Pressure-gradient singularity and production enhancement for hydraulically fractured wells

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SUMMARY

When a hydraulic fracture is modelled as a slit, the flow of the reservoir fluid towards the fracture develops a pressure-gradient singularity at the sharp fracture tips for both steady and transient flows. This pressure-gradient singularity also creates a flux-density singularity of the same type at the fracture tips. We study this pressure-gradient/flux singularity and its role in the production enhancement for hydraulically fractured wells in detail for the stabilized flow regime. Analytical solutions for slit, rectangular and elliptical shape fractures in the limit of infinite dimensionless fracture conductivity are used to analyse the flow physics. Our analyses reveal the exact mathematical nature of the pressure-gradient tip singularity and its regularization by the elliptical geometry. We show that this pressure-gradient tip singularity causes the flow from the region ahead of the fracture tip to converge and focus at the fracture tip. This flow pattern concentrates the production to the region near the fracture tip and increases the flux along the entire fracture surface. The singularity in the reservoir pressure-gradient is inherent for all fractures with sharp ends in both steady and transient flows regardless of the fracture conductivity. Our results establish pressure-gradient tip singularity as a universal and primary mechanism for enhanced productivity from hydraulically fractured wells. The large suction forces at the fracture tips induce the fluid to flow from an ultra-low permeability reservoir into a hydraulic fracture.

Key words: Hydrogeophysics; Hydrology; Fracture and flow.

1 INTRODUCTION

Hydraulic fracturing has been used as an oil well stimulation tool since the 1940s. Advances in multistage fracturing combined with horizontal drilling have made this technique the driving force for the recent spectacular success in shale gas development and production. The resulting dramatic increase in North America’s natural gas reserves and production has been viewed as a game changer for the entire North America energy industry.

Despite this huge success in the field and the extensive research efforts since the Dept. of Energy sponsored hydraulic fracturing research in the 1970s, understanding of how and why hydraulic fracturing increases production, particularly for unconventional reservoirs such as shale gas where the permeability can dip down to the nano-darcy range, is still incomplete. Prats (1961) obtained an exact solution for the pressure in the reservoir and the pressure inside an elliptic fracture under stabilized flow conditions with prescribed well production rate. His results showed that the large contact area with the reservoir as well as the high conductivity path the fracture has created are the primary reasons for the increased productivity of vertically fractured wells. There are also numerous transient pressure analyses for fractured wells with prescribed production rate, such as Prats et al. (1962), Raghavan et al. (1972), Gringarten & Ramsey (1974), Gringarten et al. (1975), Cinco-Ley et al. (1978), Agarwal et al. (1979), Cinco-Ley & Samaniego-V (1981), Wilkinson (1989), Riley (1991) and Amini et al. (2007). All of these studies, however, are geared towards well test analysis with emphasis on the type-curves for the pressure draw-down time history (Bourdet 2002). Few, if any, have tried to explore other aspects of the flow characteristics and analyse the physical mechanisms for the enhancement in production from hydraulic fractured wells. While increased contact area with the reservoir and the high-conductivity path are indeed important elements of the reason hydraulic fracturing increases production, other mechanisms are involved as well. Pressure-gradient singularity and its induced flux-density singularity at the fracture tips as shown by this study, play a significant role in promoting productivity.

Production from a hydraulically fractured well involves two flow regions, the flow from the reservoir to the fracture, and the flow along the fracture into the wellbore. Consider, for example, the stabilized flow regime, which corresponds to the steady-state solution of the governing equations for the motion of the reservoir fluid. The reservoir flow towards the fracture is a potential problem for the fluid pressure, governed by the Laplace equation which results...
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from the Darcy’s law. It is common to model the fracture as a zero-thickness plate, or a slit. In such a case, the fracture possesses sharp ends. It is well known in aerodynamics that potential flows around a thin plate develop a velocity singularity at the ends of the plate. The velocity for such a potential flow is the gradient of the velocity potential, which satisfies the Laplace equation. Analogously, when a fracture is modelled as a thin-plate/slit, the pressure-gradient of the reservoir flow becomes singular at the tips of the fracture. The Darcy velocity (flux density, or discharge density), which is proportional to the pressure-gradient, also becomes singular at the tips of the fracture. There is, however, a significant difference: in aerodynamic flow around a thin plate, the fluid is not allowed to enter the plate; as such only the tangential velocity exhibits a tip singularity while the normal velocity is zero at the plate; for production from a hydraulically fractured well, the reservoir fluid enters the fracture and both the normal and tangential components of the Darcy velocity develop tip singularity. Corner singularities in Stokes flows are well known in fluid mechanics (Moffat 1964; Pan & Acrivos 1967; Moffat & Duffy 1980; Nitsche & Parthasarathi 2012), and it is not surprising that Darcy flows also develop corner singularities since the relevant potentials obey the same differential equations. Corner singularities in Darcy flows have been studied by Lafa et al. (1980) in the pursuit of convergent numerical schemes for hydrological applications. Flux tip singularity has been well recognized in problems of hydraulic fracture propagation; oil production from a fractured well (pressure-driven flows); and flow through porous media with discrete fractures (streaming flows) by Kikani (1995), Chin (2002), Mathias & van Reeuwijk (2009), Mathias et al. (2010), Biryukov & Kuchuk (2012), Exadaktylos (2012), Pouya (2012), among others. For example, Mathias & van Reeuwijk (2009) developed a 3-D model for fluid leak-off from a propagating hydraulic fracture and showed analytically the existence of an inverse-square-root singularity in the leak-off flux at the fracture tip. They pointed out that tip singularity is a common phenomenon associated with mixed-type boundary value problems, and its existence requires extensive grid refinement in the tip region for numerical simulations. However, past attentions to tip singularity were exclusively directed towards improving accuracies of numerical schemes, rarely touching the underlying physics associated with the tip singularity. No one in the literature has studied the pressure-gradient tip singularity as a primary production enhancement mechanism for hydraulically fractured wells.

In this paper, we analyse the pressure-gradient of the reservoir flow near the tips of a hydraulic fracture for both transient and stabilized flows, and we study the fluid production mechanism in detail using analytical solutions for the stabilized flow regime in the limit of infinite dimensionless fracture conductivity for slit, rectangular and elliptical fractures. The infinite fracture conductivity limit provides the upper bound for the production from a fractured well, and it is accessible to analytical solutions. However, the singularity in the pressure-gradient of the reservoir flow is inherent for fractures with sharp ends regardless of the fracture conductivity, the flow being transient or steady. Thus, the phenomena brought by the reservoir pressure-gradient singularity are universal. The importance of the availability of analytical solutions is twofold: it offers the opportunity to explore the underlying physics in a precise manner, and it describes the exact mathematical nature of the pressure-gradient/flux tip singularity, against which accuracies of various numerical schemes can be assessed. The latter is particularly important for any flux-based numerical scheme for fractured-well simulation: any scheme incapable of accurately capturing the flux tip singularity can lead to large errors in the calculated production.

We begin the paper with a local analysis of the pressure-gradient of the reservoir flow near a sharp corner. We will then discuss the analytical solutions for the reservoir pressure for slit, rectangular and elliptical fractures in the limit of infinite dimensionless fracture conductivity, and utilize these solutions to show how pressure-gradient/flux tip singularity promotes production. We will also discuss how this singularity is regularized by the geometry of an ellipse. We end the paper with discussions on the broad implications of pressure-gradient/flux tip singularity.

2 PRESSURE-GRADIENT SINGULARITY IN RESERVOIR FLOWS AT A SHARP CORNER

Consider the reservoir flow in a region bounded by two intersecting straight lines with an included angle $2\beta$, $0 < \beta \leq \pi$ (Fig. 1). The pressure along the boundary is a constant $P_0$. For a homogeneous isotropic reservoir with permeability $K$, fluid viscosity $\mu$, porosity $\phi$ and total compressibility $\chi$, the pressure of the reservoir flow satisfies a diffusion equation (Muskat 1937):

$$\chi \frac{\partial P}{\partial t} = \nabla^2 P$$

where $\chi = \phi \mu / K$. The solution of this diffusion equation near the sharp corner is separable in the local polar coordinates with the origin located at the corner,

$$P = P_0 + e^{-r^{2\beta}t} \sum_{n=1}^{\infty} C_n J_n(\omega r) \cos \lambda_n \theta, \quad \omega > 0,$$

$$P = P_0 + \sum_{n=1}^{\infty} C_n r^{2\beta} \cos \lambda_n \theta, \quad \omega = 0,$$

where $\omega$ is an arbitrary constant, $\omega > 0$ for transient flows and $\omega = 0$ for stabilized flows (steady-state); and $C_n$ are constants determined by the far-field. The eigenvalues $\lambda_n$ are given by

$$\lambda_n = \frac{2n - 1}{2\beta} \pi, \quad n = 1, 2, \ldots,$$

![Figure 1. Local polar coordinate system at a corner for Darcy flow.](https://academic.oup.com/gji/article-abstract/195/2/923/643576)
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The smallest eigenvalue possible is $\frac{1}{2}$. The pressure-gradient near the corner is then

$$\nabla P = e^{-\omega r^2} \left[ \sum_{n=1}^{\infty} \omega C_n \frac{dJ_n(\bar{r})}{d\bar{r}} \cos \lambda_n \theta e_r \right. $$

$$\left. - \sum_{n=1}^{\infty} \lambda_n C_n \frac{J_n(\bar{r})}{r} \sin \lambda_n \theta e_r \right], \quad \omega > 0, \quad (5)$$

$$\nabla P = \sum_{n=1}^{\infty} \lambda_n C_n r^{1+\omega} \cos \lambda_n \theta e_r $$

$$- \sum_{n=1}^{\infty} \lambda_n C_n r^{1+\omega} \sin \lambda_n \theta e_r, \quad \omega = 0. \quad (6)$$

In (5), $\bar{r} = \omega r$. For $\omega > 0$ and $r \to 0$,

$$J_n(\bar{r}) \approx \left( \frac{\omega}{2} \right)^{\lambda_n} \frac{1}{\lambda_n!} r^{\lambda_n}. \quad (7)$$

Since the smallest eigenvalue for any given included angle $2\beta$ is $\lambda_1 = \frac{\pi}{2\beta}$, as $r \to 0$, we always have

$$\nabla P = O(r^{-1-\omega}) \quad (8)$$

for any $\omega > 0$. That is, the leading term in the pressure-gradient as $r \to 0$ is always of the order $r^{-1-\omega}$. Thus, when $\beta > \pi/2$, or when the included angle $2\beta > \pi$, which may be called an ‘external flow’, $\lambda_1 - 1 < 0$, and the pressure-gradient becomes singular at the corner.

Since for a Darcy flow, the flux (Darcy velocity) is proportional to the pressure-gradient

$$v = -\frac{k_n}{\mu} \nabla P, \quad (9)$$

the flux exhibits the same corner singularity as the pressure-gradient whenever the included angle $2\beta > \pi$. The strongest singularity occurs when $\beta = \pi$, that is, when the corner becomes the end of a zero-thickness plate (slit). This gives an inverse square-root singularity for the pressure-gradient and the flux at the tip,

$$v = -\frac{k_n}{\mu} \nabla P = O(r^{-1/2}). \quad (10)$$

If we replace the zero-thickness plate by a rectangle, the included angle becomes $2\beta = 3\pi/2$ (Fig. 2). The singularity of the pressure-gradient/flux at each corner is then

$$v = -\frac{k_n}{\mu} \nabla P = O \left( r^{-1/3} \right), \quad (11)$$

which is weaker than that at the tip of a zero-thickness plate. Both singularities, however, remain integrable, which give a finite total flow-rate for the fluid entering the fracture.

We conclude this section by emphasizing that the pressure-gradient/flux singularity at sharp corners in the reservoir flow arises from the fracture geometry, independent of the nature of the flow inside the fracture (infinite or finite fracture conductivity). This singularity is also inherent to all reservoir flows, transient or otherwise, as long as the fracture is modelled with sharp ends, such as a slit or a rectangle.

3 THE SLIT MODEL

When the fracture is modelled as a slit, the well is often treated as a point (Fig. 3). This geometry is commonly used when the boundary integral method is used to simulate transient flows for fractured wells in well test analysis (Cinco-Ley & Samaniego 1981). For the slit model and the stabilized flow regime, Chin (2002) provided an analytical solution to the reservoir pressure in terms of the pressure along the fracture, $P_i(x)$. Using the fracture half-length $L$ as the length scale, the well pressure $P_w$ as the pressure scale, and the dimensionless quantities

$$\bar{x} = \frac{x}{L}; \bar{y} = \frac{y}{L}; \bar{P} = \frac{P(x, y)}{P_w}; \bar{P}_i = \frac{P_i(x)}{P_w}. \quad (12)$$

the dimensionless pressure in the reservoir is given by

$$\bar{P}(\bar{x}, \bar{y}) = \frac{1}{\pi^2} \int_{-1}^{1} \frac{f(\xi) \ln \sqrt{(\bar{x} - \xi)^2 + \bar{y}^2} d\xi}{\bar{x} - \xi} + H_0, \quad (13)$$

where $H_0$ is a constant and $f(\bar{x})$ is essentially the dimensionless flux density along the fracture upper surface. $H_0$ and $f(\bar{x})$ are given by

$$H_0 = \frac{\bar{P}_i - (I_1 - I_2) \ln \bar{R}}{1 + \ln \bar{R}/\ln 2}, \quad (14)$$

$$f(\bar{x}) = \frac{1}{\pi^2} \int_{-1}^{1} \frac{\bar{P}(\xi)}{\sqrt{\bar{x}^2 - \xi^2}} \frac{d\xi}{\sqrt{1 - \bar{x}^2}} + \frac{H_0}{\pi \ln 2 \sqrt{1 - \bar{x}^2}}. \quad (15)$$

Figure 2. Rectangle and slit models for a fracture. Different singularities appear at the corners/tips due to different included angles.

Figure 3. The slit model for a fracture. The reservoir pressure at a large radius $R$ is constant $P_R$, and the well pressure at the origin is $P_w$. 

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where the constants involved are given by

\[ \tilde{P}_R = \frac{P_R}{P_w} \quad \tilde{R} = \frac{R}{L}. \tag{16} \]

\[ I_1 = \int_{-1}^{1} \frac{d\tilde{x}}{\pi \sqrt{1 - \tilde{x}^2}} \text{PV} \int_{-1}^{1} \frac{\tilde{P}_I(\tilde{\xi})}{\tilde{x} - \tilde{\xi}} \sqrt{1 - \xi^2} d\xi, \tag{17} \]

\[ I_2 = \int_{-1}^{1} \frac{d\tilde{x}}{\pi^2 \ln 2 \sqrt{1 - \tilde{x}^2}} \int_{-1}^{1} \frac{\tilde{P}_I(\tilde{\xi})}{\sqrt{1 - \xi^2}} d\xi_1. \]

In the above, ‘PV’ stands for the ‘Principal Value’ of the integral, and the reservoir pressure at a large radius \( R \) is a constant \( P_R \). While the solution (13) is incomplete since the fracture pressure \( \tilde{P}_1(\tilde{x}) \) is still unknown, it provides a glance into the flux distribution along the fracture surface, which is given by (15) explicitly in dimensionless form. For any smooth fracture pressure distribution \( \tilde{P}_1(\tilde{x}) \), the integrals in the square bracket in (15) as well as the constant \( H_p \) in (14) are all finite constants. Thus, (15) shows that the dimensionless flux along the fracture exhibits the inverse-square-root singularity at the two tips of the slit \( \tilde{x} = \pm 1 \). This singular behaviour confirms the local analysis for a slit given in Section 1 as well as the result of Mathias & van Reeuwijk (2009) for the leak-off flux during hydraulic fracture propagation.

Significant insights can be gained from a detailed study of the special case of \( \tilde{P}_1(\tilde{x}) = \text{constant} = 1 \), which corresponds to the so-called infinite dimensionless fracture conductivity. For this ideal case, the pressure loss inside the fracture is negligible, and the production calculated represents the upper bound for the production from the hydraulically fractured well. This limit, however, allows us to obtain an explicit analytical solution for the reservoir pressure, the flux density distribution along the fracture surface and the total production. For this limit, we have

\[ f(\tilde{x}) = \frac{P_R - P_w}{\pi P_w} \frac{1}{\ln(2 \tilde{R}) \sqrt{1 - \tilde{x}^2}}. \tag{18} \]

Thus, the ‘dimensional’ flux of the reservoir fluid entering the fracture through the upper surface is

\[ q(x) = \frac{\kappa_w}{\mu L} \frac{P_R - P_w}{\ln(2 \tilde{R})} \frac{1}{\sqrt{1 - x^2}}, \tag{19} \]

which clearly possesses inverse-square-root singularities at both ends of the slit \( \tilde{x} = \pm 1 \) (fracture tips). The production of the fracture is explicitly given by the relation

\[ Q_p = 2 \int_{-1}^{1} q(x) \, dx = \frac{2\pi \kappa_w}{\mu} \frac{P_R - P_w}{\ln(2 \tilde{R})}. \tag{20} \]

The simple explicit solutions for the flux and the production (19) and (20) allow us to elucidate the mechanism for enhanced production from a fractured well well analytically. It is often said that a hydraulic fracture enhances production because it creates more contact area with the reservoir. For the purpose of comparison, one can easily create a hypothetical situation with the ‘same contact area’ as the hydraulic fracture but without the end-effect (tip singularity), such as the one shown in Fig. 4. Without the end-effect, the reservoir flow towards the fracture is linear. Thus, this situation corresponds to the so-called ‘bilinear flow regime’ in the literature for flow in hydraulically fractured wells (Cinco-Ley & Samaniego 1981).

Figure 4. A fracture without the end-effect: the bilinear flow model.

Setting \( P = P_R \) on \( y = H(x) \), the solution for the pressure in the reservoir for the bilinear flow is simply

\[ P(x, y) = \frac{P_R - P_w}{H(x)} y + P_w \tag{21} \]

with \( H(x) \) given by

\[ H(x) = \sqrt{R^2 - x^2}. \tag{22} \]

The flux (Darcy velocity) of the fluid entering the fracture is

\[ q_b(x) = \frac{\kappa_w}{\mu} \frac{P_R - P_w}{\sqrt{R^2 - x^2}}, \tag{23} \]

which is finite at the fracture tips: \( q_{b,tip} = \frac{\kappa_w}{\mu} \frac{P_R - P_w}{R} \). Counting productions from both sides of the fracture, the total production is then

\[ Q_b = 2 \int_{-1}^{1} q_b(x) \, dx = 4 \frac{\kappa_w}{\mu} (P_R - P_w) \sin^{-1} \left( \frac{L}{R} \right). \tag{24} \]

Thus, the ratio between the production from a fracture with end-effect (tip singularity) and the bilinear flow is

\[ \frac{Q_p}{Q_b} = \frac{\pi}{2 \ln(2 \tilde{R})} \sin^{-1}(1/R), \tag{25} \]

\[ \tilde{R} = \frac{R}{L} \geq 1. \tag{26} \]

For large values of \( \tilde{R} \) (large reservoir, or short fracture), this ratio increases like

\[ \frac{Q_p}{Q_b} \rightarrow \frac{\pi}{2 \ln(2 \tilde{R})} \tilde{R}, \tag{27} \]

Thus, the outperformance of the slit relative to the bilinear flow increases with \( \tilde{R} \), but not quite linearly (Fig. 5). Physically, the fracture ends are exposed to a vast volume of reservoir fluid and the ends act like point sinks. This increases the production through the tip region and the over-all production significantly.

The significance of flux tip singularity is further demonstrated by computing the flux ratios between the flow with the end-effect of singularity and the bilinear flow. The flux ratio is given by

\[ \frac{q_b(x)}{q_b(x)} = \frac{1}{\ln(2 \tilde{R})} \sqrt{\frac{R^2 - x^2}{1 - x^2}}, \quad -1 \leq x \leq 1. \tag{28} \]
modelling as a rectangle, which is common when a finite difference scheme is used in numerical simulation, four sharp corners are involved. These four corners all exhibit the ‘$-1/3$’ power singularity. To gain further insight, we investigate in this section the flux distribution and the production of a rectangular fracture for the special case of infinite dimensionless fracture conductivity.

For infinite fracture conductivity, the pressure inside the rectangle is a constant $P_w$. Thus, the pressure in the reservoir is the solution to the potential problem with specified potentials at the rectangle peripheral, $P = P_w$, and $P = P_R$ on the large circle $r = R$. This general potential problem was first solved by Bickley (1934) using the conformal mapping technique (Fig. 7). We will use his solution to perform the flow analyses below. Flax & Simmons (1966) also studied a similar problem. The details of all relevant calculations are available from the authors by request.

Assume the rectangle has a length of $2a$, and a width $2b$. The region of the reservoir outside of the rectangle but bounded by the circular boundary $r = R$ on the physical $z$-plane maps to the annulus region $0 \leq \beta \leq \beta_R$ on the $\sqrt{k}/\beta$-plane (Figs 7a and b). The solution is controlled by $R/\sqrt{\beta}$, the width-to-length ratio of the rectangle $2b/a$:

$$b/a \equiv \ell = \frac{E(k) - (1 - k^2)K(k)}{E(\sqrt{1 - k^2}) - k^2K(\sqrt{1 - k^2})}. \quad (29)$$

In the above, $E$, $K$ are the elliptical integrals of the first and second kind, respectively; $k$ is the modulus of the Jacobian elliptic function. Other important parameters are defined by

$$l = E - k^2K = E(k) - k^2K(k),$$
$$l' = E' - k^2K' = E'(k) - k^2K'(k),$$
$$k = \sin \alpha,$$
$$k' = \cos \alpha = \sqrt{1 - k^2}. \quad (30)$$

The solution for the pressure on the $t$-plane is

$$P(\beta) = P_w + \frac{P_R - P_w}{\beta_R} \beta, \quad 0 \leq \beta \leq \beta_R, \quad (31)$$

where the isobars are circles $\beta$ = constant, and $\beta = 0$ corresponds to the surface of the rectangular fracture. The flux on the rectangle boundary are given by:

On $AB$: $-\alpha \leq \theta \leq \alpha$,

$$q|_{\beta=0} = \frac{\kappa_w}{\mu} \frac{P_R - P_w}{\beta_R} \frac{2l}{b(s\sin^2 \alpha - \sin^2 \theta)^{1/2}}. \quad (32)$$

On $CB$: $\alpha \leq \theta \leq \pi - \alpha$,

$$q|_{\beta=0} = \frac{\kappa_w}{\mu} \frac{P_R - P_w}{\beta_R} \frac{2l}{b(s\sin^2 \theta - \sin^2 \alpha)^{1/2}}. \quad (33)$$

Using Taylor series expansion and the properties of the Jacobian elliptic functions, it can be shown that as the corner B is approached along the side $AB$, the flux behaves as

$$q|_{\beta=0} \rightarrow \frac{\kappa_w}{\mu} \frac{P_R - P_w}{\beta_R} \frac{2l}{bk\sqrt{1 - k} \left[\frac{1}{\beta_R^2} \left|y - b\right|\right]^{1/3}}, \quad (34)$$

which is the ‘$-1/3$’ power of the distance to point B. Likewise, the same type of singularity is obtained when point B is approached along the side $CB$. This result is consistent with the local analysis of Section 1. Thus, modelling the fracture as a finite thickness rectangle weakens but does not eliminate the pressure-gradient/flux singularity at the sharp corners of the fracture.

The total production for the rectangle is

$$Q_{\text{rect}} = 4(Q_{PB} + Q_{BQ}) = \frac{2\pi \kappa_w}{\mu} \frac{P_R - P_w}{\beta_R} a. \quad (35)$$

### 4 THE RECTANGLE MODEL

It is naturally expected that the tip singularity encountered by the slit model maybe alleviated by taking into account the finite thickness of the fracture, since this reduces the sharpness of the fracture tip. Indeed ‘chopping off’ the sharp corner can reduce the severity of the singularity as Kikani (1995) has experimented, which is also evident from the local analysis given in Section 1. When a fracture is...
The dependence of the total production on the rectangle geometry is through the parameter \( \beta_R \), which is the equal-potential line (isobar) at the far-field on the \( t \)-plane where the pressure \( P_k \) is specified. The corresponding far-field equal-potential line on the physical \( z \)-plane depends on the rectangle aspect ratio, \( b/a \). It can be shown that for large values of \( \beta \) (e.g. \( \beta > 2 \)), the isobars on the \( z \)-plane can be approximated as circles with radius of \( \frac{2R}{\pi} \). Thus,

\[
\beta_R = \ln \left( \frac{2R}{b} \right) = \ln \left( \frac{2l}{a} \right) = \ln \left( \frac{2R}{a} \right) + \ln l',
\]

and the total production from a rectangle is

\[
Q_{\text{Rect}} = \frac{2\pi \kappa_m}{\mu} \frac{P_R - P_w}{\ln \frac{2R}{a} + \ln l'}.
\]

Figure 7. Potential problem outside a rectangle, with given potentials at the rectangle peripheral and a large circle \( r = R \) (after Bickley 1934): (a) the physical \( z \)-plane; (b) \( t \)-plane.

It is important to note that this formula is valid only when \( \beta_R = \ln \frac{2R}{a} + \ln l' \) is larger than, say, 2, because (36) is derived under the assumption of large \( \beta_R \), at least \( \beta_R > 2 \), to maintain a 2 per cent or less error in approximating the isobars as circles at large values of \( \beta_R \).

The term \( \ln l' \) in the denominator of (37) represents the effect of the rectangle width \( b \) on the total production. Values of \( l, l' \) are tabulated for various angles \( \alpha \) by Bickley (1934). Since \( l' \leq 1 \), we have \( \ln l' < 0 \). Thus, increasing the width of the rectangle \( b \) increases the production rate \( Q_{\text{Rect}} \). The minimum value for \( l' = 0.423607 \), \( \ln l' = -0.8589 \), which occurs for \( \alpha = 45^\circ \), or for a square, \( a = b \); while the maximum of \( l' \) is 1.0, \( \ln l' = 0 \), which occurs for a zero-thickness plate \( \alpha = 0^\circ \). Thus, for rectangles with fixed length \( a \), the maximum production is achieved for a square while the minimum production is reached for a zero-thickness plate. The ratio between the maximum production from a square \( Q_s \) and the minimum production from a plate \( Q_p \) is

\[
\frac{Q_s}{Q_p} = \frac{\ln \left( \frac{2R}{a} \right) + 0.8589}{\ln \frac{2R}{a} - 0.8589}.
\]

Again, this simplified expression requires a large \( 2R/a \). If we choose \( \beta_R = 2 \), from the denominator, we will need \( R/a \geq 8.72 \). Since \( Q_s/Q_p > 1 \), a square is always more productive than a plate when the value of \( a \) is fixed. However, \( Q_s/Q_p < 2 \) (Fig. 8). For example, at \( R/a = 8.72 (\beta_R = 2) \), \( Q_s/Q_p = 1.42 \). This value indicates that a square is only 42 per cent more productive than a plate, even though the square has twice the production area (contact area with the reservoir) as the plate. This result is due to the fact that the flux singularities at the corners of the square are weaker than the flux singularities at the ends of the plate, \( '-1/3' \) power versus \( '-1/2' \) power. In other words, a plate is a more efficient producer because it provides a stronger flux singularity at the tips. For very large reservoirs (large \( R/a \), \( Q_s/Q_p \to 1 \), that is, a plate offers the same production as a square, even though it has only half of the production area as the square.

The above exercise clearly shows that in addition to the contact area with the reservoir, the exact nature of the flux singularity has a significant influence on the productivity. Simply having a large contact area is not enough, and a strong flux tip singularity enhances production dramatically, especially for large reservoirs.

Figure 8. Ratio of the production between a square and a plate \( Q_s/Q_p \), for the same value of \( R/a \).

5 THE ELLIPSE MODEL: REGULARIZATION OF TIP SINGULARITY

Rounding off the sharp geometric corners of the fracture ends eliminates the pressure-gradient/flux singularity mathematically. However, the pressure-gradient/flux still remains very large and near singular, so long as the fracture is thin and long. This large...
pressure-gradient/flux is quantified by a detailed analysis for an elliptical fracture (Fig. 9).

In a landmark paper, Prats (1961) obtained an exact steady-state solution for the pressures in the reservoir and the fracture using Jacobian elliptic function and Fourier series. Prats (1961) modelled the fracture as a degenerate ellipse, employed elliptic coordinates and assumed the production rate at the well is prescribed. In a similar fashion, it is straightforward to show that for prescribed well pressure and infinite fracture conductivity, the pressure in the reservoir is given by

\[ P(\xi, \eta) = P_R - \frac{\xi_e - \xi}{\xi_e - \bar{\xi}_i} (P_R - P_w) \]  

(39)
in the elliptic coordinate \((\xi, \eta)\). In (39), \(\xi_e = \bar{\xi}_i\) is the limiting ellipse used to model the fracture geometry, and \(\xi = \xi_R\) is the outer ellipse used to approximate the circle \(x^2 + y^2 = R^2\), where the pressure is \(P_R\). For small \(\xi_i\),

\[ \xi_i = \frac{w_i}{2\xi} \]  

(40)

where \(w_i\) is the width of the fracture at the wellbore. In practice, \(\xi_i = O(10^{-4})\) or even smaller. For large \(R\),

\[ \xi_R = \ln \left( \frac{2R}{\xi} \right) \]  

(41)
The pressure-gradient in the reservoir is

\[ \nabla P = -\frac{1}{L\sqrt{\sinh^2 \xi + \sin^2 \eta}} \left( k_i \frac{\partial P}{\partial \xi} + k_i \frac{\partial P}{\partial \eta} \right) \]  

(42)

with the Darcy velocity

\[ v(\xi, \eta) = -\frac{\kappa_m}{\mu} \left( \frac{1}{L\sqrt{\sinh^2 \xi + \sin^2 \eta}} \frac{\partial P}{\partial \xi} \right) \]  

(43)

We define a dimensionless Darcy velocity in the reservoir as

\[ v(\xi, \eta) = \frac{\mu L}{\kappa_m} \frac{\xi_R - \bar{\xi}_i}{P_R - P_w} v(\xi, \eta) = -i \frac{1}{\sqrt{\sinh^2 \xi + \sin^2 \eta}} \]  

(44)

Eqs (42)–(44) show how the fracture tip singularity for the pressure-gradient and the flux is regularized by the rounded ellipse geometry: the pressure-gradient and the flux are large at the ends of the fracture \((\xi, \eta) = (\xi_i, 0)\), of the order \(1/\xi_i\), but remain finite.

Flow pattern in the reservoir and fracture can provide insight into the fluid production process. Fig. 10 shows the intensity of the dimensionless Darcy velocity (flux) and the streamlines of the reservoir flow towards an elliptic fracture with \(\xi_i = 0.0001\). The thin elliptic fracture is represented by the green straight-line lying between \(-1 \leq x \leq 1\). The flux is nearly singular at the fracture tips. Strong convergent flow patterns appear near the fracture tips.

At the fracture tips \(x = \pm 1\), and its magnitude (dimensionless) is \(1/\xi_i = 10^4\). The near singular behaviour of the flow around the fracture tip enhances production (Fig. 10): the fracture tips act like point sinks in the reservoir, inducing a convergent flow pattern and pulling significant fluids into the fracture through the surface area surrounding the fracture tip by virtue of the huge suction force created by the nearly singular pressure-gradient. This flow-focusing phenomenon is absent in the bilinear flow model.

The flux of the fluids entering the fracture at the fracture surface \(\xi = \xi_i\) is given by

\[ q = -i \cdot v(\xi, \eta)|_{\xi = \xi_i} = \frac{\kappa_m}{\mu} \frac{1}{L\sqrt{\sinh^2 \xi_i + \sin^2 \eta}} \frac{P_R - P_w}{\xi_R - \xi_i} \]  

(45)

When dealing with a hydraulic fracture, it is common to take the limit \(\xi_i \to 0\) while maintaining a constant value for the dimensionless fracture conductivity \(C_{fD} = \frac{\kappa_m w_i}{\kappa_m L} = \frac{2\xi_i}{\xi_i} \). If we set \(\xi_i = 0\) in (45), then the flux along the fracture surface becomes

\[ q|_{\xi_i=0} = \frac{\kappa_m}{\mu L\sqrt{\sin^2 \eta}} \frac{P_R - P_w}{\xi_R - \xi_i} = \frac{\kappa_m}{\mu L} \ln(2R) \frac{1}{\sqrt{1 - x^2}} \]  

(46)

with \(x = x/L, R = R/L\). This recovers the flux formula for the slit model, eq. (19), which has an inverse-square-root singularity at the fracture tip \(\eta = 0\), or \(\bar{x} = 1\). On the other hand, if we maintain a small but finite \(\xi_i\), then the flux at the fracture tip is

\[ q|_{\xi_i=0} = \frac{\kappa_m}{\mu L\sqrt{\sinh^2 \xi_i}} \frac{P_R - P_w}{\xi_R - \xi_i} \approx \frac{\kappa_m}{\mu L\bar{\xi}_i} \ln(2R) \frac{1}{\sqrt{1 - x^2}} \]  

(47)

which is large but finite for small values of \(\xi_i\). Thus, keeping \(\xi_i\) from becoming zero regularizes the pressure-gradient/flux tip singularity for an elliptical fracture.
The flux normalized with the flux at the fracture exit to the well is given by

\[ q_e = \frac{q(\eta)}{q_{|\eta=\pi/2}} = \frac{\sqrt{\sinh^2 \xi_1 + 1}}{\sqrt{\sinh^2 \xi_1 + \sin^2 \eta}} \tag{48} \]

(Fig. 11). At the fracture tip, the dimensionless flux is 10.03, for \( \xi_1 = 0.1 \), and 10,000, for \( \xi_1 = 0.0001 \). The area underneath the flux curve is the dimensionless total production. The flux near the fracture tip makes up a significant portion of the total production, which exemplifies the role of flux singularity in production enhancement.

The total production from the fracture is:

\[ Q = 4 \int_0^{\pi/2} q_L \sqrt{\sinh^2 \xi_1 + \sin^2 \eta} d\eta = \frac{2\pi \kappa_m P_R - P_w}{\mu \left( \frac{R}{\xi_1} - \xi_1 \right)} \]

\[ = \frac{2\pi \kappa_m}{\mu} \left( \ln(2R) - w_f/(2L) \right). \tag{49} \]

Clearly, increasing the fracture width \( w_f \) increases the total production, similar to the rectangular fracture. At small values of \( w_f/(2L) \), (49) reduces to the slit production formula (20).

6 CONCLUSIONS

The slit geometry of a hydraulic fracture creates an integrable singularity in the pressure-gradient of the reservoir flow at the fracture tips for both steady and transient conditions. For the case of infinite fracture conductivity and stabilized flows, our analyses have shown that this singularity dominates the production from a hydraulically fractured well: the pressure-gradient singularity creates a flux singularity at the fracture tip whereby flux is concentrated to the region near the fracture tip and the flux magnitude on the entire fracture surface is increased. It is through these effects that the pressure-gradient singularity increases the overall productivity from a hydraulically fractured well. For the same fracture area, a stronger but integrable singularity is always more beneficial to productivity. Modelling the fracture geometry by a nearly degenerate ellipse regularizes the pressure-gradient tip singularity. However, the pressure-gradient at the fracture tips nevertheless remains very large and nearly singular. Such a large local suction force at the fracture tips ‘sucks’ the fluids from the vast region of the low-permeability reservoir ahead of the fracture tip into the fracture via a converging and focusing flow pattern. This near-singular suction force at the fracture tips enables production from unconventional reservoirs possible: with permeability down to nano-darcy, only nearly-infinite suction force can move these fluids.

While we have used the analytical solutions for the special case of infinite fracture conductivity in the stabilized flow regime to quantify the effect of the pressure-gradient singularity on increasing productivity, the physical mechanism discussed in this study is broad and general, since it is based on the finding that the pressure-gradient singularity of the reservoir flow is inherent for all fractures with sharp ends in both steady and transient flows regardless of the fracture conductivity. Thus, our results establish pressure-gradient tip singularity as a universal and primary mechanism for enhanced production from hydraulically fractured wells. The nature of the flux singularity revealed by this study also provides a tool for assessing the accuracies of flux-based numerical schemes.

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