Radiative transfer theory for the fractal structure and power-law decay characteristics of short-period seismograms

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1 INTRODUCTION

Short-period seismic waves radiated from an earthquake source are scattered by heterogeneities distributed in the Earth medium. Their phases are complex; however, their amplitude envelopes at a given central frequency are smooth and systematic. The medium heterogeneities and absorption properties are well characterized by the ballistic term amplitude decay with traveltime and the coda amplitude decay with lapse time measured from the earthquake origin time. The radiative transfer theory has been often used for the synthesis of seismogram envelopes of a local earthquake in Japan.

SUMMARY

For short period S-wave seismograms of an earthquake, the maximum amplitude decreases according to a power of traveltime, and the coda amplitude also decreases according to a power of lapse time measured from the earthquake origin time. The radiative transfer theory has been often used for the envelope synthesis of complex seismograms composed of scattered waves due to random heterogeneities in the Earth medium; however, the conventional theory, which supposes uniform distributions of scatterers (heterogeneities) and intrinsic absorbers, predicts that both the ballistic term amplitude and the coda wave amplitude exponentially decrease with time increasing in addition to the geometrical decay. In order to explain their power-law characteristics, this paper proposes the radiative transfer theory for a fractally random and homogeneous distribution of isotropic scatterers and intrinsic absorbers with fractal dimension \( D \leq 3 \) in the 3-D space: the number density of scatterers/absorbers in a sphere of radius \( r \ll r_s \) is proportional to \( r^{D-3} \) for distance \( r \gg r_c \) but constant for \( r \ll r_c \), where the corner distance \( r_c \) is introduced to avoid divergence at a small \( r \). The case of \( D = 3 \) corresponds to the conventional uniform distribution. For the case of \( D = 2 \), the theory well predicts that the mean square (MS) amplitude of the ballistic-wave decreases according to a power of traveltime and the MS amplitude of coda waves decreases according to a power of lapse time measured from the origin time, where each power is controlled by the scattering coefficient, intrinsic absorption coefficient and corner distance. For the case of \( D = 1 \), the ballistic-wave MS amplitude decay is the inverse square of time and the coda decay is much faster. As a preliminary work, fixing \( D = 2 \) as \( a \) priori choice, we analyse S-seismogram envelopes of a local earthquake in Japan. The case study shows that the radiative transfer theory for a fractal scattering medium is useful for the study of the distribution of small-scale heterogeneities in the Earth medium.

Key words: Seismic attenuation; Theoretical seismology; Wave scattering and diffraction; Wave propagation.
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Q is equal to the Euclidean dimension of D in the 2-D space known as the Fournier universe. The area of a square is proportional to the square of its side length; however, the number of points in a square is linearly proportional to the first power of the side length. As a stochastic extension, the Levy dust is a random fractal distribution of points with a low fractal dimension compared with the Euclidean dimension. Mandelbrot (1983) precisely discusses the fractal dimension of the distribution of galaxies and the concept of fractal homogeneity, and the generation of Levy dust by the random walk model with typical examples.

We may regard the distribution of microearthquake hypocentres as a distribution of cracks of various sizes. There have been theoretical studies on scattering by distributed cracks and cavities. Yamashita (1990) proposed that the coda amplitude decay can be well represented by a power-law size distribution in order to explain the observed frequency dependence of Q for S waves. Focusing on the scattering loss and coda excitation, Benites et al. (1997) numerically studied scattering by a cluster of many cavities, and Murai et al. (1995) studied scattering by several clusters composed of thin cracks with viscous friction. From the observation in the 3-D space, Robertson et al. (1995) reported D ≈ 1.82–2.01 for the microearthquake hypocentre distribution for the distance range from 1 to 10 km in southern California. Hirata & Imoto (1991) analysed the hypocentre distribution of microearthquakes in Kanto, Japan by using the correlation integral method, and they estimated D ≈ 2.2. The region beneath this area is heterogeneous since the Pacific and Philippine Sea plates collide and subduct beneath the North American Plate. Later, precisely examining accurate hypocentre data of Hi-net in Kanto, Japan, Noguchi (2001) analysed their fractal distribution by using the box counting method. He reported D ≈ 2.3 for the distance range from 10 to 100 km for the whole data, and D ≈ 1.5–1.9 for earthquake clusters. Those observations encourage us to suppose a fractal distribution for scatterers and absorbers in the Earth medium with a fractal dimension being much smaller than the Euclidean dimension of space.

Carcolé & Sato (2010) analysed short period seismograms of local earthquakes registered by Hi-net in Japan on the basis of the multiple lapse time window analysis (Fehler et al. 1992; Hoshiba 1993). They reported that g₀ for S waves takes 5–9 × 10⁻³ km⁻¹ on average for the 4–16 Hz range. On the other hand, from the precise analysis of artificial explosion seismograms registered at Asama volcano in Japan, Yamamoto & Sato (2010) reported g₀ ≈ 0.71 km⁻¹ in the 8–16 Hz range, which is two order larger than the average value in Japan. Those observations suggest that we may expect many strong scattering spots of small sizes sparsely distributed in the Japan island arc. We may expect that the scattering coefficient can be large when it is measured in a small dimension volume at some spots but it becomes small when it is measured in other locations.
a large dimension volume. Thus, those measurements may suggest us to imagine a fractal distribution of scatterers with a lower fractal dimension compared with the Euclidian dimension of space.

Sato (1988) proposed a mathematical model for the power-law decay of amplitude with travel distance through a fractal distribution of absorbers/scatterers in a 3-D medium. Applying that model to seismic data in short periods in Italy and New Mexico, Godano et al. (1994) reported the D-value ranges from 2 to 3. Rastogi & Scheucher (1990) studied back scattered radio-coda waves by a fractal distribution of meteorological turbulences in atmospheric radar experiments. Later Sato & Fehler (1998, p. 186) formulated the radiative transfer theory in a fractal distribution of isotropic scatterers; however, their focus was put on the spatial distribution of coda energy density only and the contribution of intrinsic absorption was not considered.

In this paper, we present a formulation of the radiative transfer theory for the propagation of energy density through a fractal distribution of isotropic scatterers and intrinsic absorbers. Solving the radiative transfer theory for the instantaneous spherical radiation from a point source, we examine the space time distribution of the synthesized energy density. We show that this model for $D < 2$ well predicts a power-law decay of the ballistic- and coda-wave amplitudes with time. As a preliminary work, fixing $D = 2$ as a priori choice, we analyse observed S-seismograms of a local earthquake. Then we discuss a possible extension of this theory and analysis.

2 Radiative Transfer Theory

2.1 Fractal distribution of scatterers and absorbers

In the conventional radiative transfer theory, distributions of isotropic scatterers and intrinsic absorbers are supposed to be random and homogeneous (Sato 1977; Hoshaba 1991; Zeng et al. 1991). When the number density of scatterers is independent of volume size in the 3-D space, the total number of scatterers in a sphere of radius $r$ is proportional to $r^3$, where the power 3 is the Euclidian dimension of space. Here, we imagine a fractally random and homogeneous distribution of isotropic scatterers with the fractal dimension $D$ in the 3-D space: the total number of scatterers in a sphere of radius $r$ is proportional to $r^D$, and the number density is proportional to $r^{D-3}$, where $D$ can take a value less than or equal to 3. We may take any point as the centre of radius according to the assumption of fractal homogeneity. We can statistically realize such a fractal distribution.

We first mathematically define a function of $r$:

$$ f(r) = \frac{1}{\sqrt{1 + \left( \frac{r}{r_c} \right)^2}} \approx \frac{1}{\left( \frac{r}{r_c} \right)^{3-D}} \quad \text{for} \quad r \ll r_c, $$

where we introduce a corner distance $r_c$ to avoid divergence at $r = 0$. Function $f(r)$ shows a power-law decay at large distances as illustrated in Fig. 3. The corner distance is the lower-bound of the fractal structure and it gives the absolute value of $f(r)$ in the power-law range. Using this function with the total scattering cross-section $\sigma_{a0}$, we define the number density function $n_{\sigma}(r)$ of isotropic scatterers and the (total) scattering coefficient $g_{\sigma}(r)$:

$$ n_{\sigma}(r) = n_{\sigma0} f(r), \quad \text{(2a)} $$

$$ g_{\sigma}(r) = g_{\sigma0} f(r) = \sigma_{a0} n_{\sigma0} f(r), \quad \text{(2b)} $$

where $g_{\sigma0} = \sigma_{a0} n_{\sigma0}$ is the (total) scattering coefficient for $r \ll r_c$. The number distribution $n_{\sigma}(r)$ and $g_{\sigma}(r)$ are fractal with dimension $D$ for distances larger than $r_c$ but are uniform for distances shorter than $r_c$.

For simplicity, supposing the same fractal distribution $f(r)$ for intrinsic absorbers, we define the intrinsic absorption coefficient as

$$ g_{\sigma}(r) \equiv g_{\sigma0} f(r) = \sigma_{\sigma0} n_{\sigma0} f(r) = b_{\sigma0} g_{\sigma0}(r), \quad \text{(2c)} $$

where $b_{\sigma0} = g_{\sigma0}/g_{\sigma0}$ is the ratio of intrinsic absorption coefficient to scattering coefficient. The seismic albedo, the ratio of scattering coefficient to the sum of scattering and intrinsic absorption coefficients, is $b_{\sigma} = g_{\sigma0}/(g_{\sigma0} + g_{\sigma0}) = 1/(1 + b_{\sigma0})$.

2.2 Radiative transfer equation

As a natural extension of the conventional radiative transfer theory, for the instantaneous spherical radiation of energy from a source at the origin $W_0(t)$, the propagation of energy density $E$ in the fractal scattering medium with the background velocity $V_0$ is governed by the following convolution integral (Fig. 4):

$$ E(x, t) = W_0 G(x, t) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x - x', t - t') \times f(|x - x'|) E(x', t') \, dx' \, dt', \quad \text{(3)} $$

where $n_{\sigma0} f(|x - x'|)$ means the number density of isotropic scatterers at a distance $|x - x'|$ from the receiver at $x$. 

Figure 3. Plot of $f(r)$ against $r/r_c$ for various $D$ values, where $r_c$ is the corner distance.

Figure 4. Geometry of the scattering process for spherical radiation from the source at the origin.
2.2.1 Green’s function for the ballistic term

The propagator of the ballistic term is written by a product of geometrical factor, causality and scattering and intrinsic loss term as

\[ G(x, t) = \frac{1}{4\pi V_0 f^2} \delta \left( t - \frac{r}{V_0} \right) e^{-iU(r)}, \quad (4) \]

where \( r = |x| \). Function \( U \) on the exponent, given by a line integral
\[ t - \frac{1}{2} \int_0^t g(t' - r \omega) \, dt', \]
\[ (16b) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dke^{-i\omega} \int_0^\infty dkk \sin kr \hat{E}(k, \omega). \quad (11) \]

2.2.3 Conservation of total energy

In the case of no intrinsic absorption \( b_{\omega} = 0 \), putting \( k \to 0 \) and using (5), we have
\[ -i\omega \hat{G}(0, \omega) + V_0 g_0 \hat{G}_v(0, \omega) \]
\[ = \frac{1}{V_0} \int_0^\infty \left[ -i\omega + V_0 g_0 f(r) \right] e^{-U(r) + i\omega V_0} \, dr \]
\[ = \int_0^\infty \left[ -i\omega \frac{dU(r)}{dr} \right] e^{-U(r) + i\omega V_0} \, dr \]
\[ = -\int_0^\infty \frac{d}{dr} e^{-U(r) + i\omega V_0} \, dr \]
\[ = -e^{-U(r) + i\omega V_0} \bigg|_0^\infty = 1. \quad (12) \]

Substitution of this relation into (8) leads to
\[ \int_{-\infty}^{\infty} e^{i\omega t} \, d\bar{r} \int_{-\infty}^{\infty} E(x, \bar{r}) \, dx = W \tilde{H}(t). \quad (13) \]

where \( i/\omega \) is the Fourier transform of the step function \( H(t) \). It means the conservation of total energy: \( \int_{-\infty}^{\infty} E(x, \bar{r}) \, dx = W H(t) \).

2.3 Radiative transfer equation in the non-dimensional form

It is helpful to represent all the quantities by using normalized quantities to understand systematically the behaviour of theoretical solutions. We normalize time, length, and related quantities by \( V_0, g_0 \) and \( W \) as
\[ \tilde{r} = V_0 g_0 r, \quad \tilde{x} = g_0 x, \quad \tilde{\tau} = \frac{g_0 \tau}{V_0}, \quad \tilde{G} = \frac{G}{g_0}, \quad \tilde{E} = \frac{E}{W g_0}, \quad \text{and} \]
\[ \tilde{r}_c = g_0 r_c = \frac{r_c}{g_0}. \quad (14) \]

where an over bar means the normalized quantity. For example, \( \tilde{r}_c \) is the ratio of the corner distance to the mean free path at short distances (\( g_0^{-1} \)). The radiative transfer equation for the normalized energy density is written as
\[ \tilde{E}(\tilde{x}, \tilde{r}) = \tilde{G}(\tilde{x}, \tilde{r}) \]
\[ + \int_{-\infty}^{\infty} \tilde{G}_F(\tilde{x} - \tilde{x}', \tilde{r} - \tilde{r}') \tilde{E}(\tilde{x}', \tilde{r}') \, d\tilde{x}' \, d\tilde{r}'. \quad (15) \]

where
\[ \tilde{G}(\tilde{x}, \tilde{r}) = \frac{1}{4\pi \tilde{r}^2} \delta(\tilde{r} - \tilde{r}) e^{-iU(\tilde{r})}, \quad (16a) \]
\[ \tilde{G}_F(\tilde{x}, \tilde{r}) = \tilde{G}(\tilde{x}, \tilde{r}) f(\tilde{r}). \quad (16b) \]
and
\[
 U(\hat{r}) = (1 + b_0) \int_0^{\hat{r}} f(\hat{r}) \, d\hat{r} \\
 = (1 + b_0) \frac{2}{3} \left[ 3 - D \frac{1}{2} \right] \left( \frac{\hat{r}}{r_c} \right)^2. \tag{16c}
\]

As shown by log–log plots in Fig. 5, in the range \( \hat{r} \gg r_c \), function \( e^{-U(\hat{r})} \) is convex for \( D > 2 \) and concave for \( D < 2 \); however, it is a straight line especially for \( D = 2 \). As the \( D \) value approaches to 1, the curve becomes flat at large distances; however, the convergence to a straight line is slower than the curve of \( D = 2 \).

3 Characteristics of Solutions in Space and Time

3.1 Decay of the ballistic term with travel distance

For a delta function source, the distance dependence of the energy density of the ballistic term is \( W e^{-U(r)/(4\pi r^2 V_0)} \delta(t - r/V_0) \) from (4). For a short source duration compared to the traveltime, \( T_w \ll t \), the energy density of the ballistic term in a time window of the length \( T_w \) at \( r = V_0 t \) is \( W e^{-U(r)/(4\pi r^2 V_0)} \). In the case of \( D = 3 \), the ballistic energy density is
\[
E^0(r = V_0 t) = \frac{W e^{-\left(1 + b_0\right)\delta(\delta)} \bar{r}^2}{4\pi T_w V_0^2 t^2}. \tag{17a}
\]

In the case of \( D = 2 \), we may approximate \( U(r) \approx (1 + b_0)g_{0r}c \ln(2r/r_c) \) for \( r \gg r_c \), since \( zF_1(1/2, 1/2, 3/2; -z^2) = (\sin^{-1}z)/z = \ln(z + \sqrt{1 + z^2})/z \approx (\ln 2)/z \) for \( z \gg 1 \). At \( r = V_0 t \),
\[
E^0(r = V_0 t) = \frac{W}{\pi T_w V_0^2} \left( \frac{r_c}{2V_0 t} \right)^{2+\left(1 + b_0\right)\delta(\delta)}, \tag{17b}
\]
where the inverse power of traveltime is larger than 2 and it depends on the scattering, the absorption and the corner distance. When \( D = 1 \), \( zF_1(1, 1/2, 3/2; -z^2) = (\tan^{-1}z)/z \approx \pi/(2z) \) for \( z \gg 1 \). Then \( U(r) \approx (1 + b_0)\pi g_{0r}c, \) for \( r \gg r_c \), which leads to the inverse square of traveltime as
\[
E^0(r = V_0 t) = \frac{W e^{-(1 + b_0)\delta(\delta)}}{4\pi T_w V_0^3 t^2}. \tag{17c}
\]

In the non-dimensional form, the ballistic energy density in a time window \( T_w \ll t \) for \( r \gg r_c \),
\[
\frac{E^0(\hat{r} = \hat{t})}{4\pi T_w} = \frac{e^{-\left(1 + b_0\right)\delta}}{4\pi T_w} \frac{1}{\hat{r}^{2+\left(1 + b_0\right)\delta(\delta)}} \tag{18a}
\]
for \( D = 3 \),
\[
\frac{E^0(\hat{r} = \hat{t})}{4\pi T_w} = \frac{1}{\pi T_w c} \left( \frac{r_c}{2T_w} \right)^{2+\left(1 + b_0\right)\delta(\delta)} \tag{18b}
\]
for \( D = 2 \),
\[
\frac{E^0(\hat{r} = \hat{t})}{4\pi T_w} = e^{-\left(1 + b_0\right)\delta} \tag{18c}
\]
for \( D = 1 \).

Fig. 6 shows log–log plots of the normalized ballistic energy density \( E^0(\hat{t}) \) for various \( D \) values. The decay of the ballistic term is almost straight when \( D \leq 2 \). Thus we find that the fractal dimension \( D \) is the key parameter for the amplitude decay of the ballistic term with travel distance (traveltime). The fractal dimension \( D = 2 \) is special since the decay curve is controllable by scattering, intrinsic absorption and corner distance; however, the inverse power of travel distance for \( D = 1 \) is fixed to two independent of those parameters.

Fig. 7 shows the normalized ballistic energy density curves in a time window \( T_w = 1 \) for different \( b_0 \) values especially for the case of \( D = 2 \), where \( \hat{r}_c = 1 \). The decay gradient becomes steeper as relative intrinsic absorption increases according to the inverse power \( 2 + (1 + b_0)\delta(\delta) \).
3.2 Synthesized energy density for various fractal dimensions without intrinsic absorption

We numerically evaluate the space time distribution of energy density $E(\bar{r}, \bar{t})$ (15) by using an FFT. Fig. 8 shows log–log plots of synthesized normalized energy densities in space and time for $D = 1$ (grey lines), 2 (red lines) and 3 (black lines), where $\bar{r}_c = 1$ and $b_i = 0$. On the left-hand panel, the decay of the peak value of the ballistic term is exponential with the inverse square of traveltime, the inverse cube of traveltime, and the inverse square of traveltime for $D = 3, 2$ and 1, respectively. At a given distance, the ballistic term becomes small as $D$ increases since scattering loss becomes large. Coda energy density traces decay according to the $-1.5$th power of lapse time for $D = 3$. For $D < 3$, coda energy density traces decrease according to an inverse power of lapse time, straight lines, where the inverse power index becomes larger as $D$ becomes smaller. At a large lapse time, the coda amplitude becomes large as $D$ becomes large since the number of scatterers responsible to the coda excitation becomes large. The right-hand panel shows spatial distributions of the normalized energy density at different lapse times for three $D$ values. The concentration of energy near the source location is apparent because of multiple scattering for $D = 3$; however, the energy concentration becomes weak as $D$ becomes small.

3.3 Synthesized energy density for $D = 2$ and the single back scattering approximation

The case of $D = 2$ is interesting since the decay gradient of the ballistic energy density is controllable by the choice of parameters.

![Figure 8](https://academic.oup.com/gji/article-abstract/195/3/1831/624593)

![Figure 9](https://academic.oup.com/gji/article-abstract/195/3/1831/624593)

Figure 8. Time traces (left-hand panel) and spatial distributions (right-hand panel) of the normalized energy density for different fractal dimensions, where $b_i = 0$, $\bar{T}_w = 0.4$ and $\bar{r}_c = 1$.

Figure 9. Time traces (left-hand panel) and spatial distributions (right-hand panel) of the normalized energy density for different corner distances in a fractal scattering medium with $D = 2$, where $\bar{T}_w = 0.04$ and $b_i = 0$. 
As predicted by (17b), the ballistic energy density decreases according to an inverse power of traveltime, where the inverse power is 2 + (1 + b₀)g₀a₀. We note that parameter
\[ h = (1 + b₀)g₀a₀ = (1 + b₀)\bar{r}_c \]  
works as a constraint for b₀ and g₀a₀. Under the constraint h = constant, the normalized corner distance ñc becomes smaller as b₀ increases. Later we will study the observation for the case of 2 + (1 + b₀)g₀a₀ ≈ 3.3; therefore, taking h = 1.3, we show how this constraint affects the energy density traces by using three different kinds of lines in Fig. 10. Blue broken straight lines show the decay of the normalized ballistic term peak value against normalized travel time. As \ñc decreases b₀ increases, which lessen the coda excitation. As \ñc decreases, coda energy density curves converge to power-law curves (straight lines) at large lapse times.

Since the contribution of multiple scattering becomes small when \ñc ≪ 1 and b₀ ≫ 1, we may expect that the single scattering dominates over multiple scattering even at large lapse times. Using the asymptotic behaviours of fractal distributions of scatterers and absorbers at \( r \gg \bar{r}_c \), we have the single back scattering energy density curves at the source location for a large lapse time \( t \gg r_c/V_0 \) as

\begin{align*}
W V_0 g_0 \int_{-\infty}^{\infty} dr' \int_{-\infty}^{\infty} 4\pi r'^2 dr' \delta \left( t - t' - \frac{r'}{V_0} \right) \\
\times \left[ \frac{1}{4\pi V_0 r'^2} \left( \frac{r_c}{2r'} \right)^{1+\bar{h}_0}\delta \left( \frac{r_c}{V_0} \right) \right]^2 \delta \left( t' - \frac{r'}{V_0} \right) \\
= W g_0 V_0^2 \int_{0}^{\infty} 4\pi r'^2 dr' \frac{1}{(4\pi V_0 r'^2)^{3/4}} \\
\times \left( \frac{r_c}{2r'} \right)^{2(1+\bar{h}_0)}\delta \left( \frac{r_c}{V_0 t} \right) \\
= W g_0 \frac{2}{\pi r_c^2} \left( \frac{r_c}{V_0 t} \right)^{3+2(1+\bar{h}_0)} \\
\text{for } t \gg r_c. 
\end{align*}

In the non-dimensional form, we have

\[ \frac{E}{E^b} (\bar{r} = 0, \bar{t}) = \frac{1}{\pi r_c^2} \left( \frac{r_c}{\bar{t}} \right)^{3+2(1+\bar{h}_0)} \]  
for \( \bar{t} \gg \bar{r}_c \).

The resultant representation of the single back scattering energy density shows that the inverse power of lapse time \( 3 + 2h \) is much faster than that \( 2 + h \) for the ballistic term (18).

In Fig. 10, red straight lines show the single back scattering approximation (21), where all the lines have the common inverse power of \( 3 + 2h = 5.6 \). We see that numerical simulated coda energy density traces \( \tilde{E}^c (\bar{r} = 0, \bar{t}) \) from (15) become parallel to the single back scattering energy density trace (red straight lines) with an offset at large lapse times. We may write \( \tilde{E}^c (\bar{r} = 0, \bar{t}) \approx c E^b (\bar{r} = 0, \bar{t}) \), where \( c \approx 6.6, 2.0 \) and 1.3 at \( \bar{t} = 5 \) for \( \bar{r}_c = 0.5, 0.25 \) and 0.125, respectively. The offset is caused by multiple scattering processes mostly at short paths less than \( r_c \). The convergence is slow at \( \bar{r}_c = 0.5 (b_0 = 1.6) \) (dotted line) but it becomes fast at \( \bar{r}_c = 0.125 (b_0 = 0.94) \) (solid line).

In the log–log plot of energy density time traces, the ballistic energy density trace \( c E^b (\bar{t}) \) given by (17b) and the coda energy density trace according to the corrected single back scattering approximation \( c E^b (\bar{t}) \) intersect at lapse time \( t_b \). Equating two lines as

\[ \frac{1}{\pi r_c^2} \left( \frac{r_c}{V_0 t_b} \right)^{2+h} = c \frac{g_0}{\pi r_c^2} \left( \frac{r_c}{V_0 t_b} \right)^{3+2h} \]  
we have

\[ g_0 r_c^{1+h} = \frac{(V_0 t_b)^{1+h}}{4\pi V_0 T_0^2}. \]

**4 PRELIMINARY ANALYSIS OF SEISMOGRAMS OF A LOCAL EARTHQUAKE**

On the basis of the radiative transfer theory for a fractal scattering medium, we are able to synthesize the power-law decay of both the ballistic term amplitude and the coda amplitude with time. For the case that the inverse power of the ballistic MS amplitude decay is larger than 2, fractal dimension \( D \) should be larger than 1. It is not easy to uniquely determine the \( D \) value; however, the case of \( D = 2 \) is interesting since the power indexes are controlled by scattering and intrinsic absorption parameters. Here, fixing \( D = 2 \) as a priori choice, we make a preliminary analysis of short-period seismogram envelopes of a local earthquake. We should note that the conventional radiative transfer theory fixes \( D = 3 \) as a priori choice from the point of view of the fractal model.
5.65 leads to 2.\nThe spatial variations of MS amplitudes (top panel) and coda normalized P-S waves 5.1 s in lapse time. For the D 2 by using the grid search method in the S h h h S = 1 s. Making a log–log plot of the product of the peak (coda normalized) MS amplitude and the time width of the half maximum against traveltime, we evaluate the power-law decay of the ballistic energy density. Travetimes are limited to be less than 35 s, which is half of the lapse time 70 s used for the site amplification factor correction. Then we make log–log plots of coda MS amplitude against lapse time at 10 stations near the epicentre, where the lapse time range is taken from twice the traveltime until 70 s. We plot corrected observed ballistic MS amplitudes by blue dots against S-wave traveltime and observed coda envelopes by fine colour lines against lapse time in Fig. 14.

We perform the least-square regression analysis for the ballistic peaks (blue dots) and coda decay curves (fine colour lines) in their log–log plots. The least square regression lines are 8.75 ± 0.45 – 3.28 log10t (solid red line) for peak log MS amplitudes and 10.43 ± 0.25 – 5.65 log10t (broken black line) for coda log MS envelopes. Two regression lines intersect at t\textsubscript{s} = 5.1 s in lapse time. For the ballistic term decay, the inverse power of lapse time 2 + h = 3.28 leads to h = 1.28 in (19). Applying the corrected single back scattering approximation decay curve cE\textsuperscript{-}(r = 0, t) to the observed coda decay curve, the inverse power of lapse time 3 + 2h = 5.65 leads to h = 1.33, which is nearly the same as the estimate of ballistic term decay h = 1.28. This coincidence well supports the use of the fractal model of D = 2.

We estimate the best fit parameters of the fractal scattering medium with D = 2 by using the grid search method in the

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**Figure 11.** Epicentre (star) of an earthquake of M 4.8 in Niigata, Japan and Hi-net stations (dots) used for the analysis.

**Figure 12.** Spatial variations of MS amplitudes (top panel) and coda normalized MS amplitudes (bottom panel) in the 4–8 Hz band at different lapse times. MS amplitude at 70 s in lapse time measured from the earthquake origin time is used for the normalization.
Figure 13. Log–log plots of MS amplitudes (left-hand panel) and coda normalized MS amplitudes (right-hand panel) against lapse time measured from the earthquake origin time in the $4$–$8$ Hz band.

Figure 14. Log–log plot of observed ballistic MS amplitudes with half time width correction (blue dots) and their regression line (solid red line) against traveltime. Log–log plot of coda MS amplitudes (fine colour lines) against lapse time and their regression line (broken black line) at 10 stations near the epicentre.

$\log_{10} g_{s0} - \bar{r}_c$ space with a grid interval of 0.05 for $\bar{r}_c$ and $\log_{10} 2$ for $\log_{10} g_{s0}$. For a given set of parameters, we first calculate the residual of the observed ballistic normalized log MS amplitudes with half time-width correction $M_{\text{bal, obs}}$ from synthesized log MS amplitudes $M_{\text{bal, syn}}$ at traveltime $t$, $\text{Res}_{\text{bal}} = \frac{1}{n} \sum_{i=1}^{n} [\log_{10} M_{\text{bal, obs}}(t_i) - \log_{10} M_{\text{bal, syn}}(t_i)]^2$, where the inverse power of traveltime is set to 3.28 for the ballistic term. The estimated variance $\sigma_{\text{bal}}^2 = 0.198$. We also calculate the residual of the observed coda normalized log MS amplitudes $M_{\text{coda, obs}}$ from synthesized log MS amplitudes $M_{\text{coda, syn}}$ at lapse time $t'$, $\text{Res}_{\text{coda}} = \frac{1}{n'} \sum_{i=1}^{n'} [\log_{10} M_{\text{coda, obs}}(t'_i) - \log_{10} M_{\text{coda, syn}}(t'_i)]^2$, where we use the synthetic curve of coda envelope for lapse times from twice the traveltime to 70 s at the epicentre. Then, the estimated variance $\sigma_{\text{coda}}^2 = 0.041$. Using these variances, we search the minimum value of the weighted residual $\frac{1}{\sigma_{\text{bal}}^2} \text{Res}_{\text{bal}} + \frac{1}{\sigma_{\text{coda}}^2} \text{Res}_{\text{coda}}$ in the $\log_{10} g_{s0} - \bar{r}_c$ space. Fig. 15 shows the contour plot of the weighted residual, where the minimum residual point is not so sharp. For example, $\bar{r}_c = 0.5$ at $g_{s0} = 0.05$ km$^{-1}$ on the trough, which correspond to $r_c = 10$ km at $g_{s0}^{-1} = 20$ km. If we use the corrected single back scattering approximation, the cross point $t_x = 5.1$ s in Fig. 14 does not uniquely determine $g_{s0}$ and $\bar{r}_c$ values but gives a constraint between them. We may rewrite (23) as $\log_{10} g_{s0} = \frac{1}{h} \log_{10} \{4c V_0 T_0 2^h [\frac{1}{r_c}]^{1+h}\}$ with $h = 1.28$. Substitution of the ratio of the best-fitting coda MS envelope to the power-law curve of the corrected single back scattering approximation for $c = 3.0$ into this relation gives the thick broken blue line, which well explains the trade off for the range of...

Figure 15. Grid search of the minimum value (red) of the weighted residual $\frac{1}{\sigma_{\text{bal}}^2} \text{Res}_{\text{bal}} + \frac{1}{\sigma_{\text{coda}}^2} \text{Res}_{\text{coda}}$ in the $\log_{10} g_{s0} - \bar{r}_c$ space, where numerals for contours are weighted residuals. The red dotted contour shows twice the minimum residual. Thick broken blue line shows the estimate from the cross point of two power-law lines in Fig. 14 by using the single back scattering approximation (23).

In Fig. 15, the red dotted contour line shows twice the minimum residual as a reference. We find that a trough runs through the dotted contour, which indicates a trade off between $g_{s0}$ and $\bar{r}_c$, where the minimum residual point is not so sharp. For example, $\bar{r}_c = 0.5$ at $g_{s0} = 0.05$ km$^{-1}$ on the trough, which correspond to $r_c = 10$ km at $g_{s0}^{-1} = 20$ km. If we use the corrected single back scattering approximation, the cross point $t_x = 5.1$ s in Fig. 14 does not uniquely determine $g_{s0}$ and $\bar{r}_c$ values but gives a constraint between them. We may rewrite (23) as $\log_{10} g_{s0} = \frac{1}{h} \log_{10} \{4c V_0 T_0 2^h [\frac{1}{r_c}]^{1+h}\}$ with $h = 1.28$. Substitution of the ratio of the best-fitting coda MS envelope to the power-law curve of the corrected single back scattering approximation for $c = 3.0$ into this relation gives the thick broken blue line, which well explains the trade off for the range of...
radiative transfer theory supposing $D = 3$. Their measurement is done for the dimension of the order of 100 km. As shown in Fig. 17, our estimation $g_{0} = 0.02$ km$^{-1}$ (solid red curve), the flat level of solid red curve at short distances for the fractal scattering medium with $D = 2$, is three to four times larger than their estimation in the framework of $D = 3$; however, our distribution function $g_{0}(r)$ rapidly decreases less than their estimate as distances larger than 50 km. Our estimate of intrinsic absorption $g_{0}$, the flat level of the broken blue curve, is larger than their estimate of $g_{0}$ using the $D = 3$ model. Our estimate of $g_{0}$ is still weaker than $g_{0} \approx 0.71$ km$^{-1}$ in the 8–16 Hz range measured at Asama volcano in Honshu, Japan in the framework of $D = 3$ (Yamamoto & Sato 2010); however, we note their measurement was done in a small area of about 5 km in dimension. Our estimate of intrinsic absorption $g_{0}$ is about seven times smaller than their estimate $0.46$ km$^{-1}$ ($Q_{i} = 100$) at a volcano using the $D = 3$ model.

We have a priori applied the fractal scattering model for $D = 2$ to observed S-seismograms as a preliminary analysis since the choice of $D = 2$ is fortunately appropriate to explain simultaneously the observed ballistic amplitude decay and coda amplitude decay with time. However in general, the difference of their power indexes is not always explainable by the $D = 2$ model. In some cases, observed ballistic amplitude decay and coda amplitude decay do not exactly show the power-law decay. In such cases, we need to imagine more flexible fractal structures for the distributions of scatterers and intrinsic absorbers.

On the basis of the radiative transfer theory, we have formulated the multiple scattering process through a fractal distribution of isotropic scatterers and intrinsic absorbers. The distributions of scatterers and absorbers are supposed to have the same fractal dimension and the corner distance for mathematical simplicity; however, we can easily extend the model to adopt different fractal dimensions and different corner distances for them. It can be accomplished by replacing $g_{0}(r)$ of $B(r)$ in (5) with $\sigma_{0}Q_{i}\sqrt{1 + (r/r_{c})^{D-3}}$ with fractal dimension $D$, and corner distance $r_{c}$ for intrinsic absorbers.

We have evaluated the energy propagation process solving the radiative transfer integral equation; however, there is a way to simulate it by using a Monte Carlo method. The fractal distribution of scatterers can be statistically realized by using the Monte Carlo simulation as the Levy dust and the energy propagation can be simulated as a stochastic scattering process for many particles shot from a point source by the fractal distribution of many scatterers. It will be important to measure the variance from the mean since a large fluctuation is expected for ballistic term amplitudes. The fractal distribution in space has been supposed to be isotropic. It is extendable to adopt a non-isotropic fractal structure having different fractal dimensions in the vertical direction and in the horizontal space, which is in harmony with the decrease of scattering strength and intrinsic absorption with depth increasing as Gusev (1995) supposed.

The model here developed based on the isotropic scattering assumption is not able to explain the observed broadening of an $S$ wavelet with travel distance increasing; therefore, multiplying the time width by the peak value of MS amplitude, we have corrected the ballistic energy density at each travel distance in the preliminary analysis. In short periods, forward scattering due to velocity inhomogeneities is thought to be larger than wide angle scattering in lithospheric heterogeneities. It is important to develop the radiative transfer theory for the fractal distribution of such nonisotropic scatterers.

We have not proposed physical models of scatterers and intrinsic absorbers yet. For example, however, we may imagine a fractally random distribution of randomly oriented cracks containing viscous

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**Figure 16.** Best-fitting curves (solid black lines) for the minimum residual parameter set (red spot in Fig. 15) with corrected observed ballistic term energy densities (blue dots) and coda energy density traces (fine colour lines). The thick broken grey line is the synthesized coda envelope at the epicentre.

**Figure 17.** Log–log plots of the scattering coefficient (solid red line) $g_{i}(r)$ and the intrinsic absorption coefficient (broken blue line) $g_{0}(r)$ against $r$ for the best fit parameters for $D = 2$ in the 4–8 Hz band. Light colour boxes show measurements of $g_{0i}$ and $g_{0}$ in the Niigata area based on the $D = 3$ model for references.
fluid with the power-law size distribution as an extension of the uniform distribution crack model proposed by Yamashita (1990). Measurements of envelopes in a single frequency band give information about the model parameters for their fractal distribution in space. If we measure envelopes in various frequency bands, we may have more information about the size distribution of those cracks. There have been many studies on the relation among the radiative transfer theory in a scattering medium and the stochastic treatment of wave propagation in random media; however, their link is established only when the fractal dimension is equal to the Euclidean dimension. We note that Gusev & Abubakirov (1996) studied the fractal nature of power-law spectra of velocity inhomogeneities, which causes frequency dependent non-isotropic scattering. It will be important to establish a link among the power-law spectra of random velocity inhomogeneities in the wave theory and the fractal distribution of scatterers in the radiative transfer theory especially for cases of $D < 3$.

6 CONCLUSION

For short period seismograms of an earthquake, the maximum amplitude decreases according to a power of traveltime (travel distance), and the coda amplitude also decreases according to a power of lapse time measured from the earthquake origin time. In order to explain their power-law characteristics, we have proposed the radiative transfer theory in the 3-D space when the spatial distribution of isotropic scatterers and that of intrinsic absorbers are fractally random and homogeneous with fractal dimension $D$. Introducing the corner distance $r_c$ to avoid divergence at $r = 0$, we have supposed that their distribution densities are proportional to $r^{D-3}$ for $r \gg r_c$, but take constant values for $r \ll r_c$. The case of $D = 3$ corresponds to the conventional uniform distribution. For the case of $D = 2$, the theory well predicts that the ballistic wave and coda-wave MS amplitudes decrease according to a power of traveltime and lapse time measured from the origin time, respectively, where each power is controlled by the scattering coefficient, intrinsic absorption coefficient, and corner distance. For the case of $D = 1$, the ballistic-wave amplitude decay is the inverse square of time and the coda decay is much faster.

As a preliminary work, fixing $D = 2$ as a priori choice, we have shown a way how to analyse S-seismogram envelopes in short periods on the basis of the fractal model. From the practical analysis of a crustal earthquake of $M \approx 4.8$ in Japan, we have shown that the fractal model well explains the characteristics of observed seismogram envelopes at the 4–8 Hz band. Our estimation of $g_{s0}$ at short distances is larger than the reported average $g_{s0}$ in this area but smaller than $g_{s0}$ beneath a volcano estimated on the bases of the conventional radiative transfer theory with $D = 3$. This case study shows that the radiative transfer theory for a fractal scattering medium is useful for the study of small-scale heterogeneities in the Earth.

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