[
\sigma_{mn} = \lambda e_{ij} \left( \frac{\partial u_m}{\partial x_i} + \frac{\partial u_n}{\partial x_j} \right) + 2\mu \left( e_{mn} + e_{nj} \frac{\partial u_l}{\partial x_j} \right)
+ A e_{ij} \left( e_{mn} + e_{nj} \frac{\partial u_l}{\partial x_j} \right) + C e_{ij} e_{nl} \frac{\partial u_m}{\partial x_n}.
\]

(2)

The stress tensor depends on five elastic constants \( \lambda, N, A, B, C \) and it takes into account second order non-linear corrections to the standard linear stress tensor. Such corrections are also included in the strain tensor \( e_{mn} \),

\[
e_{mn} = \frac{1}{2} \left( \frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} + \frac{\partial u_k}{\partial x_m} + \frac{\partial u_k}{\partial x_n} \right).
\]

(3)

The stress tensor is written explicitly in terms of components of \( \vec{u} \) by substituting eq. (3) into eq. (2). The corresponding expression has been derived by Gol’dberg (1960).

### 2.2 Linear limit

Retaining only the linear terms in eq. (1) we obtain

\[
\frac{\partial^2 \vec{u}}{\partial t^2} - c_p^2 \nabla \cdot (\nabla \cdot \vec{u}) + c_s^2 \nabla \times \nabla \times \vec{u} = \vec{f}/\rho.
\]

(4)

Here, \( c_p = (H/\rho)^{1/2} \) and \( c_s = (N/\rho)^{1/2} \) are the compressional and the shear wave velocities. Eq. (4) is solved by the method of Green’s tensors. The Green’s tensor’s \( G_{mn} \) is equal to the component \( u_m \) of the solid displacement for the case where \( \vec{f} = 4\pi \rho \vec{e}_0 \nabla (\vec{r} - \vec{r'}) \), and \( \vec{e}_0 \) is the unit vector along the nth coordinate. The Green’s tensor is split in two parts, \( G_{mn} = G^{(1)}_{mn} + G^{(2)}_{mn} \), where \( G^{(1)}_{mn} \) represents longitudinal (compressional) deformations with \( \nabla \times \vec{u} = 0 \) and \( G^{(2)}_{mn} \) represents transversal (shear) deformations with \( \nabla \cdot \vec{u} = 0 \). We consider periodic deformations that depend on time as \( \exp(-i\omega t) \). The purposes of this study we restrict ourselves to an infinite medium.

In this case

\[
G^{(2)}_{mn} = \left[ \frac{1}{k^2} \frac{1}{R} \delta_{mn} - \left( \frac{3}{k^2} \frac{1}{R^2} \frac{R_m R_n}{R^2} \right) \right] e^{i k R} c_p^2
\]

(5)

\[
G^{(2)}_{mn} = \left[ - \frac{1}{k^2} \frac{1}{R} \frac{R_m R_n}{R^2} \right] e^{i k R} c_p^2
\]

(6)

where \( R = |\vec{r} - \vec{r'}|, k = \omega/c \) is the wave vector, \( c = c_p \) for longitudinal waves and \( c = c_s \) for transversal waves. The signs plus and minus in eqs (5) and (6) correspond to convergent and divergent waves. In what follows we are dealing with divergent waves only, that is, with the waves propagating outward from the source. Note that the expressions for the elastic Green’s tensors \( G^{(1)}_{mn} \) in the book by Morse & Feshbach (1953), which have been also used by Jones & Kobett (1963) and by Beresnev (1993) contain a misprint.

### 2.3 Method of successive approximations

We restrict ourselves to the case of small non-linearity, where eqs (1) and (2) can be solved using the method of successive approximations. The stress tensor is represented in the form \( \vec{\sigma} = \vec{\sigma'} + \vec{\sigma''} \), where \( \vec{\sigma'} \) and \( \vec{\sigma''} \) are the linear and non-linear parts, respectively. At the first step we neglect the non-linear part \( \vec{\sigma''} \), which results in eq. (4) for the linear solid displacement \( \vec{u}' \). Solution of eq. (4) has the form

\[
u_m^{(1)}(\vec{r}) = \frac{1}{4\pi \rho} \int G^{(1)}_{mn}(\vec{r}, \vec{r'}) f_m(\vec{r'}) d\vec{r}'.
\]

(7)

Here, \( u_m^{(1)}(\vec{r}) \) are displacements in the compressional and shear waves, Green’s functions \( G^{(1)}_{mn} \) are given by eqs (5) and (6), and \( f_m \) is the force created by a seismic source. The linear displacement \( \vec{u}' \) is used to calculate the non-linear part of the stress tensor \( \vec{\sigma'} \).

The non-linear solid displacement \( \vec{u}^{(2)} \) is described by eq. (4) with \( \vec{f} = \vec{f}' \) where \( \vec{f}'(\vec{r}) = \nabla \cdot \vec{\sigma} \). The value \( \vec{u}^{(2)} \) is also found from eq. (7) after the substitution \( f_m = f_m^{(2)} \). The non-linear force \( \vec{f}'(\vec{r}) \) is derived in Appendix A for the case of plane waves.

### 3 Symmetries of the Non-linear Force

The integrand in eq. (7) varies rapidly in space. Contributions to the integral from different spatial domains tend to cancel each other, and the net integral almost vanishes. Under certain conditions the integrand does not oscillate, and the net integral is relatively large. Such conditions are called the resonant conditions. The non-linear interactions under these conditions are called the resonant interactions, and they have attracted significant attention in the literature (Jones & Kobett 1963; Childress & Hambrick 1964; Taylor & Rollins 1964; Korneev et al. 1989). In this section, we show that the non-linear elastic force possesses certain symmetries. The resonant wave interactions are naturally split in triplets. Every interaction in a given triplet is described by the same coupling coefficient. This observation significantly simplifies our further analysis, and it has not been utilized in the literature previously.

Eq. (7) involves three modes. In case of plane waves the solid displacements in these modes are equal to

\[
\vec{u}_j = a_j \vec{e}_j \exp(-i\omega_j t + i\vec{k}_j \cdot \vec{r}) + \text{complex conjugate}.
\]

(8)

Here, \( j = 1, 2, 3 \), \( \omega_j \) are the frequencies, \( \vec{k}_j \) are the wave vectors, and \( \vec{e}_j \) are unit vector along \( \vec{u}_j \), which are called the polarization vectors. The index \( j \) labels all modes without distinction whether they represent the linear or non-linear parts of the deformation. The modes interact resonantly if they are coherent both in time and in space. Coherence occurs if the mode frequencies and wave vectors satisfy the frequency and the phase matching condition,

\[
\omega_1 = \omega_2 + \omega_3, \quad \vec{k}_1 = \vec{k}_2 + \vec{k}_3.
\]

(9)

In eqs (9) we label modes in such a way that the subscript \( j = 1 \) refers to the mode with the highest frequency. According to the phase matching condition the wave vectors \( \vec{k}_j \) lie in one plane and they form a triangle as shown in Fig. 1.

Due to mode interaction their amplitudes \( a_j \) change in space and/or in time. The non-linear interactions are small, and the spatial...
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Figure 1. Wave vectors \( \vec{k}_1, \vec{k}_2 \) and \( \vec{k}_3 \) of resonantly interacting modes satisfy the phase matching condition and hence they form a triangle. We use the convention that the subscript '1' refers to the mode with the highest frequency in the resonant triplet, so that \( \omega_1 > (\omega_2, \omega_3) \). The subscript '3' refers to the mode with the lower phase velocity \( c_1 < c_2 \). The angles between the wave vectors \( \vec{k}_1, \vec{k}_2, \vec{k}_3 \) and \( \vec{k}_2 \) and \( \vec{k}_3 \) are denoted as \( \theta \), \( \phi \) and \( \psi \).

and temporal dependences of their amplitudes are weak, \( \omega_j \) and \( |\nabla a_j| \ll \omega_1 |a_j| \) and \( \omega_1 \). Substituting eq. (8) into eq. (4) yields

\[
\left( \frac{\partial}{\partial t} + \vec{\omega} \cdot \nabla \right) a_j = \frac{i}{2\omega_1} \vec{f}_j \cdot \vec{e}_j, \tag{10}
\]

where \( \vec{e}_j = \vec{k}_j \omega_j / k_j^2 \) and \( \vec{f}_j = \vec{f}_j^{(2)} \exp(i\omega_j t - ik_j \cdot \vec{r}) \). The values \( \vec{f}_j^{(2)} \) are resonant parts of non-linear forces acting on \( j \)th mode, which vary in time and space as \( \vec{f}_j^{(2)} \propto \exp(-i\omega_j t + ik_j \cdot \vec{r}) \). Explicit expressions for \( \vec{f}_j^{(2)} \) are presented in Appendix A, but they are not necessary here. One only needs to know that the non-linear force is proportional to the product of amplitudes of interacting waves. Consequently the values \( \vec{f}_j \) have the form

\[
\vec{f}_1 \cdot \vec{e}_1 = -iV_j a_3 a_3, \quad \vec{f}_2 \cdot \vec{e}_2 = -iV_j a_3 a_3, \quad \vec{f}_3 \cdot \vec{e}_3 = -iV_j a_3 a_3. \tag{11}
\]

Here, \( V_j \) are the coupling coefficients, which are real functions of frequency and of the wave vectors, and the star denotes the complex conjugate. Equations describing the generation of double frequencies due to self-interactions of seismic waves are obtained as a limiting case of eq. (10), where a wave with the frequency \( \omega_1 = 2\omega_3 \) interacts with two waves which have frequency \( \omega_2 \) and amplitude \( a_2 / \sqrt{2} \).

The energy density of the \( j \)th mode is proportional to \( \omega_j^2 |a_j|^2 \). By analogy with quantum mechanics, one can treat the values \( n_j = \omega_j |a_j|^2 \) as the number of phonons in the \( j \)th mode normalized on the Planck constant (Waldov & Rollins 1963; Taylor & Rollins 1964). Eq. (10) can then be represented in the form of continuity equations

\[
\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \vec{e}_j) = \frac{V_j}{\rho} \Im (a_3^* a_3 a_3), \tag{12}
\]

where \( \text{Re} \) denotes the real part. We consider processes where the disappearance of a phonon in mode '1' produces one phonon in mode '2'. In such a way that mode '3' is always a compressional mode. If the amplitudes of the compressional modes are originally large, then their non-linear interaction will generate the shear mode with a difference frequency, \( \omega_1 - \omega_2 = \omega_3 \). If the second compressional mode and the shear mode are originally large, then they will generate the compressional mode with the sum frequency, \( \omega_2 + \omega_3 = \omega_1 \). Finally, if the first compressional mode and the shear mode are large, then they generate the compressional mode with the difference frequency, \( \omega_1 - \omega_3 = \omega_2 \). Each of these three processes is a counterpart of a single resonant interaction between three harmonics. Once one of these processes exists, the other two should also be possible, and they are described after a proper choice of variables by the same coupling constant. These conclusions are confirmed by direct calculations presented in Appendix A.

4 Resonant Tripletts of Plane Waves

Previous authors distinguished different numbers of possible pairwise resonant interactions depending on the type and polarization of the elastic waves involved. According to Zarembo & Krasil’nikov (1971), Childress & Hambrick (1964), Taylor & Rollins (1964) and Korneev et al. (1998) there exist 18, 8, 4 and 10 such interactions, respectively. Properties of pairwise resonant interactions of plane waves in a uniform isotropic medium are described in more details in Appendix B, where we distinguish nine such interactions. As is explained in Appendix B, the number 18 given by Zarembo & Krasil’nikov (1971) is erroneous. Childress & Hambrick (1964), Taylor & Rollins (1964) and Korneev et al. (1998) describe actually the same pairwise interactions as we do. However these authors used different ways of counting which resulted in different numbers. In this section, we show that nine pairwise interactions are naturally split in four resonant tripletts. Consequently, there exist only four independent resonant processes. We describe the wave polarization according to orientation of their displacement vector with respect to the plane where the wave vectors of the interacting waves lie, see Fig. 1. This plane is referred to as the 'k-plane'. Elastic deformations in a uniform and isotropic medium are naturally separated into compressional waves and shear waves. We denote compressional waves by the letter P (primary). The polarization vector \( \vec{e}_P \) of compressional waves is parallel to its wave vector \( \vec{k}_P \). Shear waves with the polarization vector \( \vec{e}_S \) in the k-plane are called the waves with vertical polarization and denoted as SV. Waves with horizontal polarization, denoted as SH, are waves whose polarization vector \( \vec{e}_h \) is normal to the k-plane.

We label modes '2' and '3' in Fig. 1 as 1 such a way that mode '3' has the lower phase velocity, \( c_3 \geq c_1 \). Taking into account that the frequency and phase matching conditions (9) are satisfied we have \( k_2 + k_3 = \omega_2 / c_2 + \omega_3 / c_3 \leq (\omega_2 + \omega_3) / c_3 = \omega_1 / c_3 = (c_1 / c_3) k_1, \) or \( k_2 + k_3 \geq (c_1 / c_3) (k_2 + k_3) \). Since the sum of two side lengths of a triangle is larger than the third side length, that is, \( k_2 + k_3 \geq (c_1 / c_3) (k_2 + k_3) \), the resonant interaction between non-collinear waves is possible only if \( c_1 > c_3 \). The phase velocity of compressional waves is larger than the phase velocity of shear waves. Hence, in an isotropic medium where velocities of SV and SH modes are equal, mode '1' is always a compressional mode. The two remaining modes could be \((P, SV), (P, SH), (SV, SV), (SV, SH)\) and \((SH, SH)\). It can be verified that the non-linear elastic force created by two interacting waves expands along their wave vectors and polarization vectors. In case of interacting compressional waves and shear waves with vertical
polarization, the non-linear force lies in the $k$-plane. This force cannot generate solid displacements in the direction perpendicular to the $k$-plane, and shear waves with the horizontal polarization cannot be excited. A resonant triplet needs to contain at least two $SH$ waves with horizontal polarization, or it does not contain such waves at all. Consequently the triplets ($P$, $SV$, $SH$), and ($P$, $P$, $SH$) are not possible. The three remaining triplets ($P$, $P$, $SV$), ($P$, $SV$, $SV$) and ($P$, $SH$, $SH$ containing two $P$, two $SV$ and two $SH$ modes respectively. We call them $P$, $V$- and $H$-triplet.

Triangles formed by a resonant triplet can be described by using the angle $\theta$ between the vectors $\mathbf{k}_1$ and $\mathbf{k}_3$, see Fig. 1, as an independent parameter. The other parameters are recovered using the law of cosines, $k_1^2 + k_2^2 - 2k_1k_2\cos\theta = k_3^2$, the law of sines, $k_1/\sin\psi = k_2/\sin\phi = k_3/\sin\theta$, and the frequency matching condition, $k_1c_1 = k_2c_2 + k_3c_3$, where the velocities $c_j$ are assumed to be known. Since the resonant conditions do not depend on directions of the polarization vectors, they are identical for the triplets $V$ and $S$. In both cases, $\theta$ can vary in the range from 0 (parallel vectors) to $2\pi$ (antiparallel vectors). Substituting the mode velocities, we obtain the ratios $\omega_{2j}/\omega_{1j}$, which are summarized in Table 2 in terms of $\beta = c_1/c_3$ and $\gamma = (1 - \beta^4\cos^2\theta)/(1 - \beta^4)$. Table 2 also presents minimum and maximum values of $\omega_{2j}/\omega_{1j}$ and of the angles $\phi$ and $\psi$. The angle $\phi$ tends to zero if $\theta \to 0$, and is maximum at $\theta \to 0$. The angle $\psi$ tends to $\pi$ at $\theta \to 0$. In usual materials with positive Poisson ratio, $c_1^2/c_3^2 = \beta^2 < 1/2$ (Landau & Lifshitz 1986). For this reason the angle $\psi$ can never reach zero. The $V$- and $H$-triplet, $\psi$ is minimum if $\omega_2 = \omega_3$.

The coupling coefficients for the resonant triplets are calculated in Appendix A, and they are equal to

$$\begin{align*}
V_P &= (\lambda + 3N + A + 2B) \cos(\theta) \sin(\psi + \varphi) k_1k_2k_3, \quad (14) \\
V_V &= \left[ (\lambda + N) \cos^2\psi + \left( N + \frac{A}{2} + B \right) \cos(2\psi) \right] k_1k_2k_3, \quad (15) \\
V_H &= \left[ (\lambda + B) \cos\psi + 2 \left( N + \frac{A}{4} \right) \cos\theta \cos\phi \right] k_1k_2k_3. \quad (16)
\end{align*}$$

These coupling coefficients are proportional to the matrix elements of the elastic Hamiltonian found by Taylor & Rollins (1964). Eqs (14)–(16) together with the geometric relations shown in Table 2 provide a complete characterization of mode interaction in resonant triplets in the absence of attenuation. In Section 5, we show that the increment of the parametric instability, where mode ‘1’ decays into modes ‘2’ and ‘3’ is proportional in the absence of attenuation to $V\sqrt{\omega_{2j}\omega_{1j}}$, see eq. (21). Since mode ‘1’ in all the triplets is the compressional mode, it is convenient to normalize the frequencies and wave vectors to $k_1$ and $\omega_1$: $k_j = k_1/k_1$, $\omega_j = \omega_1/\omega_1$. Then the coupling in triplets is characterized by dimensionless coefficients of the type $\tilde{k}_j\tilde{k}_3\Psi/(\omega_2\omega_1)^{1/2}$, where $\Psi = \Psi(\theta, \phi, \varphi)$ is the angular part of coupling coefficients in eqs (14)–(16). Figs 2–4 show these coefficients together with the frequency ratio $\omega_{2j}/\omega_{1j}$ and the angle $\phi$. The calculations were made for a Poisson ratio of $\nu = 0.2$, which corresponds to $c_3/c_\rho \simeq 0.61$.

From Table 2 we see that all the resonant triplets could exist in the limit $\psi \to \pi$, ($\theta \to 0$, $\phi = \pi$), where they become collinear. The coupling coefficient $V_P$ is zero in this limit, so that a collinear $P$-triplet is not observable. The coupling coefficients $V_V$ and $V_H$ remain finite. They have equal amplitudes and opposite signs, so that

$$(V_V, -V_H) \to \left( \lambda + 2N + \frac{A}{2} + B \right) k_1k_2k_3. \quad (17)$$

Collinear $V$- and $H$-triplet are essentially the same. They correspond to the interaction of copropagating compressional and shear waves with frequencies $\omega_{2j}$ and wave vectors $k_0s$, and a counterpropagating shear wave whose frequency and wave vectors are equal

<table>
<thead>
<tr>
<th>Triplet</th>
<th>$\omega_{2j}/\omega_{1j}$</th>
<th>Range of $\omega_{2j}/\omega_{1j}$</th>
<th>Range of $\phi$</th>
<th>Range of $\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>$\gamma - \sqrt{\gamma^2 - 1}$</td>
<td>$[\frac{1}{1 - \beta^2}, 1]$</td>
<td>$[0, \arccos \beta]$</td>
<td>$[\arccos \beta, \pi]$</td>
</tr>
<tr>
<td>$V, H$</td>
<td>$\frac{1 - n^2}{1 + n^2}$</td>
<td>$[\frac{1}{2}, 1 + \frac{n}{2}]$</td>
<td>$[0, 2\pi]$</td>
<td>$[\arccos(1 - 2\beta^2), \pi]$</td>
</tr>
</tbody>
</table>

Figure 2. Dependence of the normalized coupling coefficient $\cos \theta \sin(\psi + \phi) k_1k_2k_3/(\omega_2\omega_1)^{1/2}$ (see eq. (14)), of the frequency ratio $\omega_{2j}/\omega_{1j}$, and of the angle $\phi$ on the angle $\theta$ in the $P$-triplet. Here, $\tilde{k}_j = k_j/k_1$, $\tilde{\omega}_j = \omega_j/\omega_1$. The normalized coupling coefficient is proportional to the increment of the parametric instability of mode ‘1’ in absence of attenuation. The calculations are made for the Poisson ratio $\nu = 0.2$.

Figure 3. Dependence of the frequency ratio $\omega_{2j}/\omega_{1j}$ and of the angle $\phi$ on the angle $\theta$ in the $V$- and $H$-triplet, see eqs (15) and (16). The calculations are made for the Poisson ratio $\nu = 0.2$. 

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to \( \omega_p - \omega_s \) and \( k_3 - k_p \), respectively. The signs of \( V_I \) and \( V_H \) are opposite, because according to out convention concerning directions of the polarization vectors explained in Appendix (see eq. A11), the polarization vectors of counter-propagating SH waves are collinear, while the polarization vectors of counter-propagating SV waves are anti-collinear. For this reason the sign of the product \( u_2 u_3 \) changes depending of whether the shear waves are considered as SV or SH waves. The resonant condition under which a collinear \( V \)- or \( H \)-triplet exists has the form \( \omega_0 - \omega_s = (\omega_p - \omega_s)/c_s = \omega_p/c_p \), or

\[
\frac{\omega_p}{\omega_s} = \frac{2c_p}{c_p + c_s}.
\]

(18)

Generation of difference-frequency harmonics under resonant condition (18) has been observed experimentally by Liu et al. (2011), who investigated interaction of counter-propagating compressional and shear waves.

In addition to the above three resonant triplets, there exists a triplet with three collinear compressional modes. This triplet is the same as the parametric array by Westervelt (1963). We call it the ‘acoustic’ triplet and label it by the letter ‘A’. If one of conditions (9) for the acoustic triplet is satisfied, the second of these conditions is satisfied too. Thus, in contrast to collinear \( V \)- and \( H \)-triplets that satisfy condition (18) the acoustic triplet can exist for arbitrary ratios \( \omega_0/\omega_2 \). For the sake of generality we admit a weak medium anisotropy. Then the speed of compressional waves propagating in different directions might be different and as a consequence the acoustic triplet might be non-collinear. Small anisotropic corrections to velocities can be neglected in evaluating the non-linear force. Then the calculations of Appendix A are still applicable and they show that the coupling coefficient of the acoustic triplet \( V_A \) is equal to

\[
V_A = \left[ \lambda + 2B + 2C + 2(\lambda + 3N + A + 2B) \cos \theta \cos \phi \cos \psi \right]
× k_1 k_2 k_3.
\]

(19)

Resonant triplets \((SV, SH, SH)\) formed by the shear mode with vertical polarization and two shear modes with the horizontal polarization can also exist in a weakly anisotropic medium. The coupling coefficient \( V_S \) of these ‘shear’ triplets is equal to

\[
V_{\pm S} = \left( N + \frac{A}{4} \right) \sin(\theta \pm \phi)k_1 k_2 k_3.
\]

(20)

The signs ‘\( \pm \)’ in eq. (20) correspond to the interactions \( SH(\omega_1) + SH(\omega_2) \rightarrow SV(\omega_1 \pm \omega_2) \), while the \( SV \) mode is labelled as mode ‘1’ in both cases. In isotropic media the speeds of \( SV \) and \( SH \) waves coincide. In this case \( \theta = 0, \phi = 0 \) and the coupling coefficient \( V_S \) vanishes.

In Appendix A, we show that not only resonant but also non-resonant pairwise interactions of plane waves are completely described by the five coupling coefficients (14)–(19).

5 EVOLUTION OF RESONANTLY COUPLED MODES

In the previous section we calculated coupling coefficients for resonantly interacting modes, which is a static problem. Below we analyse how the wave amplitudes change in time. This allows us to estimate the maximum achievable amplitudes of non-linear signals in rocks.

Suppose that only mode ‘1’ is excited originally. Assuming that the initial amplitudes of modes ‘2’ and ‘3’ are identically zero, one would conclude that they will never be excited. Such an assumption is however not physical, because vanishingly small but still non-zero fluctuations are always present. Then due to non-linear coupling mode ‘1’ starts generating modes ‘2’ and ‘3’. The energies of modes ‘2’ and ‘3’ remain small at the initial stage of the process. The amplitude \( a_1 \) of mode ‘1’ is approximately constant, and eq. (10) reduces to two linear coupled equations for \( a_2 \) and \( a_3 \). We solve linearized eq. (10) taking into account wave attenuation. This is done by renormalizing the amplitudes as \( a_j \rightarrow a_j \exp(-\nu_j t) \), where \( \nu_j \) is the damping rate of the \( j \)-mode. In addition, we allow a deviation from the frequency matching condition, so that \( \delta \omega = \omega_1 - \omega_2 - \omega_3 \neq 0 \). Introducing the variables \( \tilde{a}_j = \sqrt{\nu_j} a_j \), assuming that \( \tilde{a}_2, \tilde{a}_3 \) are proportional to \( \exp[\pm i\delta \omega/2t] \) and neglecting the spatial derivatives one derives the dispersion relation,

\[
y = -\frac{v_2 + v_3}{2} + \sqrt{\left(\frac{v_2 - v_3 + i\delta \omega}{2}\right)^2 + W^2|\tilde{a}_1|^2}.
\]

(21)

Here, \( W = V/(2\rho \sqrt{\omega_1 \omega_2 \omega_3}) \). Modes ‘2’ and ‘3’ grow exponentially if \( \text{Re}(\gamma) > 0 \). This phenomenon is called the parametric instability.

For an exact resonance where \( \delta \omega = 0 \), the parametric instability condition reduces to \( W^2|\tilde{a}_1|^2 > v_2 v_3 \), which in the order of magnitude reads as

\[
k|a_1|/\Gamma > 1/Q.
\]

(22)

Here, \( k = \omega/c \) is the wave vector, \( c \) is the characteristic wave velocity, and \( \Gamma \) is the non-linear parameter,

\[
\Gamma = \frac{A}{\rho c^2}.
\]

(23)

The value \( A \) represents any of non-linear elastic moduli or their combinations, and \( Q \) is the characteristic quality factor of interacting waves. According to measurements by Sams et al. (1997) (see also Pride et al. 2003), the quality factor for sedimentary rocks scales approximately as \( 1/Q = 10^{-5} \omega_s \), where the frequency \( \omega_s \) is measured in inverse seconds \((1 \text{ s}^{-1} = 2\pi \text{ Hz})\). Hence inequality (22) can be written as \( |a_1| > 10^{-5} c/\Gamma \). Taking as an estimate \( \Gamma \approx 10^3 \) (Bakulin...
Non-linear effects can be significant if two modes with different frequencies exist originally. This happens during propagation of packages of plane waves. Due to the source non-monochromaticity and wavefront curvature wave packages contain harmonics distributed within some frequency band, \( \omega_0 \pm \delta \omega_0 \), and within some range of wave vectors, \( \vec{k}_0 \pm \delta \vec{k} \). As one can see from Table 2, two compressional waves with close wave vectors generate shear waves, which propagate at angle \( \phi = \arccos \vec{\nu} \) with respect to the average wave vector \( \vec{k}_0 \) of wave package. Bercoff et al. (2004) observed generation of shear waves by propagating compressional waves in experiments with ultrasound. They called this phenomenon the elastic Cherenkov effect by analogy with electromagnetic radiation emitted by high energy charged particles. It can also be compared with the ‘sonic boom’ created by supersonic aircrafts. The compressional wave package propagates with a speed higher than the speed of shear waves, while the non-linear effects make the interaction between the compressional and shear modes possible. The compressional wave package plays the role of a supersonic aircraft that emits ‘sonic-shear’ waves at the Cherenkov angle.

Using eqs (10) and (11) we can also evaluate the maximum amplitude of the non-linearly generated harmonic. The properties of this system of equations have been studied by Zakharov & Manakov (1973) and by Kaup (1981). In the absence of spatial dependencies eqs (10) and (11) possess three conservation laws

\[
\rho \rho_1 \rho_2 \sin \phi = U, \quad \rho_1^2 + \rho_2^2 = m, \quad \rho_1^2 + \rho_2^2 = n. \tag{24}
\]

Here, \( \rho_j = |\vec{a}_j| = n^{1/2} \), and \( U, m \) and \( n \) are constants. Using eqs (24) one reduces eqs (10) and (11) to a single equation for \( \rho_j \). The solution of the resulting equation with the initial condition \( \rho_1(0) = 0 \) is

\[
\rho_1 = m^{1/2} \sin(\tau |k|^2), \tag{25}
\]

while the amplitudes of other waves are equal to \( \rho_2 = m^{1/2} \cos(\tau |k|^2) \) and \( \rho_3 = n^{1/2} \tan(\tau |k|^2) \). Here, \( m, c, n, c_1 \) and \( m_0 \) are Jacobian elliptic functions, and \( \tau = Wm^{1/2}t \). Jacobian elliptic functions are periodic. The period is \( T = 2K(\kappa^2) \), where \( K \) is the complete elliptic integral of the first kind, and in the order of magnitude it is equal to \( 1/(Wm^{1/2}) \). The condition that modes do not decay significantly during one period, \( \tau T < 1 \), coincides with inequality (22). In this case non-linearly generated modes can grow to amplitudes comparable to original amplitudes of linear modes. If \( \tau T > 1 \), then growth of resonant interaction force \( f^{(2)} \) between such waves can be found using the equations given in Appendix A. Substituting \( f^{(2)} \) into eq. (7) one then calculates the displacement in combined harmonics. We evaluate integral (7) by three different methods. First we use a qualitative analysis, then apply the saddle-point (stationary phase) method, and finally perform a numerical integration.

In geophysical settings sources can be considered as points, and explored targets are mostly located in the far field of these sources, where \( kR \gg 1 \). Waves generated by seismic sources can be locally approximated in the far field by plane waves, and the non-linear interaction force \( f^{(2)} \) between such waves can be found using the equations given in Appendix A. Substituting \( f^{(2)} \) into eq. (7) one then calculates the displacement in combined harmonics. We evaluate integral (7) by three different methods. First we use a qualitative analysis, then apply the saddle-point (stationary phase) method, and finally perform a numerical integration.

The integrand in eq. (7) contains an oscillating exponent \( \exp(i\omega t) \), where \( \psi \) is the phase of the non-linear signal at the receiver \( \psi = k_1|\vec{r} - \vec{r}_1| \pm k_2|\vec{r} - \vec{r}_2| + k_0|\vec{r} - \vec{r}_0| \). Amplitudes of weak microseismic noise are about \( 10^{-3} \) m in the exploration seismic frequency band. Hence, the amplitude of the interacting linear waves \( |a_0| \) should be at least \( 10^{-7} \) m for non-linear effects to be observable.
point \( \mathbf{r} \) of the non-linear layer arrive at the receiver with the same phase \( \phi \), if \( \nabla \cdot \varphi = 0 \), or

\[
\frac{k_1 \mathbf{r}_{10} - \mathbf{r}}{|\mathbf{r}_0 - \mathbf{r}|} = k_2 \frac{\mathbf{r} - \mathbf{r}_{10}}{|\mathbf{r} - \mathbf{r}_1|} \pm \frac{k_2 - \mathbf{r} - \mathbf{r}_{10}}{|\mathbf{r} - \mathbf{r}_1|}.
\]

(27)

Here, the subscript ‘\( \perp \)’ denotes components of vectors in the horizontal plane. Eq. (27) is the same as the resonance condition in two dimensions, \( k_0 = k_{10} + k_1 \). A neighbourhood of the point \( \mathbf{r} = \mathbf{r}_1 \) where condition (27) is satisfied gives the dominant contribution to the signal observed at point \( \mathbf{r}_0 \). This neighbourhood is an analogue of the first Fresnel zone. Non-linear waves generated beyond this neighbourhood mostly cancel each other and they do not influence the signal measured. Although the non-linear domain might be infinite, the non-linear signals are effectively generated in a finite area whose volume is equal to the volume of the ‘first Fresnel zone’.

To estimate integral (7) we assume that \( k_{1,2} \) and \( k \) have the same order of magnitude. The non-linear force \( f^{(2)} \) scales as \( f^{(2)} \approx Ak^2u^2 \). The solution of the Navier equation in the far field can be approximately replaced by the solution of the Poisson equation

\[
\nabla^2 u^{(2)} = \frac{f^{(2)}}{\rho c^2},
\]

so that

\[
u^{(2)} \approx \frac{1}{\rho c^2} \int \frac{f^{(2)} d^2}{4\pi |\mathbf{r} - \mathbf{r}'|} \approx \Gamma \frac{k^2 u^2}{4\pi H} h_s,
\]

where \( u_s \) is the volume of the ‘non-linear Fresnel zone’ where stationary phase condition (27) approximately holds. We assume that the distance between the sources and the receiver does not essentially exceed the non-linear layer depth \( H \), so that the characteristic travel distances are about \( H \). The magnitude of the solid displacement in linear waves is equal to \( u = F/(2\pi \rho c^2 H) \), where \( F \) is the vibroseis force. Assuming that the layer depth \( H \) is larger than the lateral extension \( L_0 \) of the ‘non-linear Fresnel zone’, \( H \gg L_0 \), we conclude that the wave path from sources to the receiver changes by about \( \sqrt{H^2 + L_0^2} \approx H \) when the non-linear wave is generated at opposite horizontal boundaries of the ‘non-linear Fresnel zone’. The requirement that the reflected waves are in phase implies \( L_0^2/(2H) \) is about a half wavelength, or \( L_0^2 \approx 2\pi H/k \). Substituting the above results in eq. (28) we get

\[
u^{(2)} \approx \frac{\Gamma k^2}{2H^2} \left( \frac{F}{2\pi \rho c^2} \right)^2 h_s,
\]

(29)

where \( h_s \approx \pi/k \) is the height of \( u_s \).

Integral (7) can be calculated more accurately using the saddle-point method. A 2-D integral in the saddle-point method is evaluated as (see e.g. Chapman 2004),

\[
\int_{-\infty}^{\infty} e^{-i\varphi} g(x, y, \mathbf{r}_0) dx dy = 2\pi \frac{g(\mathbf{r}_0) e^{i\varphi(\mathbf{r}_0)}}{\sqrt{|\Lambda_1 \Lambda_2/4|}} e^{i(\phi(x, y, \mathbf{r}_0) - \sigma/2)}. \]

(30)

Here, \( g = g(x_1, x_2), \varphi = \phi(x_1, x_2), \mathbf{r}_p \) is the point where \( \nabla \varphi = 0 \), \( \Lambda_{1,2} \) are the eigenvalues of the matrix \( \partial^2 \varphi/\partial x_1 \partial x_2 \), with \( j, k = 1, 2 \) at the point \( \mathbf{r}_p \), and \( \sigma \) is the sign of \( \Lambda_j \). In the case considered \( \varphi = (k_1 \pm k_2)x_0 - k_0x \), where \( c_0 \) is the speed of the combined-frequency wave generated in the layer, the function \( g \) is obtained by skipping the phase factor \( \exp(i\varphi) \) in the integrand of eq. (7), and \( \mathbf{r}_p \) is found by solving eq. (27). In this calculation we use Green’s tensors in their exact form (5) and (6). Thus, our far-field approximation refers only to linear waves, but not to non-linearly generated signals. After the 2-D integral is found, one determines the response from the layer by multiplying the result by the layer width, or by the width of the ‘first Fresnel zone’ \( h_s \), if the layer width exceeds \( h_s \).

We also calculate integral (7) numerically for infinitely thin layers and compare the result with predictions from the saddle-point method. The frequencies of the sources are specified as \( \omega_{1,j}/(2\pi) \approx 7 \) Hz and \( \omega_{2,j}/(2\pi) \approx 5.25 \) Hz. We assume that the sources generate \( P \) modes. The formation density is \( \rho = 2000 \text{ kg m}^{-1} \), the speed of compressional waves is \( 2000 \text{ m s}^{-1} \) and the speed of shear waves is \( 1000 \text{ m s}^{-1} \). The depth of the non-linear layer in our examples is \( 1000 \) m. The non-linear layer is a square with sides of \( 4000 \) m and with a small thickness. The distance between sources is \( 500 \) m. The middle point between the sources is located above the layer centre and its coordinate is \( x = 0 \). According to eq. (A12), the non-linear responses depend on two combinations of third order elastic moduli of the layer, which we denote \( C_1 = \lambda + 2B + 2C \) and \( C_2 = \lambda + 3N + 4 + 2B \).

Figs 6 and 7 show the normalized surface displacement in the sum- and difference-frequency harmonics that are generated inside the non-linear layer for the case \( C_1 = 1 \) and \( C_2 = 0 \). Circles represent numerical calculations and lines represent the saddle-point method. Using the saddle point method we calculate separately displacements in the generated compressional mode (dash–dotted lines) and in the shear mode (dash lines) as well as the total displacement (solid lines). In the case \( C_1 = 0 \) and \( C_2 = 1 \) we obtained...
B. N. Kuvshinov, T. J. H. Smit and X. H. Campman

1 and in eq. (29). According to with angular frequency \( \omega = \bar{\omega} \approx (31) \approx u_{500} \) kN, so that \( \omega W \) \( \pi \) to the second harmonic. The energy density of the \( \bar{\omega} \) \( \approx 10 \) generates the sec-

\[ \bar{u}_{\omega}(2) \approx \alpha \frac{c}{c_{w}} \]

\[ \bar{u}(1) / \bar{u}(1) \]

\[ \bar{u}(1) \]

\[ \alpha = \Gamma k^2 \bar{u}(12) \]

\[ \nabla \cdot ( \bar{u}(2) \bar{e} ) = \alpha \bar{u}(2) ( \frac{\bar{u}(1)}{\bar{u}(1)} )^2 \] (31)

\[ \nabla \cdot \bar{u}(2) = W(2) \]

7 ANALYSIS OF FIELD TESTS

The calculations carried out in Section 6 suggest that non-linear signals generated at depths of more than several hundred metres cannot be observed in field tests with active sources. This result warrants a closer look at field tests, where such signals were supposedly registered.

We are interested in those non-linear signals that can provide a better formation characterization compared to conventional seismic methods. Detection of such signals is not a trivial task. Non-linearities can be of different origin. They can be generated by detectors and sources. A source can also interact non-linearly with the ground roll excited by itself or by another source. Wave attenuation causes additional problems in non-linear measurements. To separate non-linear signals generated by interacting seismic waves inside rocks from other types of non-linear signals it is convenient to measure the ratio of the amplitudes of non-linear and linear (source) harmonics rather than the absolute values of these amplitudes. A growth of such ratio would indicate presence of the non-linear effects we are looking for.

We start with the analysis of data by Beresnev & Nikolaev (1988), who measured the ratio of the non-linear to the source harmonics at various distances from a seismic source. The source consists of a vibrator emitting a monochromatic wavefield at 19 Hz. The surface wave velocity and the compressional wave velocity in the upper layer are 200 and 2000 m s\(^{-1}\), respectively. The wavefield was measured at different locations from the source, and for each location the ratio between the source harmonic and the double-frequency harmonic was calculated. Solov’ev (1990) summarized the corresponding numbers in a table, which we use in our analysis. Solov’ev (1990) criticized Beresnev & Nikolaev (1988), arguing that they supported their conclusions by giving the amplitude ratios for one out of the 24 channels only (the best result was obtained for the 12th channel) and that no systematic pattern can be observed when any other of the channels is studied nor when an average over 24 recording channels is taken (see also a response from Beresnev 1995). Having considered the data presented by Solov’ev (1990) we concluded that the non-linearities observed by Beresnev & Nikolaev (1988) are most probably associated with surface effects.

We estimate the growth behaviour of the non-linear harmonic amplitude using the balance for energy flux and transmitted power. The linear displacement \( u^{(1)} \) with angular frequency \( \omega \) generates the second (non-linear) harmonic \( u^{(2)} \) with frequency 2\( \omega \). The non-linear force \( f^{(2)} \approx \rho k^2 u^{(12)} \) transmits a power \( W^{(2)} \approx f^{(2)} (du^{(2)}/dt) \approx A \omega k^2 u^{(12)} u^{(2)} \) to the second harmonic. The energy density of the non-linear harmonic is equal by an order of magnitude to \( E^{(2)} \approx \rho \omega^2 u^{(12)} \). The corresponding energy flux \( \bar{S}^{(2)} \) can be estimated as \( \bar{S}^{(2)} \approx \varepsilon E^{(2)} \bar{e} \), where \( \varepsilon \) is the wave velocity and \( \bar{e} \) is the unit vector in the direction of wave propagation. Substituting the above estimates for the energy flux and the transmitted power in the energy balance equation \( \nabla \cdot \bar{S}^{(2)} = W^{(2)} \), we obtain

\[ \nabla \cdot ( \bar{u}^{(2)} \bar{e} ) = \alpha \bar{u}^{(2)} ( \frac{\bar{u}^{(1)}}{\bar{u}^{(1)}} )^2 \] (31)

Here, \( \bar{u}^{(1)} \) is the displacement of the linear harmonic in some reference point \( \bar{r} \). Propagation of surface waves is a 2-D process. Assuming the radial symmetry one writes the divergence operator

Figure 7. Normalized vertical and horizontal surface displacement in the combined harmonic with the difference-frequency. Notations are the same as in Fig. 6.
in cylindrical coordinates as ∇ · (u^2r^2) = (1/r)(d/dr)(ru^{2/2}). Surface
waves decay inversely proportional to the square root of distance from the source. Then u_r(1)/u_r(1) = (r_f/r)^1/2 and solution of
eq (31) can be written in the form
\[ \frac{u_r(2)}{u_r(1)} = \left( \frac{\vec{r}}{r_f} \right)^{1/2}. \] (32)
Substituting in eq. (32) parameters of the tests by Beresnev &
Nikolaev (1988), Ω/(2π) = 19 Hz, c = 200 m s^{-1}, Λ ≃ 10^3 and
u_r(1) = 10^{-5} m for r_f = 1 m, we get u_r(2)/u_r(1) ≃ 4 × 10^{-3}/r, where
the distance r is measured in metres. Our theoretical estimate for
u_r(2)/u_r(1) is shown in Fig. 8 by the solid line. The same figure shows
experimental data by Beresnev & Nikolaev (1988). The data from the
12th channel as well as an average over 24 channels are in a
good agreement with our estimate. This suggests that Beresnev &
Nikolaev (1988) indeed observed formation non-linearity, although
this non-linearity is related to surface waves rather than to body
waves.

In 2009, Shell and CGGV carried out an experiment to test var-
ious low-frequency sweeps and study the propagation character-
istics of low-frequency signals (Baeten et al. 2010). During the
experiment, time was allocated for the testing of the generation
of combined harmonics by simultaneously driving two vibrators at
different frequencies. Geophones were placed on the surface and
in a wellbore. Non-linear harmonics were present in both surface
and downhole data. However, the ratio between the amplitudes of
non-linear and linear harmonics did not increase with the distance
from the sources in contrast to the observations by Beresnev &
Nikolaev (1988). This is illustrated in Fig. 9 which shows the rela-
tive amplitude of double-frequency harmonic. We average spectra
from 5 geophones at the surface at 25 distances along the line. In
case of downhole measurements, we did not average over neighbour-
ning traces as we only have a limited offset range. For this reason the
downhole data is strongly scattered. The amplitude ratios of the fun-
damental and second harmonic did not appear to give the desired
growth. Moreover we note a decreasing linear trend, presumably
caused by exponential attenuation, which is faster for the second
harmonic.

Differences between data measured by Beresnev & Nikolaev
(1988) and (Baeten et al. 2010) can be attributed to the wave
attenuation that was neglected in the above analysis. The wave
amplitude decreases due to dissipation effects proportionally to
exp \{-π(2x)/λΩ\}. Here, x is the travelled distance, λ is the wave-
length and Q is the quality factor. The second harmonic attenuates
faster because is has shorter wavelength, while the quality factor Q
varies slow with the wave frequency. According to the above rela-
tion, the logarithmic ratio of the amplitudes of the second to first
harmonic in absence of non-linear effects decays linearly with the
distance as log [u_2/u_1] = const − πx/λΩ, where λ is the wave-
length of the first harmonic. The quality factor Q is the smallest
in unconsolidated sediments, and shallow formations are less con-
solidated than deeper formations. Hence, one can expect that the
amplitude ratio decays faster in case of surface waves. Comparison
of the trends in Fig. 9 shows the opposite: the surface trend has a
lower slope and hence the amplitude ratio of surface waves (upper
figure) decays slower with the distance than that of the bulk waves
travelling downwards along the borehole (lower figure). This might
indicate the presence of non-linear effects at the surface, which gen-
erate the second harmonic and partially compensate its attenuation.
However, the quality of field data is not sufficient to confirm this.
The difference between the surface and the borehole trends can be
due to higher wave attenuation in the deeper subsurface, due to
presence of fluids or scattering at the borehole surface.

We conclude that the measurements made by Baeten et al. (2010)
could not reproduce the results of Beresnev & Nikolaev (1988). This
can be explained by different conditions in the tests by Beresnev &

![Figure 8](https://academic.oup.com/gji/article-abstract/194/3/1920/653074/1929)

**Figure 8.** Ratio of amplitudes of non-linear and linear harmonics in tests by
Beresnev & Nikolaev (1988). Solid line shows theoretical prediction (32),
circles and triangles shows data presented by Solov’ev (1990).

![Figure 9](https://academic.oup.com/gji/article-abstract/194/3/1920/653074/1929)

**Figure 9.** Ratio of amplitudes of non-linear (30 Hz) and linear (15 Hz)
harmonics in tests by Baeten et al. (2010) versus the distance along the
line. Upper figure:- surface measurements, lower figure: downhole mea-
surements. The zero distance in downhole measurements corresponds to the
depth of 1370 m.
The vibro-acoustography measurements are qualitatively the same as in Fig. 5. Sources with frequencies \( \omega_1 \) and \( \omega_2 \) are focused on a target. If the target elastic properties are different from the elastic properties of the surrounding medium, then a non-linear response in the acoustic range is observed, whose frequency is equal to \( \omega_1 - \omega_2 \). Otherwise, the non-linear response is absent.

The task of the vibro-acoustography measurements is to detect harder targets (tumors) inside a relatively soft surrounding medium (fluid-like tissue). Elastic moduli of both targets and surrounding medium are non-linear, and they both can generate non-linear responses. For correct interpretation of the measured signals it is essential to understand where they come form, and whether they are governed by the non-linear properties of the target or of the surrounding media. In spite of extensive research, there is no agreement concerning the physics behind the vibro-acoustography method.

Fatemi & Greenleaf (1998, 1999) explained the observed effect by the dynamic radiation force. It has been also suggested that this effect is mainly governed by the target non-linearity (Alison Malcolm, private communication), as we assumed in Section 7. Thierman (2004) repeated the tests by Fatemi & Greenleaf (1998, 1999) and put forward an alternative point of view. He argued that the non-linear harmonics are generated in the outer medium due to the parametric array effect, which appears after double reflections of ultrasound beams from the target and transducers. A similar explanation has been proposed by Silva et al. (2008) and Silva & Mitri (2011), with the difference that a secondary parametric array is formed by nearly spherical ultrasound waves scattered by a small target. Compressional non-scattered ultrasonic beams can also interact and generate non-linear harmonics, which subsequently scatter from the target and amplify the radiation force (Silva et al. 2006).

The radiation force acting on a light sphere of radius \( r_0 \) placed in a fluid of density \( \rho_f \) is equal to (Chen et al. 2002) \( F_{\text{rad}} = \pi r_0 E Y \).

Here, \( E = p^2 / (2 \rho_f c^2) \) is the energy density of the incident wave, and \( Y \) is the dimensionless radiation force function that characterizes material properties of the target. In the order of magnitude \( Y \sim 1 \).

The difference-frequency displacement \( u_{\omega-\omega} \) induced by \( F_{\text{rad}} \) is about \( u_{\omega-\omega} \sim F_{\text{rad}} / (4 \pi \rho_f c^2 r) \). Here, \( \rho_f \) is the density of the surrounding fluid. Taking into account the relation \( u_{\omega-\omega} \sim \rho_f k^2 c^2 k_\perp \), where \( p \) is the pressure and \( k = \omega / c \) is the wave vector of the difference-frequency harmonic, we get \( u_{\omega-\omega} \sim k \rho_f p^2 / (8 \rho_f c^2 r^2) \).

For a more accurate calculation one needs to take into account the resistance force between the sphere and fluid and use the exact expression for the fluid pressure generated by an oscillating sphere (Pierce 1981). This gives

\[
F_{\omega-\omega} = \frac{3p^2}{8 \rho_f c^2} \sqrt{1 + k_r^2 r^2} r_0^2 \cos \theta, \tag{33}
\]

where \( \theta \) is angular coordinate between the line connecting the sphere with the observation point and the direction of the propagation of ultrasonic waves. Eq. (33) coincides with the relation given by Silva & Mitri (2011) for \( k - r \gg 1 \).

Fatemi & Greenleaf (1999) performed tests with a glass sphere, where \( r_0 = 225 \mu m, \rho_0 = 2.5 \text{ g cm}^{-3}, r = 5 \text{ cm}, \omega_{\omega} / (2\pi) = 40 \text{ kHz}. \)

For an ultrasound beam with intensity of \( I = 1 \text{ W cm}^{-2} \) the wave pressure is about \( p = (2pcI)^{1/2} = 1.7 \cdot 10^{10} \text{ Pa}, \) and eq. (33) gives \( p_{\omega-\omega} = 1.4 \times 10^{-5} \text{ Pa}. \) According to fig. 3 of the paper by Fatemi & Greenleaf (1999), an acoustic intensity \( 5 \cdot 10^{-14} \text{ W cm}^{-2} \) was observed under above conditions. This corresponds to an acoustic wave pressure \( p_{\omega-\omega} = 4 \times 10^{-2} \text{ Pa}, \) which exceeds the theoretical value by two orders of magnitude.
The above discrepancy between theory and experiments requires a more accurate analysis of the vibro-acoustography method. Fig. 10 shows the considered configuration of acoustic sources. Since exact modelling of non-linear ultrasound tests is extremely difficult, one needs to develop approximate approaches (Malcolm et al. 2008). In our analysis we use the result by Wen & Breazeale (1988) who have proposed to represent arbitrary sources as sums of Gaussian sources. Although Gaussian functions do not constitute a complete orthogonal system, one can achieve a reasonably good accuracy using them as an expansion basis. The method of Wen & Breazeale (1988) has been further developed in Huang & Breazeale (1999), Ding et al. (2003) and Ding & Zhang (2004). If the wave equation is replaced by its parabolic approximation (Zabolotskaya & Khokhlov 1969; Kuznetsov 1971) then in case of a Gaussian source both linear and double harmonics can be calculated analytically (Naugolnykh & Ostrovsky 1998). Hence, the Gaussian-beam expansion together with the parabolic approximation provides a powerful analytical approach to study non-linear acoustic waves. In previous studies of Ding et al. (1996), Ding (2000, 2004) and Huang et al. (2009) based on this approach, amplitudes of combined harmonics were represented in the form of an integral that has to be calculated numerically. We note that the corresponding result can be written in a fully analytical form using the solution by Darvennes & Hamilton (1990). The details of this solution are given in Appendix C.

We applied the developed method to model tests by Fatemi & Greenleaf (1999). The calculations were performed for two confocal sources driven at frequencies $\nu_0 \pm \Delta \nu/2$ with $\nu_0 = 3$ mHz. The value of $\Delta \nu$ was either 7 or 40 Hz. The inner transducer has a circular shape with radius 14.8 mm. The outer transducer has a ring shape with radii 16.8 and 22.5 mm. Both transducers have approximately the same area of 7 cm$^2$. The focal distance of both transducers is 70 mm. Fig. 11 shows the pressure distribution in the difference-frequency 40 Hz harmonic. The transducer pressures were set as $p_0^{(out)} = 8.35$ kPa and $p_0^{(in)} = 8.7$ kPa, respectively. With this choice, each of the transducers creates the required focus intensity of about $10^4$ W m$^{-2}$. According to our modelling both transducers have the focal gain of about 20, which is the ratio of the focal pressure to the transducer pressure. The difference-frequency harmonic generated by compressional ultrasound beams has a distinct peak in the focal area. At a distance of several centimetres from the focus, its amplitude becomes too low to be measured. Scattering of the difference-frequency harmonic by a small target, which is essentially the same as the effect considered by Silva et al. (2006), is also very weak, and it cannot contribute to the signal measured by Fatemi & Greenleaf (1999). We also found that the effects of acoustic streaming and of co-propagating non-linear waves are very small. The non-linear signals generated at the boundaries or inside small targets such as in tests by Fatemi & Greenleaf (1999) are also insignificant. Such signals can be measured only if the target non-linearity is unrealistically large.

9 INTERACTION OF SCATTERED HARMONICS

The ‘dynamic radiation force’ effect introduced by Fatemi & Greenleaf (1999) is in fact the Doppler effect. In their tests the scattering occurs from a moving target. The target oscillates in the wavefield of the first beam with frequency $\omega_1$. Scattering of the second beam results in a frequency shift so that the scattered wave contains the difference frequency harmonics $\omega_1 - \omega_2$.

Generation of combined harmonics during scattering from oscillating targets is a well known phenomenon. It was investigated in particular in a series of papers by Censor (1972, 1984, 1986, 2004). Rogers (1973) and Piquette & Buren (1984, 1986a,b, 1993) argued that the effect discussed by Censor is small compared to the generation of non-linear harmonics by reflected waves in the outer medium. The Doppler effect can be dominant at relatively short distances from the target to the observation point (Mujica et al. 2003). Silva et al. (2008) and Silva & Mitri (2011) reached essentially the same conclusion as Rogers (1973) and Piquette & Buren (1984, 1986a,b, 1993), by noting that the sound-to-sound interaction of scattered waves is larger than the ‘dynamic radiation force’. In our analysis we also arrived at this conclusion.

If the target is small compared to the ultrasonic wavelength then the scattered waves are approximately spherical, and their amplitude varies as $1/r$, where $r$ is the distance from the target. The amplitude of the difference frequency harmonic can be estimated using eq. (31), where we replace $u^{(2)}$ by $u_-$ and take into account the amplitudes of the wave displacement and pressure are related as $ku \approx \rho/(pc^2)$. Similarly to eq. (32) we get $p_- = [\Gamma k R_0^2 p^2/(2c^2)] \ln r/r_0$, where $r_0$ is the target radius. A more accurate result is obtained by solving eq. (C1) in spherical
coordinates, which gives
\[ p_\perp = -\frac{i\Gamma p^2 k_r r^2}{2\rho c^2} g \perp, \quad g \perp = C_2 + e^{-2i\xi} E_1(-2i\xi) + \ln(2i\xi). \]  

Here, \( \xi = k_r r, \xi_0 = k_r r_0 \) and the constant \( C_2 = -e^{-2i\xi_0} E_1(-2i\xi_0) - \ln(2i\xi_0) \) is found from the condition \( p_\perp (r_0) = 0 \). Using asymptotics for the exponential integral one can introduce the approximation \( g \perp = \ln(2i\xi) + y - i\pi/2 \), which is valid at \( \xi_0 \to 0 \) and \( \xi \gg 1 \).

The scattered waves in laboratory tests are not exactly spherical. As a result the parametric array effect is weaker and the observed non-linear pressure should be smaller than predicted by eq. (34). This signal reduction can be taken into account by applying an expansion in spherical Bessel functions as has been done by Silva et al. (2008) and Silva & Mitri (2011). Alternatively one can multiply \( p_\perp \) by an empirical reduction factor, which is approximately equal to 0.1 for the vibro-acoustography tests. Applying the above formulas to tests by Silva & Mitri (2011) we obtain results similar to those that have been measured. For the parameters of the tests by Fatemi & Greenleaf (1999), eq. (33) predicts a difference-frequency pressure by about an order of magnitude larger than eq. (34), which is still significantly smaller than the pressures reported by Fatemi & Greenleaf (1999). We expect that Fatemi & Greenleaf (1999) either observed some artefacts or their paper contains a misprint. Note that the presence of artefacts is common in tests of this type (Piquette & Buren 1993; Calil et al. 2002). Piquette & Buren (1993) confirmed that the measured difference-frequency pressure in their case was primarily due to hydrophone non-linearity. They observed this effect for a wide range of hydrophones, even in cases where data from the Naval Research Laboratory suggested insignificant hydrophone non-linearity.

The vibro-acoustography was developed for medical imaging where the analysed objects consist mainly of water. The wavelength of sound in rocks is higher than wavelength of sound in water. For this reason, sound waves in rocks are more difficult to focus. On the other hand, in contrast to medical diagnostics where the ultrasound intensity should be less than about 1 W cm\(^{-2}\) in order not to damage tissues, one does not need to limit the source power in experiments in rocks. Moreover, the non-linearity of porous materials is about two orders of magnitude higher than the non-linearity of liquids. Hence the vibro-acoustography method might be useful to analyse the internal structure of rock samples.

We analysed waves reflected from a small scatterer inside a rock. Fig. 12 shows the pressure in the reflected linear and non-linear harmonics. In these calculations we considered two ultrasound waves with frequencies \( \nu_1 = 3.05 \) MHz and \( \nu_2 = 2.95 \) MHz, so that the difference frequency is \( \nu_1 - \nu_2 = 100 \) kHz. The scatterer size is \( r_0 = 100 \) \( \mu \)m and the signal is observed at distance of 10 cm from the target. The difference-frequency harmonic is generated in the spherical parametric array formed by reflected linear harmonics. Parameters of the solid medium are \( c = 3500 \) m s\(^{-1}\), \( \rho = 2500 \) kg m\(^{-3}\) and \( \Gamma = 10^3 \). Fig. 12 suggests that the non-linear responses become measurable for focal pressures above 1 MPa. Since we describe the scattering process approximately, this conclusion should be used only as an indication.

10 DISCUSSION AND CONCLUSIONS

We have shown that interactions of plane waves, both resonant and non-resonant, are completely described by five coupling coefficients given by eqs (14)–(20). The coupling coefficients calculated by previous authors (Jones & Kobett 1963; Childress & Hambrick 1964; Taylor & Rollins 1964; Zarembo & Krasil’nikov 1971; Korneev et al. 1998) can be recovered from these equations after a proper relabelling of angles in the triangle shown in Fig. 1. Instead of considering 54 types of interactions distinguished by Zarembo & Krasil’nikov (1971) it is sufficient to investigate behaviour of the above five coupling coefficients. Since all these coefficients have the same order of magnitude, the feasibility of non-linear seismic exploration can be assessed by analysing a pairwise wave interaction of any particular type.

In Section 6, we considered scattering of compressional waves generated by point-like sources from a non-linear layer. The wave attenuation was neglected. Contrary to Beresnev (1993), we have concluded that non-linear signals from deep formations cannot be measured at the current level of technology. Our conclusion is confirmed by both direct calculation of integral (7) and its approximate evaluation using the saddle point method. The main factors restricting the non-linear signals are small amplitudes of interacting linear harmonics at large depths and the finite size of the volume where the non-linear signal is generated. In case of resonantly interacting plane waves the growth of non-linear harmonics is restricted by the inelastic dissipation of wave energy. Estimations presented in Section 5 show that for typical rock parameters the amplitude of linear harmonics should be at least in the micron range to generated observable non-linear responses.

In our calculations we use the five-constant model by Gol’dberg (1960) with the characteristic value of the non-linear parameter \( \Gamma \simeq 10^3 \). The maximum value of the non-linear parameter in microinhomogeneous materials lies in the range \( \Gamma \simeq 10^3–10^5 \) (Belyaeva & Zaitsev 1998; Zaitsev et al. 2009). Even for such value of \( \Gamma \) the non-linear signals from deep formations should be significantly below the seismic noise level.

It is not excluded that non-linear signals are generated by some unknown physical effect that is not covered by the five-constant model. Presence of such an effect can be confirmed only in field tests. However, by analysing field data available we could not find convincing evidences of non-linear signals coming from deep formations. Seemingly, such signals are generated either in the shallow subsurface as in Fig. 8, or in the vicinity of sources as in Fig. 9.

Our analysis of field data in Section 7 also reveals problems in the interpretation of non-linear signals. Such signals can be of three types: (1) signals generated by sources (by controlled sources in used seismic acquisition or by areas around hypocentres of
earthquakes); (2) signals generated by local areas in the subsurface (playing a role of secondary sources), whose size does not exceed the characteristic wavelength and that have strong non-linear properties and (3) signals accumulated due to the parametric array effect along wave paths much larger than the characteristic wavelength. The signals of the second type present the most interest for us, because they provide information concerning the formation, which supplements standard linear seismic data. Signals of the third type could also be useful. However, their practical significance is limited, since they have a poor spatial resolution. The signals of the first type might be important in seismology. In geophysical prospecting they present an obstacle to correct interpretation of experimental data. Since the non-linear signals of all three types are very small, it is very difficult to discriminate between them. For this purpose one should analyse the spatial dependence of the signals. However, such an analysis is hindered by the wave attenuation effect, which can introduce additional uncertainties. Moreover, most of the energy of seismic sources is converted into surfaces waves that are sensitive to shallow (<200 m) underground structure. Shallow subsurface has usually stronger non-linear properties than more consolidated deeper layers. As a result, signals generated locally in the shallow subsurface can give a dominant contribution to the non-linear response, thus making invisible the informative signals coming from a large depth. As we see, separation of the source, path and site effects is a highly non-trivial problem that seemingly cannot be always solved.

We have also considered the possibility to increase non-linear signals using the wave focusing. Since all non-linear interactions of elastic waves have comparable strength we restricted ourselves to analysis of compressional waves, describing them by model eqs (C1) and (C2). We have developed an analytical scheme to solve this equation based on results by Wen & Breazeale (1988) and Darvennes & Hamilton (1990). Our analysis confirmed the conclusion by Silva et al. (2008) and Silva & Mitri (2011) that the dominant effect in the vibro-acoustography tests is the second parametric array formed by scattered linear beams.

The above finding implies that the observed difference-frequency response in experiments with focused beam characterizes the target only as a linear reflector. In the ultrasound imaging, non-linear techniques can be advantageous due to large noise at high frequencies. The low-frequency noise is small, so that the signal-to-noise ratio might be larger for the difference-frequency response. In geophysical settings, the noise at low frequencies is at least as large as the noise at high frequencies. There is no reason to expect that the signal to noise ratio will be small for the low-frequency non-linear harmonics. Another problem is creation of large sound pressures. In case of ultrasound imaging, one is able to focus on targets wave pressures of several mega Pascals. For typical exploration parameters, one can achieve pressures of tens of Pascals at target depths. Generation of pressures that are similar to the pressures in the ultrasound tests one needs to use thousands of vibrators, which does not look practical.

On the other hand, the vibro-acoustography method by Fatemi & Greenleaf (1999) might be used to analyse rock samples. Similar to medical imaging, the low-frequency sound range should be less noisy, which can provide a higher signal-to-noise ratio for the difference-frequency responses.

ACKNOWLEDGEMENT

We would like to thank Shell Global Solutions International for permission to publish this paper.

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APPENDIX A: SECOND ORDER ELASTIC FORCE

Elastic stress tensor (2) is split in two parts, \( \bar{\sigma} = \bar{\sigma}^{(1)} + \bar{\sigma}^{(2)} \). The superscripts ‘(1)’ and ‘(2)’ denote the linear and the second-order terms correspondingly. The second order elastic force \( f^{(2)} \) is found by substituting linear solid displacements \( \bar{u}_j = \bar{c}_j \exp(-i\omega_j t + i\bar{k}_j \cdot \bar{r}) \) with \( j = 1, 2 \) in \( \nabla \cdot \bar{\sigma} \),

\[
\bar{f}^{(2)} = \nabla \cdot [\bar{\sigma}^{(2)} (\bar{u}_1, \bar{u}_2)].
\]  

(A1)

We represent \( \bar{f}^{(2)} \) in the form

\[
\bar{f}^{(2)} = f^{(2)}_+ e^{-i(\omega_1 + \omega_2)t + (\bar{k}_1 + \bar{k}_2) \cdot \bar{r}} + f^{(2)}_- e^{-i(\omega_1 - \omega_2)t + (\bar{k}_1 - \bar{k}_2) \cdot \bar{r}} + \ldots.
\]  

(A2)

Here, ‘\ldots’ denote terms with double frequencies. From eqs (A1) and (A2) it follows \( \bar{f}^{(2)}_\pm = -i \bar{I}_\pm \),

\[
\bar{I}_\pm = \left( N + \frac{A}{4} \right) \left[ (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_2 \pm (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) + (\bar{\epsilon}_2 \cdot \bar{\epsilon}_1)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_1 \right]
\]

\[
+ \left( \lambda + N + \frac{A}{4} + B \right) \left[ (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_2 \pm (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_1 + (\bar{\epsilon}_2 \cdot \bar{\epsilon}_1)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_1 \right]
\]

\[
+ \left( \frac{A}{4} + B \right) \left[ (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_2 \pm (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_1 + (\bar{\epsilon}_2 \cdot \bar{\epsilon}_1)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_1 \right]
\]

\[
+ (B + 2C) \left[ (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_2 \pm (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_1 + (\bar{\epsilon}_2 \cdot \bar{\epsilon}_1)(\bar{k}_1 \cdot \bar{k}_2) \bar{\epsilon}_1 \right].
\]  

(A3)

Eq. (A3) has been previously derived by Jones & Kobett (1963), Childress & Hambrock (1964) and Korneev et al. (1998). In the above references the non-linear force was written in terms of sines and cosines, while we write it in the exponential form. For this reason we have the additional factor of 2i in the expression for \( \bar{f}^{(2)}_\pm \). By regrouping terms in eq. (A3) we arrive at

\[
\bar{f}^{(2)}_+ = -i \left\{ (\lambda + B) \bar{I}_s + \left( N + \frac{A}{4} \right) \bar{I}_s \right\} + \frac{A}{4} \bar{I}_d + B \bar{I}_d + 2C \bar{I}_c.
\]  

(A4)

where

\[
\bar{I}_s = (\bar{k}_1 \cdot \bar{k}_2)(\bar{\epsilon}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) + (\bar{k}_2 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{k}_2) \left[ (\bar{k}_1 + \bar{k}_2) \cdot \bar{\epsilon}_1 + (\bar{k}_1 \cdot \bar{\epsilon}_1) \left[ (\bar{k}_1 + \bar{k}_2) \cdot \bar{k}_2 \right] \right] \bar{\epsilon}_2.
\]  

(A5)

\[
\bar{I}_d = (\bar{k}_1 \cdot \bar{k}_2) \left[ (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2) \bar{k}_1 + (\bar{\epsilon}_2 \cdot \bar{\epsilon}_1) \bar{k}_2 \right] + (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2) \bar{k}_2 + (\bar{\epsilon}_2 \cdot \bar{\epsilon}_1) \bar{k}_1.
\]  

(A6)

\[
\bar{I}_c = (\bar{k}_1 \cdot \bar{\epsilon}_2)(\bar{k}_1 \cdot \bar{\epsilon}_2) \bar{\epsilon}_2 + (\bar{k}_1 \cdot \bar{k}_2) \left[ (\bar{\epsilon}_1 \cdot \bar{\epsilon}_2) \bar{\epsilon}_2 + (\bar{\epsilon}_2 \cdot \bar{\epsilon}_1) \bar{\epsilon}_2 \right].
\]  

(A7)
The difference-frequency force $\mathbf{f}^{(2)}$ is given by the same equations as above with the substitution $\mathbf{k}_2 \rightarrow -\mathbf{k}_2$. Each term in the expression for the non-linear force written in the form of eqs (A4)–(A9) depends linearly on $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_\pm = \mathbf{k}_1 \pm \mathbf{k}_2$. Hence, $\mathbf{f}^{(2)}_\pm$ scales as the product of amplitudes of these vectors and it can be written as

$$\mathbf{f}^{(2)}_\pm = -i(k_1 k_2 k_3) \mathbf{\hat{p}}_\pm.$$

The values $\mathbf{\hat{V}}_\pm$ introduced in such a way depend only on the elastic constants and on the angles between the vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_\pm$.

Depending on the polarization of the interacting waves we have 6 different cases, which are referred to as $PP$, $VV$, $HH$, $PV$, $PH$- and $HH$-interactions. The letters "P", "V" and "H" denote compressional waves, shear waves with vertical polarization, and shear waves with horizontal polarization. The values $\mathbf{\hat{V}}_\pm$ representing interacting modes of type 'M1' (mode with the wave vector $\mathbf{k}_1$) and 'M2' (mode with the wave vector $\mathbf{k}_2$) are denoted as $\mathbf{\hat{V}}_\pm^{M1,M2}$. We expand $\mathbf{\hat{V}}_\pm$ along the set of mutually orthogonal unit vectors

$$\mathbf{\hat{V}}_\pm = \mathbf{\hat{V}}_{\pm,1} \mathbf{k}_{\pm,1} + \mathbf{\hat{V}}_{\pm,2} \mathbf{k}_{\pm,2}, \quad \mathbf{\hat{V}}_{\pm,1} \cdot \mathbf{\hat{V}}_{\pm,2} = 0.$$

The polarization vector $\mathbf{\hat{e}}_j$ of a wave with the wave vector $\mathbf{k}_j$ is specified as $\mathbf{\hat{e}}_j = \mathbf{k}_j/|\mathbf{k}_j|$, $\mathbf{\hat{e}}_1 = \mathbf{\hat{e}}_b$, and $\mathbf{\hat{e}}_2 = \mathbf{\hat{e}}_b$ for $P$, $SV$ and $SH$ waves, respectively. After some algebra we find

$$\mathbf{\hat{V}}_\pm^{PP}(\theta, \phi, \psi) = (\lambda + 3N + A + 2B) \cos \theta \left[ \pm \cos \phi \cos \psi \mathbf{\hat{e}}_\pm - \sin (\phi \mp \psi) \mathbf{\hat{e}}_\pm \right] \pm (\lambda + 2B + 2C) \mathbf{\hat{e}}_\pm,$$

$$\mathbf{\hat{V}}_\pm^{PV}(\theta, \phi, \psi) = \left[ (\lambda + 3N + A + 2B) \cos \theta - \left( \frac{N + A}{2} + B \right) \right] \mathbf{\hat{e}}_\pm,$$

$$\mathbf{\hat{V}}_\pm^{PH}(\theta, \phi, \psi) = \pm \left( \lambda + 3N + A + 2B \right) \cos \phi \sin (\theta + \psi) \mathbf{\hat{e}}_\pm \pm \left[ (\lambda + 3N + A + 2B) \cos^2 \psi - \left( \frac{N + A}{2} + B \right) \right] \mathbf{\hat{e}}_\pm,$$

$$\mathbf{\hat{V}}_\pm^{VV}(\theta, \phi, \psi) = \left( \frac{N + A}{4} \right) \sin (\phi \pm \theta) \mathbf{\hat{e}}_\pm.$$

Here, $\theta$ is the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$, $\phi$ is the angle between $\mathbf{k}_1$ and $\mathbf{k}_\pm$ and $\psi$ is the angle between $\mathbf{k}_2$ and $\mathbf{k}_\pm$. Since the wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_\pm$ form a triangle, only two of the angles $\theta$, $\phi$ and $\psi$ are independent. The third angle can be expressed in terms of two other angles using the equation $\theta - \psi = \pm \phi$.

The values $\mathbf{\hat{V}}_\pm$ satisfy the reciprocity relations

$$\mathbf{\hat{V}}_\pm^{PP}(\theta, \phi, \psi) \cdot \mathbf{\hat{e}}_\pm = \mathbf{\hat{V}}_\pm^{PP}(\psi, \theta, \phi) \cdot \mathbf{\hat{e}}_\pm = - \mathbf{V}_\pm^{PP}(\theta, \phi, \psi),$$

$$\mathbf{\hat{V}}_\pm^{PV}(\theta, \phi, \psi) \cdot \mathbf{\hat{e}}_\pm = - \mathbf{\hat{V}}_\pm^{PV}(\psi, \theta, \phi) \cdot \mathbf{\hat{e}}_\pm = - \mathbf{V}_\pm^{PV}(\theta, \phi, \psi),$$

$$\mathbf{\hat{V}}_\pm^{PH}(\theta, \phi, \psi) \cdot \mathbf{\hat{e}}_\pm = - \mathbf{\hat{V}}_\pm^{PH}(\psi, \theta, \phi) \cdot \mathbf{\hat{e}}_\pm = - \mathbf{V}_\pm^{PH}(\theta, \phi, \psi).$$
The validity of eqs (A18)–(A22) can be checked directly. These equations reflect conservation of phonons in three-mode interactions and they can be derived using qualitative arguments as in Section 3. Eqs (A18)–(A22) show that the second order non-linear interaction of plane waves is completely described in terms of five functions: $V_A, V_P, V_P, V_P$ and $V_{±}.$

Expressions for the non-linear force presented in the literature usually contain only one angle $\theta$ (see e.g. Childress & Hambrick 1964; Taylor & Rollins 1964; Korneev et al. 1998). To exclude the angles $\phi$ and $\psi$ from eqs (A12) to (A17) one applies the sine theorem to the triangle formed by the wave vectors $\mathbf{k}_1, \mathbf{k}_2$ and $\mathbf{k}_3$ and considers projections of the wave vectors on $\mathbf{k}_1$ and $\mathbf{k}_2$. This gives

$$\sin \phi = \frac{k_2}{k_±} \sin \theta, \ \sin \psi = \frac{k_1}{k_±} \sin \theta,$$

(A23)

and

$$\cos \phi = \frac{1}{k_±} (k_1 \pm k_2 \cos \theta), \ \cos \psi = \frac{1}{k_±} (k_1 \cos \theta \pm k_2).$$

(A24)

Using eqs (A23) one writes eqs (A12)–(A17) in the conventional form, which allows to compare our formulas with formulas derived by previous authors. In our view the conventional representation of the non-linear force is not convenient because it results in bulky expressions and hides the reciprocal properties of three-mode interactions.

General expressions for the second order non-linear forces generated by plane waves have been presented by Childress & Hambrick (1964, see eqs (10)–(18) therein). Childress & Hambrick (1964) used the coordinate frame ($\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$), where the vector $\mathbf{n}_n$ is the same as our vector $\mathbf{e}_n$. Two other coordinate vectors $\mathbf{n}_{1,2}$ are rotated with respect to our vectors $\mathbf{e}_n$ and $\mathbf{e}_m$ (A11) by the angle $\phi$ so that $\mathbf{n}_1 = \mathbf{e}_n \cos \phi \mp \mathbf{e}_m \sin \phi$ and $\mathbf{n}_2 = \pm \mathbf{e}_n \sin \phi + \mathbf{e}_m \cos \phi$. Using these relations together with eqs (A23) and (A24) we found that our formulas differ from formulas of Childress & Hambrick (1964). The elastic parameters $\lambda$ and $B$ appear in eq. (A14) for the force generated by two $SH$ modes as the sum $\lambda + B$. The corresponding expression (14) by Childress & Hambrick (1964) contains $\lambda$ and $B$ only in the combination $\lambda + 3B$. Applying the first of eqs (A24)–(A18) given by Childress & Hambrick (1964) we see that their interaction force between $SV$ and $SH$ modes has an angular dependence proportional to $\pm \cos \phi \sin \theta$. According to our calculations this force scales as $\sin (\phi \pm \theta) = \sin \phi \cos \theta \pm \cos \phi \sin \theta$, see eq. (A17). We also found discrepancies with Childress & Hambrick (1964) in other cases. We believe that the equations presented by Childress & Hambrick (1964) are incorrect.

**APPENDIX B: RESONANT INTERACTION OF PLANE WAVES**

In Appendix A, we have considered general, not necessarily resonant, non-linear interaction of plane waves. The magnitude of the non-linear force is determined by eqs (A10) and (A12)–(A17). Substituting these equations in eq. (7) together with expressions for the Green's tensors of Childress & Hambrick (1964, see eqs (10)–(18) therein). Childress & Hambrick (1964) used the coordinate frame ($\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$), where the vector $\mathbf{n}_n$ is the same as our vector $\mathbf{e}_n$. Two other coordinate vectors $\mathbf{n}_{1,2}$ are rotated with respect to our vectors $\mathbf{e}_n$ and $\mathbf{e}_m$ (A11) by the angle $\phi$ so that $\mathbf{n}_1 = \mathbf{e}_n \cos \phi \mp \mathbf{e}_m \sin \phi$ and $\mathbf{n}_2 = \pm \mathbf{e}_n \sin \phi + \mathbf{e}_m \cos \phi$. Using these relations together with eqs (A23) and (A24) we found that our formulas differ from formulas of Childress & Hambrick (1964). The elastic parameters $\lambda$ and $B$ appear in eq. (A14) for the force generated by two $SH$ modes as the sum $\lambda + B$. The corresponding expression (14) by Childress & Hambrick (1964) contains $\lambda$ and $B$ only in the combination $\lambda + 3B$. Applying the first of eqs (A24)–(A18) given by Childress & Hambrick (1964) we see that their interaction force between $SV$ and $SH$ modes has an angular dependence proportional to $\pm \cos \phi \sin \theta$. According to our calculations this force scales as $\sin (\phi \pm \theta) = \sin \phi \cos \theta \pm \cos \phi \sin \theta$, see eq. (A17). We also found discrepancies with Childress & Hambrick (1964) in other cases. We believe that the equations presented by Childress & Hambrick (1964) are incorrect.

**Table B1.** Allowed resonant interaction processes in isotropic solids are labelled by letters. Letters ‘$P$, ‘$V$’ and ‘$H$’ label pairwise interaction of modes in the corresponding triplet. The letter ‘A’ denotes the resonant acoustic triplet (the parametric array). Cells are empty if the second order non-linear interaction (also non-resonant) is absent. The sign ‘$\times$’ indicates that non-resonant interaction is present, but the resonance conditions cannot be satisfied. The sign ‘0’ indicates that non-resonant interaction is present and the resonance conditions can be satisfied, but the non-linear force vanishes at the resonance.

<table>
<thead>
<tr>
<th>No</th>
<th>Interacting waves $\omega_1$</th>
<th>Interacting waves $\omega_2$</th>
<th>Scattered waves $\omega_1 + \omega_2$</th>
<th>Scattered waves $\omega_1 - \omega_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P$</td>
<td>$P$</td>
<td>$A$</td>
<td>$A$</td>
</tr>
<tr>
<td>1</td>
<td>$P$</td>
<td>$P$</td>
<td>$\times$</td>
<td>$A$</td>
</tr>
<tr>
<td>2</td>
<td>$SV$</td>
<td>$SV$</td>
<td>$V$</td>
<td>$\times$</td>
</tr>
<tr>
<td>3</td>
<td>$SH$</td>
<td>$SH$</td>
<td>$H$</td>
<td>$0$</td>
</tr>
<tr>
<td>4</td>
<td>$P$</td>
<td>$SV$</td>
<td>$P$</td>
<td>$\times$</td>
</tr>
<tr>
<td>5</td>
<td>$P$</td>
<td>$SH$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>6</td>
<td>$SV$</td>
<td>$P$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>7</td>
<td>$SH$</td>
<td>$P$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>8</td>
<td>$SV$</td>
<td>$SH$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>9</td>
<td>$SH$</td>
<td>$SV$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
There are three types of cases where the resonant interaction is absent. First, the value \( \hat{V}_{\pm}^{M_1 M_2} \cdot \hat{c}_3 \) always vanishes, even if the mode frequencies and wave vectors are not related by eq. (9). Then the modes \( M_1 \), \( M_2 \) and \( M_3 \) interact neither resonantly nor non-resonantly. Such cases are indicated by empty cells in Table B1. Second, the value \( \hat{V}_{\pm}^{M_1 M_2} \cdot \hat{c}_3 \) is not zero, but eq. (9) cannot be satisfied. Then the modes \( M_1 \), \( M_2 \) and \( M_3 \) can interact non-resonantly, while the resonant interaction is forbidden. These cases are indicated by the symbol ‘×’. We also distinguish the cases of interaction between two SH modes and one \( SV \) mode, which are indicated by the symbol ‘0’. The peculiarity of these cases is that there exist non-resonant interaction between the above modes and condition (9) is satisfied when the modes are collinear. The non-linear force however tends to zero as one approaches the resonance.

There are 10 cells in Table B1 labelled by letters ‘A’, ‘P’, ‘V’ and ‘H’. They correspond to the 10 interactions described by Korneev et al. (1998). Two of these interactions represent the parametric array by Westervelt (1963) formed by three collinear \( P \) modes (cases labelled by letter ‘A’ in Table B1). In addition to the above 10 interactions, Table V of the paper by Zarembo & Krasil’nikov (1971) presents another eight allowed resonant interactions between collinear waves, labelled by the letter ‘C’. As has been noted by Korneev et al. (1998), this result is apparently an error.

Childress & Hambrick (1964) considered only non-collinear interactions (eight cases labelled by the letters ‘P’, ‘V’ and ‘H’ in Table B1). Correspondingly, they counted 8 allowed processes. The cases \( P(\omega_1) + SV(\omega_2) \rightarrow P(\omega_1 + \omega_2) \) and \( P(\omega_2) + SV(\omega_1) \rightarrow P(\omega_1 + \omega_2) \) in Table B1 describe the same process, which can be represented in different forms depending on whether the frequency of the shear mode \( SV \) is labelled as \( \omega_2 \) or \( \omega_1 \). While Childress & Hambrick (1964), Zarembo & Krasil’nikov (1971) and Korneev et al. (1998) counted two of the above cases as separate interactions, Taylor & Rollins (1964) did not discriminate between them. Taylor & Rollins (1964), similar to Childress & Hambrick (1964), studied only non-collinear interactions. In addition Taylor & Rollins (1964) combined the interactions \( SV(\omega_1) + SV(\omega_2) \rightarrow P(\omega_1 + \omega_2) \) and \( SH(\omega_1) + SH(\omega_2) \rightarrow P(\omega_1 + \omega_2) \) in a single case, as well as the interactions \( P(\omega_1) + SV(\omega_2) \rightarrow SV(\omega_1 - \omega_2) \) and \( P(\omega_2) + SH(\omega_1) \rightarrow SH(\omega_1 - \omega_2) \). As a result, Taylor & Rollins (1964) counted only five possible interactions, although they have described the same non-collinear interactions as Childress & Hambrick (1964), Zarembo & Krasil’nikov (1971) and Korneev et al. (1998).

Table B2 shows properties of our pairwise resonant interactions. We do not discriminate between the cases \( P(\omega_1) + SV(\omega_2) \rightarrow P(\omega_1 + \omega_2) \) and \( P(\omega_2) + SV(\omega_1) \rightarrow P(\omega_1 + \omega_2) \). This gives nine possible interactions, which are split into four groups. Each group is labelled as ‘A’, ‘P’, ‘V’ or ‘H’ represents a single three-mode process (triplet) that is characterized by the same coupling constant \( V_{\text{cpl}} \). Calculating pairwise interaction forces between modes in a triplet one can get apparently different expressions. These differences are consequences of the convention to label the modes in the considered triplet as mode ‘1’ and mode ‘2’, and to denote the angle between these modes by \( \theta \). If labelling of the angles in a triangle formed by the modes forming a triplet is fixed, then the magnitude of the pairwise interaction force between any two of these modes with unit amplitudes is equal to the absolute value of the coupling constant, \( f_{ij} = |V_{\text{cpl}}| \). For example, two pairwise interactions in the triplet ‘V’ are characterized by the normalized forces \( \hat{V}_{\pm} \cdot \hat{k}_3 \) and \( \hat{V}_{\mp} \cdot \hat{k}_4 \). These forces satisfy reciprocity relation (A20), so that they are derived from the same coupling constant \( V_{\text{cpl}} \) by relabelling wave vectors \( (k_1, k_2, k_3) \rightarrow (k_1, k_3, k_2) \) and angles \( (\theta, \phi, \psi) \rightarrow (\psi, \theta, \phi) \). Interaction forces between two modes in the same triplet, which generate sum- and difference-frequency harmonics have opposite sign, see eq. (13). The force in reciprocity relations (A18)–(A21) does not always change its sign because relabelling of modes also changes coordinate frame (A11). This might result in reversion of the polarization vectors of \( SV \) and \( SH \) modes and hence in additional reversion of the sign of the interaction forces. Table B2 also gives the cosine of the angle between modes that are labelled as ‘1’ and ‘2’, which is expressed via ratios \( \beta = c_i/c_p \) and \( d = \omega_1/\omega_2 \). Depending on the mode labelling, the angle \( \theta \) is one of the angles \( \theta, \phi \) or \( \psi \) in the triangle shown in Fig. 1.

<table>
<thead>
<tr>
<th>Triplet</th>
<th>Coupling ( V_{\text{cpl}} ) and reciprocity</th>
<th>( \cos \theta )</th>
<th>Pairwise process</th>
<th>Non-linear force</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( V_A ) (19)</td>
<td>1</td>
<td>( P(\omega_1) + P(\omega_2) \rightarrow P(\omega_1 - \omega_2) )</td>
<td>( \hat{V}^{PP} \cdot \hat{k}_- )</td>
</tr>
<tr>
<td></td>
<td>( (A18) )</td>
<td>1</td>
<td>( P(\omega_1) + P(\omega_2) \rightarrow P(\omega_1 + \omega_2) )</td>
<td>( \hat{V}^{PP} \cdot \hat{k}_+ )</td>
</tr>
<tr>
<td>P</td>
<td>( V_P ) (14)</td>
<td>( \beta )</td>
<td>( P(\omega_1) + SV(\omega_2) \rightarrow SV(\omega_1 - \omega_2) )</td>
<td>( \hat{V}^{PP} \cdot \hat{k}_h )</td>
</tr>
<tr>
<td></td>
<td>( (A19) )</td>
<td>( \beta )</td>
<td>( P(\omega_1) + SV(\omega_2) \rightarrow P(\omega_1 - \omega_2) )</td>
<td>( \hat{V}^{PP} \cdot \hat{k}_+ )</td>
</tr>
<tr>
<td></td>
<td>( \beta )</td>
<td>( \beta )</td>
<td>( P(\omega_1) + SV(\omega_2) \rightarrow P(\omega_1 + \omega_2) )</td>
<td>( \hat{V}^{PP} \cdot \hat{k}_+ )</td>
</tr>
<tr>
<td>V</td>
<td>( V_V ) (15)</td>
<td>( \beta )</td>
<td>( SV(\omega_1) + SV(\omega_2) \rightarrow P(\omega_1 + \omega_2) )</td>
<td>( \hat{V}^{PV} \cdot \hat{k}_+ )</td>
</tr>
<tr>
<td></td>
<td>( (A20) )</td>
<td>( \beta )</td>
<td>( SV(\omega_1) + SV(\omega_2) \rightarrow SV(\omega_1 - \omega_2) )</td>
<td>( \hat{V}^{PV} \cdot \hat{k}_+ )</td>
</tr>
<tr>
<td>H</td>
<td>( V_H ) (16)</td>
<td>( \beta )</td>
<td>( SH(\omega_1) + SH(\omega_2) \rightarrow SH(\omega_1 - \omega_2) )</td>
<td>( \hat{V}^{PH} \cdot \hat{k}_h )</td>
</tr>
<tr>
<td></td>
<td>( (A21) )</td>
<td>( \beta )</td>
<td>( SH(\omega_1) + SH(\omega_2) \rightarrow P(\omega_1 + \omega_2) )</td>
<td>( \hat{V}^{PH} \cdot \hat{k}_+ )</td>
</tr>
</tbody>
</table>
Suppose that plane waves ‘1’ and ‘2’ resonantly interact inside an area with characteristic size much smaller than the wavelengths. Then the interaction force can be assumed to be constant, which significantly simplifies the calculations. The amplitude of the generated mode in the far field is equal to

\[ U_{3,b} = \frac{k_s k_b E_3}{s t \rho c^3} \left[ \hat{\bar{V}}_{b_3} \cdot \hat{e} \right] U_1 U_2 v. \]  

(B1)

Here, \( c_s \) is the velocity of the generated mode, \( U_i \) are physical mode amplitudes that are related to complex mode amplitudes \( a_i \) (see eq. 8) as \( u_i = 2|a_i| \), and \( \nu \) is the volume of the interaction area. We have compared eq. (B1) with similar eq. (25) from the paper by Taylor & Rollins (1964). To make such a comparison one should use eqs (A23), (A24) and expressions for \( \cos \theta \) in Table B2. Our results coincide with the results by Taylor & Rollins (1964) except for case IV in their Table I, which refers to the interaction \( P(a_01) + SV(a_02) \rightarrow P(a_01 - a_02) \). In this case we reconstruct the function \( \Delta \) introduced by Taylor & Rollins (1964) with the factor \( 1 - c^2 \) in the denominator, while Taylor & Rollins (1964) have the factor \( (1 - c^2)^2 \). [In notations used by Taylor & Rollins (1964) \( c = a_01/a_02 \) is the same as our \( \beta \).] Comparison with Korneev et al. (1998) also reveals only a minor discrepancy in a single case. This is case 6 from their Table 2 referring to the interaction \( P(a_01) + SV(a_02) \rightarrow SV(a_01 - a_02) \), where compared to Korneev et al. (1998) we have an addition multiplication factor of \((c_s/c_p)/(1 - \beta)\) in the resulting force.

**APPENDIX C: INTERACTION OF GAUSSIAN BEAMS**

The Westervelt (1963) equation describes propagation of acoustic waves in non-linear fluids, and it has the form

\[
\left( \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \right) p = \frac{\Gamma}{2 \rho c^2} \frac{\partial^2 p^2}{\partial t^2}.
\]

(C1)

Here, \( \rho \) is the fluid density, \( p \) is the fluid pressure, \( c \) is the speed of sound. In case of ideal gases the non-linear parameter \( \Gamma \) is equal to \( \gamma + 1 \), where \( \gamma \) is the adiabatic factor. If we take

\[
\Gamma = \frac{3}{2} + \frac{A + 3B + C}{\kappa + 2N}
\]

(C2)

then in the 1-D limit eq. (C1) is identical to eqs (1)-(3) with \( \nabla \times \vec{u} = 0 \) (Goldberg 1960). We use eqs (C1) and (C2) as model equations to describe interaction of compressional waves in solids. Making transition to variables \((\tau = t - z/c, \vec{r})\) and assuming that the wave is mainly propagating in one direction, along the \( z \)-axis, and hence has a strong dependence on the variable \( \tau = t - z/c \), we cast eq. (C1) in the Khokhlov–Zabolotskaya–Kuznetsov (KZK) form (Zabolotskaya & Khokhlov 1969; Kuznetsov 1971),

\[
2 \frac{\partial^2 p}{\partial \tau \partial z} = c \nabla^2 p + \frac{\Gamma}{2 \rho c^2} \frac{\partial^2 p^2}{\partial \tau^2}.
\]

(C3)

Darvennes & Hamilton (1990) solved eq. (C3) for the source configuration shown in Fig. 10. The sources are located in the plane \( z = 0 \) and the generate on oscillatory pressure whose amplitudes \( \tilde{p}_{aw} \) have the radial distribution \( \tilde{p}_{aw} = \tilde{p}_{aw0} \exp \left[ -b_a (\vec{r}_\perp - \vec{r}_{aw})^2 / R_a^2 \right] \). Here, \( \alpha = m, n \), the pressure amplitudes \( \tilde{p}_{aw} \) are normalized on the value \( \rho c^2, R_a \) is the characteristic source radius, \( \tilde{p}_{aw0} \) are the maximal source amplitudes, and \( b_a \) are dimensionless parameters characterizing the width of the pressure distribution. The above description describes unfocused sources. Suppose that sources are focused at points \( \Omega_a \). If the focal distances of the sources are \( d_{f_a} \), then the distance between \( \Omega_a \) and the point with coordinates \( \vec{r}_\perp - \vec{r}_{aw} \) is approximately equal to \( d_{f_a} - \vec{r}_{aw} \cdot (\vec{r}_\perp - \vec{r}_{aw}) / d_{f_a} \). Here, we assumed that \( |\vec{r}_\perp - \vec{r}_{aw}| \ll d_{f_a} \). In a focused source, the change in the distance between \( \Omega_a \) and \( \vec{r}_\perp - \vec{r}_{aw} \) is compensated by a proper distribution of the pressure phase across the source surface, so that signals from different parts of the source surface arrive at \( \Omega_a \) with the same phase. From this condition we conclude that focused Gaussian sources can be modelled by the equation

\[
\tilde{p}_{aw} = \tilde{p}_{aw0} \exp \left[ -b_a \left( \tilde{p}_{aw} - \tilde{p}_{aw0} \right)^2 - \frac{\kappa_a^2 \eta_{aw}^2}{4 b_a} \right].
\]

(C4)

Here, \( \tilde{p}_{aw} = \vec{\tilde{r}} / R_a \),

\[
\hat{b}_a = b_a + \frac{2i(k_a R_a)}{d_{f_a}}, \quad \tilde{p}_{aw} = \frac{\vec{r}_{aw} + \frac{i k_a}{2b_a} \vec{\eta}_{aw}}{R_a},
\]

(C5)

\( \kappa_a = (\omega_a/c) R_a \) is the normalized wave vector, \( \omega_a \) is the source frequency, and \( \vec{\eta}_{aw} \) is the projection of the vector \( \vec{\eta}_a \) on the \((x, y)\)-plane. Solution of eq. (C3) with source function (C5) in a linear medium is (Naugolnykh & Ostrovsky 1998)

\[
\tilde{p}_{aw}(\vec{r}) = \frac{\tilde{p}_{aw0}}{1 + 2i b_a \xi / \kappa_a} \exp \left[ -b_a \left( \tilde{p}_{aw} - \tilde{p}_{aw0} \right)^2 - \frac{\kappa_a^2 \eta_{aw}^2}{4 b_a} \right].
\]

(C6)

where \( \xi = z / R_a \). Substituting eq. (C6) into eq. (C3) to evaluate the non-linear term and solving the resulting equation one finds the amplitude of the sum-frequency harmonic,

\[
\tilde{p}_{aw} = \frac{\Gamma}{8} \frac{k_s k_b \kappa_a}{b_a b_b F} \tilde{v}^{E_0 - \nu} \left[ E_1(\xi_1) - E_1(\xi_2) \right] \tilde{p}_{aw0} \tilde{p}_{aw0}.
\]

(C7)
Here, $\kappa$ is the normalized wave vector of the combined harmonic, $\kappa = \kappa_m + \kappa_n$, $E_1$ is the elliptic integral,

$$\zeta_1 = \frac{\kappa \mathbf{\hat{w}}^2}{FG} \cdot \zeta_2 = \frac{\kappa \mathbf{\hat{w}}^2}{FG} \left( 1 + \frac{2iF\xi}{\kappa_m \kappa_n G} \right)^{-1}, \varphi_{0\alpha} = -\frac{\kappa_m^2 \eta_{\perp m}}{4\delta_m} - \frac{\kappa_n^2 \eta_{\perp n}}{4\delta_n},$$

$$F = \frac{\kappa_m^2}{b_m} + \frac{\kappa_n^2}{b_n} + 2i\xi, \quad G = \frac{\kappa}{b_m b_n} + 2i \left( \frac{1}{b_m} + \frac{1}{b_n} \right) \xi, \quad \mathbf{\bar{w}} = \kappa \mathbf{\hat{p}}_{\perp m} - \kappa_m \mathbf{\hat{p}}_{\perp 0m} - \kappa_n \mathbf{\hat{p}}_{\perp 0n},$$

$$\mathbf{\bar{w}} = \left( \frac{\kappa_m}{b_m} - \frac{\kappa_n}{b_n} \right) \mathbf{\hat{p}}_{\perp} + \left( \frac{\kappa_m}{b_m} + 2i\xi \right) \mathbf{\hat{p}}_{\perp 0} - \left( \frac{\kappa_n}{b_n} + 2i\xi \right) \mathbf{\hat{p}}_{\perp 0}.$$

(C8)

In the case where the difference-frequency harmonic is generated, one applies the same equations with the reversed sign of the lower primary frequency, $\omega_n \rightarrow -\omega_n$ (without loss of generality we have assumed that $\omega_m > \omega_n$). Correspondingly, the wave vector $\kappa_n$ changes the sign, $\kappa_n \rightarrow -\kappa_n$.

Eq. (C7) differs from equations given by Darvennes & Hamilton (1990) by a constant phase shift, which does not play a role. Another difference is that the arguments $\zeta_{1,2}$ of the exponential integral $E_1$ in eq. (C7) are four times larger than the corresponding arguments in eq. (9) of Darvennes & Hamilton (1990). We have repeated calculations of Darvennes & Hamilton (1990) using derived equations, and we reproduced figures presented in their paper. By reducing the arguments of the exponential integrals four times we got essentially different results. This suggests, that Darvennes & Hamilton (1990) used correct formulas in their analysis but they published them with a misprint. Note that the paper by Ji et al. (2008) reproduces the same misprint.

In most ultrasonic laboratory tests rigid transducers are used. Such transducers generate a wave with a constant pressure amplitude across their surface, $|p_{0\text{rigid}}(r_\perp)| = p_0$ for $r_\perp < r_s$ and $|p_{0\text{rigid}}(r_\perp)| = 0$ for $r_\perp > r_s$. Wen & Breazeale (1988) showed that the above step function can be approximated by a series of Gaussian functions, $|p_{0\text{rigid}}(r_\perp)| = a_j \exp \left(-b_j r_\perp^2/r_s^2\right)$, where $a_j$ and $b_j$ are some constant coefficients. The source focusing can be taking into account in the same way as in eq. (C4). As a result, the interaction of waves generated by rigid sources is represented as a sum of pairwise interaction of Gaussian harmonics. In a similar way, the Gaussian expansion can be applied to other source functions (Wen & Breazeale 1988).