Prediction of rocks thermal conductivity from elastic wave velocities, mineralogy and microstructure

Lucas Pimienta,* Joel Sarout, Lionel Esteban and Claudio Delle Piane

CSIRO Earth Science and Resource Engineering, Perth, Australia. E-mail: pimienta@geologie.ens.fr

SUMMARY
While knowledge on Thermal Conductivity (TC) of rocks is of interest in many fields, determining this property remains challenging. In this paper, a modelling approach for TC prediction from Elastic Wave Velocity (EWV) measurements is reported. To this end, a new effective TC model for a typical sedimentary rock is introduced that explicitly accounts for the presence of pores, pressure-sensitive microcracks (or grain contacts) and formation fluids. A model of effective elasticity is also devised for this same rock that links its microstructural characteristics to the velocity of elastic waves. The two models are based on the same effective medium approach and involve the same microstructural parameters. A workflow based on this explicit modelling approach is devised that allows for the prediction of the TC of a reservoir rock using (i) the elastic waves velocities, (ii) the dominant mineral content and (iii) the bulk porosity. This workflow is validated using experimental data reported in the literature for dry and water-saturated Fontainebleau and Berea sandstones. The datasets include measurements of TC and EWV as a function of effective pressure. In addition, it is shown that the dependence of TC on the rock microstructure is formally and practically similar to that of EWV. It is also demonstrated that the accuracy of TC predictions from EWV increases with effective pressure (burial depth). The underlying assumptions and limitations of the present approach together with the effect of burial are discussed.

Key words: Heat flow; Microstructures; Defects; Acoustic properties; Mechanics, theory, and modelling.

1 INTRODUCTION
Prospection for and exploitation of geothermal energy rely on the proper assessment of underground temperatures. While reliable formation temperature estimates can be obtained locally from borehole measurements (up to a certain depth limit), no existing tool allows for large-scale estimation of rock formations temperature away from available boreholes. Knowledge on heat transport phenomenon is thus needed to determine it. Heat transport in sedimentary rocks at the depths studied (i.e. $T \leq 150$ °C and $P \leq 50\,\text{MPa}$) can occur through two physically distinct mechanisms: (i) conduction, controlled by the effective Thermal Conductivity (TC) of the fluid-saturated rock; and (ii) convection, controlled by the effective permeability of the rock and the calorific capacity of the fluid moving in the pore space. Both heat transport properties are expected to be affected by the rock microstructure, although not in the same manner. In this contribution, we focus only on the conduction aspect of heat transfer in the subsurface reservoirs. A non-invasive downhole tool for in situ TC measurement was recently designed, yet its ability to record accurately the thermal properties still needs to be demonstrated (Freifeld et al. 2008) and TC needs to be predicted. Estimation of this physical property is also of direct interest in a variety of other application fields such as civil engineering (Midttomme & Roaldset 1998) or the disposal of underground radioactive waste (Giraud et al. 2007).

Crustal rocks are polymineralic, porous, microcracked and macrofractured media saturated by various fluids. Understanding and predicting physical properties on such media is thus challenging. As for Elastic Wave Velocities (EWVs) and electrical resistivity, TC depends on mineral composition (Torquato 1998), grain size (e.g. Midttomme & Roaldset 1998) and fluid saturation (Walsh & Decker 1966; Tong et al. 2009). Studies also emphasized the important effect of grain size (e.g. Midttomme & Roaldset 1998), grain geometry (e.g. Côe & Konrad 2009) and grain contacts (e.g. Gegenhuber & Schoen 2012) on TC, and it has been experimentally demonstrated (Woodside & Messmer 1961; Abdulagatova et al. 2009, 2010; Lin et al. 2011) that the TC of sedimentary rocks strongly depends on pressure. Yet, most of the existing models accounting for this microstructural aspect are semi-empirical and based on over-simplistic parameters, which limits their predictive power (Wang et al. 2006). To the authors’ knowledge, there are no models built from first principles that explicitly relate TC to rock’s...
microstructure, and in particular to pressure-sensitive features such as microcracks or grain contacts.

To predict TC in a cost-effective way, some authors related this property to other potentially available physical measurements such as wireline logs (Griffiths et al. 1992), electrical resistivity (Revil 2000) or EWV (Zamora et al. 1993; Ozkrahman et al. 2004; Kazatchenko et al. 2006; Gegenhuber & Schoen 2012). Yet, electrical resistivity and permeability in sedimentary rocks are controlled by the connectivity of the pore space (Guéguen & Palciauskas 1994), and are therefore expected to be related to convection more than to conduction. On the other hand, elastic stiffnesses (and EWV) are known to depend strongly on pressure-dependent microstructures rather than by the connectivity of the pore space (e.g. Guéguen et al. 2009). To the authors’ knowledge, approaches relating TC and EWV remain however either empirical (e.g. Zamora et al. 1993) or based on statistical correlations (e.g. Gegenhuber & Schoen 2012).

In this paper, an approach for TC prediction in porous and microcracked rocks is reported that involves Effective Medium Theory (EMT) applied to both TC and elasticity. Extending the Torquato (2001) and Giraud et al. (2007) approach to the case of a microcracked rock, a new effective TC model is devised that explicitly accounts for the presence of microcracks (or grain contacts) in a rock. Using published experimental data on dry and water-saturated Berea sandstones, it is first shown that (i) TC and EWV bear similar dependency to pressure; and (ii) the TC model for microcracked rocks and its analogue for elasticity (e.g. Kachanov 1993; Adelinet et al. 2011) compare well in their dependency to microstructural features. A second TC model is then built that aims at representing a porous and microcracked rock, and its analogue for elasticity (e.g. Kachanov 1993; Fortin et al. 2007; Adelinet et al. 2011) is also reported. A workflow based on this approach is devised that allows for the prediction of the TC of a rock using (i) the velocities of elastic waves, (ii) the dominant mineral content and (iii) the total porosity. This workflow is tested using published experimental data on sandstones, and a reasonable fit is obtained between predictions and the data. In addition, possible deviations between TC and EWV models are analysed in the light of the models’ assumptions and the used data set.

2 LITERATURE DATA: PRESSURE DEPENDENCE OF ELASTICITY AND TC

Data on both TC and EWV as a function of effective pressure reported in the literature are used to validate the new models and the associated TC prediction workflow. The laboratory-derived data sets and the petrophysical characteristics of the associated lithologies are presented in Appendix A (Sections A1 and A2), along with a discussion on experimental uncertainties (Section A3).

Before defining the modelling approaches, let us introduce first in Fig. 1 a comparison between published measurements of TC (Woodside & Messmer 1961a; Lin et al. 2011) and EWV (Christensen & Wang 1985; Sayers et al. 1990; Tao et al. 1995; Prasad & Manghnani 1997; Pagoulatos & Sondergeld 2004; Mavko & Vanorio 2010) on the Berea sandstone at different confining pressures. Scaling TC and P-wave velocities measurements at different confining pressures by their value at maximum confining pressure (i.e. $P_c \sim 50$ MPa) reveals a systematic relation between both properties. In the particular case of TC data of Woodside & Messmer (1961a), that were measured only up to a pressure of $P_c \sim 25$ MPa, the maximum ratio value at $P_c \sim 25$ MPa is taken as equal to scaled P-wave values on similar samples (i.e. same porosity) at this pressure.

![Figure 1. TC and P-wave velocity in Berea sandstone, scaled by their maximum value at $P_{eff} = 50$ MPa, as a function of effective confining pressure. Data reported here are found in Lin et al. (2011), Tao et al. (1995), Mavko & Vanorio (2010) and Pagoulatos & Sondergeld (2004). Note that $\lambda$ stands for TC and $V_p$ for P-wave velocity.](https://academic.oup.com/gji/article-abstract/197/2/860/617871)

The similar sensitivity of these two physical quantities to confining pressure supports our attempt at (i) using a similar microstructure-based effective medium approach for modelling TC and EWV; and (ii) predicting TC from EWV at in situ conditions. Similarly to EWV, TC is not only controlled by mineral content and porosity, but also by the presence of stress-sensitive features such as inter- or intragranular microcracks. Although this fact has long been established for EWV (e.g. Guéguen & Kachanov 2011), it is not fully recognized for TC and many models recently published are still ignoring it.

3 MODELLING EFFECTIVE ELASTICITY AND TC

The porous rock is considered as a homogeneous solid embedding (i) stress-insensitive equant pores representing the bulk porosity of the rock; and (ii) stress-sensitive flat cavities representing microcracks or grain contacts. Pores and microcracks are assumed to be randomly located in space such that the rock is homogeneous at a scale larger than the considered heterogeneities, and a Representative Elementary Volume (REV) exists for physical modelling purposes. Only clean isotropic rocks are considered, which implies that (i) the background matrix is mono-mineralic and made of randomly shaped and oriented crystal grains, and that (ii) the microcracks are randomly oriented in space. The effective elastic and TC properties of such a rock are predicted in a similar manner using the EMT: bulk porosity is modelled as spherical cavities, and microcracks are modelled as oblate spheroidal cavities. This Eshelby-like approach is well known for elasticity (e.g. Guéguen & Kachanov 2011). While it is also reported for effective TC of a porous rock with various minerals inclusions (Giraud et al. 2007), it is uncommon in the case of microcracked rocks. The present approach uses Giraud et al. (2007) approach to model porosity, and extends it to rocks with microcracks. Since it is relatively uncommon to apply EMT to TC, details of the mathematical derivations are provided in Appendix B (i.e. Section B1).

In the following sections, a multistep methodology is introduced to model effective TC and EWV resulting from various cavity shapes. The final modelling approach is schematized in Fig. 2.
3.1 Model 1: porous rock

To predict the effect of bulk porosity, spherical pores are randomly embedded in an isotropic background solid with known properties using the Mori-Tanaka upscaling scheme that accounts for the possible interactions between pores (stress and temperature fields). After addition of spherical inclusions into the solid phase, the effective medium remains isotropic. Details of the TC and EWV models for the porous rock are given in Section B2 (Appendix B).

The effective elasticity model for a dry or water-saturated sphere embedded in a homogeneous and elastic solid has been derived for instance by Shafiro & Kachanov (1997), based on the original work of Kachanov (1993). The upscaling of this solution to account for the presence of several spherical pores using the Mori-Tanaka scheme was recently reported by Fortin et al. (2007), and is reported in Section B2.1 (i.e. eqs B12, B14 and B15). The same microstructural approach is used by Giraud et al. (2007) to compute the effective TC of the fluid-saturated porous rock (eq. B16). Note that this equation is often referred to as the Eucken–Maxwell equation (e.g. Wang & Yi 2004). The TC model for the dry situation (eq. B18) is obtained by a Taylor expansion in $\varepsilon = \lambda / \lambda_m$ for the model of the fluid-saturated situation. The final $\lambda_{\text{sat}}$ and $\lambda_{\text{dry}}$ of the fluid-saturated and dry porous rock are (eqs B17 and B18):

$$\frac{\lambda_m}{\lambda_{\text{sat}}} = 1 + \frac{3\phi_b (1 - \alpha_{th})}{2 (1 + \alpha_{th})} \frac{\lambda_{\text{sat}}}{\lambda_{\text{dry}}}$$

and

$$\frac{\lambda_m}{\lambda_{\text{dry}}} = 1 + \frac{3\phi_b}{2 (1 - \phi_b)},$$

where $\alpha_{th} = \lambda / \lambda_m$ is the fluid to solid TC ratio, $\phi_b$ is the porosity, and $\lambda_m$ is the TC of the host medium.

Note that this porosity is assumed relatively stress-insensitive within the pressure range dealt with (i.e. $P \approx 50$ MPa). The physical properties (TC or EWV) of such hypothetic rock can be measured on natural rocks at confining pressures high enough so that all microcracks are closed (Dvorkin et al. 1999).

3.2 Model 2: rock with microcracks

To simulate the presence of microcracks (or grain contacts), randomly located and oriented flat (oblate) spheroids are embedded into an isotropic background solid with known properties. These cavities are characterized by two parameters, namely (i) their volumetric concentration (also known as crack density), defined as $\rho = \sum R^2 / V$ (Walsh 1965), where $R$ is the radius of the spheroids and $V$ is the REV; and (ii) their geometry or aspect ratio defined as the ratio of the half-opening $a$ to the radius $R$, that is, $\varepsilon = a / R \ll 1$. The Non-Interaction Approximation (NIA) is used to upscale the effect of several spheroidal cavities. In other words, the interactions between the stress (or temperature) fields around neighbour spheroidal cavities are neglected. The validity of this assumption has been verified numerically by Grechka & Kachanov (2006) for the elastic field, provided that the aspect ratio (opening over diameter) of the spheroidal cavities is sufficiently small and the amount of cavities per unit volume does not exceed a certain threshold. As a first approximation, this approach is assumed to be valid under similar conditions in the case of the temperature field. The rock modelled in this way remains isotropic and homogeneous at the scale of an REV.

Details of the microcracked models are given in Section B3. The EMT applied to microcracked media using the NIA scheme has been extensively studied in the literature for effective elasticity (e.g. Kachanov 1993; Ravalec et al. 1996; Shafiro & Kachanov 1997; Fortin et al. 2007; Adelinet et al. 2011). The resulting elastic model equations obtained from Adelinet et al. (2011) for spheroids embedded using the NIA scheme are reported in Section B3.1 (i.e. eqs B19 and B21). In Section B3.2, the effective TC of a microcracked rock is obtained by extending Giraud et al. (2007) approach to the case of flat stress-sensitive spheroids embedded using the NIA scheme. Introducing an analogous parameter as in elasticity $\delta_{th} = \frac{\pi}{4} \left( 1 - \frac{a}{R} \right)$, the final $\lambda_{\text{sat}}^{th}$ and $\lambda_{\text{dry}}^{th}$ of the dry or fluid-saturated microcracked TC model are (eqs B22 and B24):

$$\frac{\lambda_m}{\lambda_{\text{sat}}^{th}} = 1 + \frac{16}{3} \rho \left[ 2 \left( 4 + 3\delta_{th} (3 - 4\pi \varepsilon) \right) \frac{\delta_{th}}{4 \delta_{th}^2 + 3\pi \varepsilon (1 + \delta_{th})(3 - 4\pi \varepsilon) \delta_{th}} \right]^{-1}$$

and

$$\frac{\lambda_m}{\lambda_{\text{dry}}^{th}} = 1 + \frac{8}{9} \rho \left( 4 + 3\pi \varepsilon \frac{3 - 4\pi \varepsilon}{4 - \pi \varepsilon} \right),$$

where $\lambda_m$ and $\lambda_f$ are the TC of host medium and fluid, and $\rho$ and $\varepsilon$ are the microcracks density and aspect ratio.

Eqs (B21) and (B24) highlight a fundamental difference between elasticity and TC of the dry microcracked rock. While the effective elastic moduli depend only on the density of cavities $\rho$ (e.g. Kachanov 1993), the effective TC depends on both $\rho$ and the cavities aspect ratio $\varepsilon$. This difference is thought to be rooted in the fundamental differences between Fourier’s law of thermal conduction and Hooke’s law of elastic deformation: (i) tensorial rank or (ii) diffusion versus wave propagation mechanisms.

Note that these cavities are inherently stress-sensitive due to their anisotropic geometry, and thus close under confining (Walsh 1965). The background medium into which the cavities are embedded has the properties of the rock at high confining pressures such that all microcracks are closed (Dvorkin et al. 1999), and can thus be addressed as the High Confining Pressure Medium (HCPM).

3.3 Model 3: porous rock with microcracks

Let us now consider a more realistic porous rock subjected to subsurface stresses that may not be high enough to close all microcracks. In order to model such a porous and microcracked rock, the two previous models are combined using a two-step pseudo-differential effective medium scheme (Fig. 2). As detailed in Section B4, the final Model 3 is obtained by introducing first spherical pores (Model 1) in a homogeneous background and calculating the resulting
effective properties. This porous rock (i.e., Model 1) is then used as a host for embedding microcracks (i.e., Model 2).

Similar approaches as in Section B.1 were defined in the literature (e.g. Fortin et al. 2007; Adelinet et al. 2011) to predict elasticity of porous and microcracked rocks. For effective TC, as this approach is uncommon, it is developed in Section B.2. While explicit expressions for Model 3 in the saturated situation are rather lengthy, it remains possible to obtain an explicit equation for TC in dry conditions (eq. B36):

$$\lambda_{MM}^\text{dry} = \left(1 + \phi_p \left\{ \frac{3}{2} + \frac{4 + \varepsilon}{\phi_p} \left(\frac{4 + 3 \pi \varepsilon}{4 - 4 \pi \varepsilon}\right) \right\} \right).$$

For reference, systematic predictions of effective elastic moduli and TC for various values of porosity $\phi_p$, density of flat cavities $\rho$ and their aspect ratio $\varepsilon$ for Model 3 are shown in Section B4 (Fig. B1).

4 APPLICABILITY OF THE TC MODELS

Since the introduced models accounting for microcracks in rocks (i.e., Model 2 and Model 3) are uncommon in TC usual approaches, they are tested using specific data sets available in the literature with known porosity $\phi_p$ and physical property (i.e. EWV or TC). Under the knowledge that EWV models for microcracked rocks apply well to the reality (e.g., Guégueu & Kachanov 2011), these are used as references to test TC models’ applicability. The model testing is here done by inverting the microcracks density $\rho$ and aspect ratio $\varepsilon$ for both EWV and TC using either Model 2 or Model 3.

4.1 Methodology

Each one of the two models (i.e. Model 2 and Model 3) is in fact associated to a specific approach.

In the first approach, aiming at solely testing Model 2, microcracks are embedded into a host medium for which the elastic and thermal properties are directly measured in the laboratory at high confining pressure (i.e. HCPM). As a consequence, the HCPM host medium properties (i.e. $K_m = K_{\text{HCPM}}, G_m = G_{\text{HCPM}}$ and $\lambda_m = \lambda_{\text{HCPM}}$) are introduced in Model 2 to compute the relevant effective properties. The microcracks density and aspect ratio inverted using this approach will be addressed, respectively, as $\rho_{\text{HCPM}}$ and $\varepsilon_{\text{HCPM}}$.

In the second approach, aiming at testing Model 3, the initial host medium is composed of a single mineral (e.g. quartz in a clean sandstone) in which pores then microcracks are introduced. Therefore, $K_m = K_{\text{MM}}, G_m = G_{\text{MM}}$ and $\lambda_m = \lambda_{\text{MM}}$, where the subscript MM stands for mono-mineral, are introduced in Model 3 to compute the effective properties. The microcracks density and aspect ratio inverted using this second approach will be addressed, respectively, as $\rho_{\text{MM}}$ and $\varepsilon_{\text{MM}}$.

Since the properties of the porous rock with no open microcracks are directly measured at high confining pressure rather than modelled, the first approach (and Model 2) is intuitively expected to reduce uncertainties inherently associated with an extra modelling step and related assumptions (e.g. mono-mineral composition in the second approach). Models 2 and 3 and its related approaches bear in fact differing intrinsic modelling assumptions:

1. All microcracks are closed at $P_{\text{eff}} = 50$ MPa in the HCPM (Model 2);
2. The host medium is effectively a pure isotropic mineral (Model 3);
3. Bulk porosity is made of spherical cavities (Model 3);
4. Bulk porosity is stress-insensitive (Model 3);
5. The Mori-Tanaka upscaling scheme is used to account for the presence of many pores (Model 3);
6. Microcracks are modelled as thin oblate spheroidal cavities (Models 2 and 3);
7. Microcracks are stress-sensitive (Models 2 and 3);
8. The non-interaction upscaling scheme is used to account for the presence of many microcracks (Models 2 and 3).

Model 2 relies on assumptions 1, 6, 7 and 8, whereas Model 3 relies on assumptions 2, 3, 4, 5, 6, 7 and 8. Since Model 2 (used to extract $\rho_{\text{MM}}$) involves less assumptions, it is expected to be more robust (i.e. reliable) than Model 3 (used to extract $\rho_{\text{MM}}$). Before testing the latter, Model 2 is therefore considered first to verify the consistency in modelling effective TC of a microcracked rock. A comparison between the two models-related approaches is introduced in the discussion in order to evaluate the validity of the mono-mineral assumption used for the final prediction.

4.2 Inversion method

A standard inversion method is used to identify the model parameters from experimental data reported in the literature. The solution of a given inverse problem is obtained through an iterative mini-

mization of a least-square cost-function evaluating the difference between the data and the relevant model predictions. The degrees of freedom of this cost-function are the sought model parameters. For a given rock property $P$ (P-wave velocity or TC), the cost-function $f$ is defined as the difference (in the least-square sense) between the measured and predicted values $P_{\text{meas}}$ and $P_{\text{mod}}$ of that property. Some of the model parameters (i.e. background properties and $\phi_p$) are known (directly measured) and the remaining ones (i.e. $\rho$ and $\varepsilon$) are sought such that $P_{\text{mod}}(\rho, \varepsilon) \sim P_{\text{meas}}$. Therefore, the least-square cost-function $f$ can be written as

$$f(\rho, \varepsilon) = \left[ P_{\text{mod}}(\rho, \varepsilon) - P_{\text{meas}} \right]^2.$$

This cost-function is minimized using the iterative Levenberg-Marquardt algorithm in Mathematica software package, requiring an initial set of boundary value parameters. Existence and uniqueness of the solution are not guaranteed for the non-linear inverse problem at hand. Preliminary inversion tests are therefore performed on synthetic data generated by the relevant forward model for a particular set of sought parameters. This additional test shows that, although the inverse problems are non-linear in the pursued parameters, inverting for two parameters using two measurements lead to constrained values of the parameters. In particular, it is observed that the inversion algorithm with either Model 2 or Model 3 converges to the sought set of $\rho$ and $\varepsilon$ if either ($V_{\text{p}0}^{\text{dry}}, V_{\text{p}0}^{\text{wet}}$) or ($\lambda_{\text{dry}}, \lambda_{\text{wet}}$) are known. If only $V_{\text{p}0}^{\text{dry}}$ is available, then only $\rho$ can reliably be inverted for. Note that experimental data of S-wave measurements are not used for models validation as they are usually less reliable than P-wave data due to larger uncertainties in the determination of the arrival time.

4.3 Model 2: rock with microcracks

To test Model 2, the microcracks density $\rho_{\text{HCPM}}$ and aspect ratio $\varepsilon_{\text{HCPM}}$ are obtained following the first approach (Section 4.1) and assuming all microcracks closed (i.e. HCPM) at an effective confining pressure of $P_{\text{eff}} = 50$ MPa. Using actual laboratory measurements of TC and EWV at various effective confining pressures (Tao et al. 1995; Pagoulatos & Sondergeld 2004; Mavko & Vanorio...
L. Pimienta et al.

It also demonstrates (i) how similar is the effect of these features on elastic and heat conduction properties; and (ii) the consistency of using a same EMT approach to predict both elastic and TC properties of microcracked rocks.

4.4 Model 3: porous rock with microcracks

Model 2 being assessed, Model 3 is then tested using the same experimental data set as in Section 4.3. Microcracks density $\rho_{MM}$ and aspect ratio $\varepsilon_{MM}$ are inverted for following the second approach (Section 4.1), assuming that (i) the host medium for embedding spherical pores and microcracks is a pure solid quartz; (ii) the notoriously anisotropic properties of quartz are spatially averaged, assuming random crystal and grain orientations such that the host solid is isotropic; (iii) the bulk porosity is insensitive to pressure variations, is known and is required as input. Fig. 4 reports the inversion results of: (i) $\rho_{MM}$ and $\varepsilon_{MM}$ derived from $(\lambda^{dry}, \lambda^{sat})$; (ii) $\rho_{MM}$ and $\varepsilon_{MM}$ derived from $(V_p^{dry}, V_p^{sat})$; and (iii) $\rho_{MM}$ derived from $V_p^{dry}$ alone.

Most of the observations reported for $\rho_{HCPM}$ and $\varepsilon_{HCPM}$, derived using Model 2 in Section 4.2, also hold for $\rho_{MM}$ and $\varepsilon_{MM}$ derived using Model 3. Few exceptions are worth noting though: (i) In contrast with $\rho_{HCPM}$, $\rho_{MM}$ does not cancel at the maximum pressure available $P_{eff} \sim 50$ MPa, and $\rho_{MM} > \rho_{HCPM}$ for all pressures; (ii) There is less scatter in the $\varepsilon_{MM}$ derived from the available data sets as compared to the $\varepsilon_{HCPM}$ derived from the same data sets; and (iii) While the pressure dependency of $\varepsilon_{HCPM}$ derived from either $(\lambda^{dry}, V_p^{dry})$ and aspect ratio $\lambda^{eff}$ with increasing effective pressure (e.g. Prasad & Manghnani (1995), Tao et al. (1995), Mavko & Vanorio (2010) and Pagoulatos & Sondergeld (2004). Inversions from dry and water-saturated TC (i.e. $\lambda$) and $P$-wave velocity (i.e. $V_p$) are represented as stars and dots, respectively. The squares in (B) represent the mean $\rho_{HCPM}$ inverted from dry $P$-wave velocity (i.e. $V_p$) of 13 samples, and the error-bar is the standard deviation between the inversions on those samples.

2010; Gomez et al. 2010; Lin et al. 2011) up to $P_{eff} = 50$ MPa, microcracks density $\rho_{HCPM}$ and aspect ratio $\varepsilon_{HCPM}$ are inverted for and plotted in Figs 3(A) and (B) as a function of effective pressure: (i) $\rho_{HCPM}$ and $\varepsilon_{HCPM}$ derived from $(\lambda^{dry}, \lambda^{sat})$; (ii) $\rho_{HCPM}$ and $\varepsilon_{HCPM}$ derived from $(V_p^{dry}, V_p^{sat})$; and (iii) $\rho_{HCPM}$ derived from $V_p^{dry}$ alone.

The stress-dependency of the aspect ratio $\varepsilon_{HCPM}$ as inverted from either $(\lambda^{dry}, \lambda^{sat})$ or $(V_p^{dry}, V_p^{sat})$ is introduced in Fig. 3(A). $\varepsilon_{HCPM}$ derived independently from any of the two types of data (either TC or EWV) at various pressures are of similar order of magnitude and follow similar trends with pressure. Note that at room conditions $\varepsilon_{HCPM} \sim 10^{-2}$, a value consistent with that reported by Prasad & Manghnani (1997) for this sandstone. Note also that the derived $\varepsilon_{HCPM}$ falls in the range $\varepsilon_{HCPM} \sim 10^{-2} \sim 10^{-1}$ over the available pressure range, which is similar to the range derived by Gomez et al. (2010) for Fontainebleau sandstones at similar pressures.

The inversions based on each of the two types of data $[(\lambda^{dry}, \lambda^{sat})$ on one hand, and $(V_p^{dry}, V_p^{sat})$ or $V_p^{dry}$ alone on the other hand] yield similar results in terms of $\rho_{HCPM}$, regardless of the porosity of the samples. In addition, the inverted $\rho_{HCPM}$ follows a well-documented trend with effective confining pressure, that is, (i) $\rho_{HCPM}$ decrease with increasing effective pressure (e.g. Prasad & Manghnani 1997); (ii) $\rho_{HCPM}$ cancels at high pressure ($P_{eff} \sim 50$ MPa here); and (iii) $\rho_{HCPM}$ seems to reach a plateau with increasing pressure (at $P_{eff} \sim 30$ MPa here).

These results strongly support the critical importance of tiny defects such as open microcracks on the effective properties of rocks.
and is also required as an \( \lambda \) or \((V_{p}^{\text{dry}}, V_{p}^{\text{sat}})\) are similar (i.e. stars versus dots in Fig. 3B), this is no longer true for \( \varepsilon_{\text{MM}} \) (i.e. stars versus dots in Fig. 4B). In particular, \( \varepsilon_{\text{MM}} \) derived from \((\lambda^{\text{dry}}, \lambda^{\text{sat}})\) slightly increases from \(10^{-2}\) to \(3.10^{-2}\) in the low pressure range \((P_{\text{eff}} \leq 10 \text{ MPa})\), then remains constant, whereas \( \varepsilon_{\text{MM}} \) derived from \((V_{p}^{\text{dry}}, V_{p}^{\text{sat}})\) increases by one order of magnitude from \(10^{-2}\) at atmospheric pressure to \(10^{-1}\) at \(P_{\text{eff}} \sim 30 \text{ MPa}\), then also reaches a plateau.

As for Model 2, Model 3 thus proves to behave similarly for both elasticity and TC. In particular, a similar dependence to the stress-dependent cavities is observed that can be modelled with the same underlying assumptions for both properties.

### 5 TC PREDICTION WORKFLOW

As shown in Section 4, elastic and heat conduction properties of rocks seem to bear a similar dependency on bulk porosity \( \phi \) and effective confining pressure (i.e. microcracks density \( \rho \)). This similarity appears to slightly breakdown for the aspect ratio \( \varepsilon \) (see eqs B21 versus B24). However, this difference does not seem numerically significant when comparing EWV and TC predictions using the same set of input parameters (see Fig. B1 from Section B4). This overall similarity is advantageous used in this section to build a TC prediction workflow based on the widely available EWV data from both field and laboratory investigations.

The structure of the TC prediction workflow is schematized in Fig. 5. As discussed in Section 4.4, microcracks density \( \rho_{\text{MM}} \) and aspect ratio \( \varepsilon_{\text{MM}} \) (or \( \rho_{\text{MM}} \)) are inverted for using the relevant elastic model (i.e. Model 3) from known \((P\)-wave velocities \( V_{p}^{\text{dry}} \) and \( V_{p}^{\text{sat}} \) (or \( V_{p}^{\text{dry}} \)) only); (ii) bulk porosity \( \phi_{p} \); and (iii) major mineral component (e.g. quartz) and saturating fluid (e.g. water) properties. These parameters are subsequently input into the relevant TC model (i.e. Model 3), and effective TC is calculated \((\lambda^{\text{dry}} \text{ or } \lambda^{\text{sat}})\). To assess the validity of this approach, the predicted TC is compared to direct TC measurements for a given rock under given pressure conditions. In the following, this approach is applied to quartz-rich sandstones saturated with water or air (dry).

More precisely, \( V_{p}^{\text{dry}} \) measurements are inverted to identify \( \rho_{\text{MM}} \) assuming that the sandstones’ solid phase is composed of pure quartz (mono-mineral assumption). As \( \varepsilon_{\text{MM}} \) is also required as an input for TC predictions, it is derived as follows: (i) For Berea sandstone (Fig. 4), \( \varepsilon_{\text{MM}} \) trend at atmospheric conditions yields \( \varepsilon_{\text{MM}} \sim 1.10^{-1} \); (ii) For Fontainebleau sandstone, Gomez et al. (2010) inverted electrical resistivity data for aspect ratio using an effective medium model (Fig. 12 in Gomez et al. (2010)), and found \( \varepsilon \sim 2 - 5.10^{-2} \). Since TC data in the literature are mainly measured at room pressure conditions, only values of \( \varepsilon \) under similar conditions are used.

The results of this comparison between measured and predicted TC are introduced in Fig. 6. \( \lambda_{\text{pred}} \) and \( \lambda_{\text{meas}} \) are plotted against porosity for dry and water-saturated Fontainebleau and Berea sandstones. In addition, predictions from well-known semi-empirical TC models are also plotted, namely Eucken–Maxwell (e.g. Wang & Yi 2004), geometric mean (Brigaud & Vasseur 1989), Zimmerman (Zimmerman 1989; Côté & Konrad 2009) and Coté & Konrad models (Côté & Konrad 2009).

In the dry case (Fig. 6A), Model 3 predictions fit the measurements reasonably well for Berea and quartz-rich (i.e. Fontainebleau, Berkeley, St Peters and Tensleep) sandstones. The water-saturated case (Fig. 6B) shows even better fit between predictions and measurements. Model 3 predictions thus prove to be reasonably good for both dry and water-saturated conditions knowing that the slight discrepancy observed could very well be related to (i) the use of different samples (heterogeneities); (ii) the variable measurement protocols employed in different laboratories or (iii) the measurements uncertainties.

### 6 DISCUSSION

#### 6.1 Assessment of the model assumptions

The stiffer nature of the bulk porosity as compared to the microcracks makes assumptions 4 and 7 reasonable and realistic for
sandstones. The choices of a spherical shape for modelling the stiff porosity, and a thin spheroidal shape for modelling stress-sensitive microcracks only allow the models to remain analytically and bear simple parameters. Therefore, assumptions 3 and 6 are considered as reasonable for a first-order approach. Note that assumption 8 is common to both models, and as such cannot be used to discriminate the relative robustness (i.e. reliability) of both Models 2 and 3. Assumptions 1, 2 and 5 thus remain to be verified.

### 6.1.1 Ubiquity of microcracks

In Figs 3 and 4, both $\rho_{HCPM}$ and $\rho_{MM}$ display a non-linear evolution with $P_{eff}$ in the range $0 < P_{eff} < 30$ MPa, followed by a more or less linear asymptotic trend in the range $30 \lesssim P_{eff} \lesssim 50$ MPa. The initial non-linear trend can be interpreted as an expression of thin microcracks closure, corresponding to the initial non-linear increase in the measured $P$-wave velocity (see Fig. 1). This closure threshold is similar for both models and is therefore independent from the composition of the host solid. As already pointed out by many authors (e.g. Mavko et al. 2003), the existence of a subsequent linear increase in $P$-wave velocity with pressure highlights the fact that most microcracks have closed and only stiffer pores keep deforming at a significantly smaller rate with increasing pressure. It is therefore reasonable to consider assumption 1 as rather realistic.

At this stage, let us introduce an additional set of data related to the Fontainebleau sandstone for which $P$-wave velocities are reported by Gomez et al. (2010). Fontainebleau sandstone being made of ~99 per cent of quartz, as opposed to the Berea sandstone made of ‘only’ ~89 per cent quartz, the aim is to use the former as a reference sandstone for which assumption 2 is more accurately applicable. $P$-wave velocities versus pressure reported by Gomez et al. (2010) suggest that two families of Fontainebleau sandstone samples were in fact tested: (i) samples displaying relatively moderate decrease in porosity $\Delta \phi_p$ and significant increase in $P$-wave velocity $\Delta V_p/V_p$; and (ii) samples displaying relatively significant decrease in porosity $\Delta \phi_p$ and moderate increase in $P$-wave velocity $\Delta V_p/V_p$. Fig. 7 summarizes these observations and shows the two groups of samples in the $\Delta \phi_p$ versus $\Delta V_p/V_p$ plane.

Fig. 8 shows the evolution of $\rho_{HCPM}$ as inverted using Model 2 from $P$-wave velocity measurements on Fontainebleau sandstone, and from $P$-wave velocity or TC measurements on Berea sandstone. While the inversions for Berea sandstone using ($V_p$ or $\lambda$) yield very similar values of $\rho_{HCPM}$ for all rocks, this is not the case for Fontainebleau samples. The inversions for Fontainebleau samples yield two different trends in terms of $\rho_{HCPM}$, supporting the observation reported in Fig. 7 of two existing subsets. While subset

![Figure 7](https://example.com/figure7.png)

**Figure 7.** Relative variation in $P$-wave velocity versus porosity change for dry Fontainebleau sandstone. Data from Gomez et al. (2010).

![Figure 8](https://example.com/figure8.png)

**Figure 8.** Microcracks density $\rho_{HCPM}$ inverted from either TC (i.e. stars) or EWV (i.e. dots and squares) versus effective confining pressure for Fontainebleau (i.e. dots) and Berea (i.e. squares) sandstones. Data reported here are retrieved from Lin et al. (2011), Gomez et al. (2010), Mavko & Vanorio (2010) and Pagoulatos & Sondergeld (2004).

2 (i.e. light grey dots in Fig. 7) displays a significantly lower stress sensitivity, subset 1 (i.e. dark grey dots in Fig. 7) yields a $\rho_{HCPM}$ comparable with that of Berea sandstone (i.e. larger stress sensitivity of $V_p$ and therefore of $\rho_{HCPM}$). Model 2 thus proves to apply well even in the poly-mineralic Berea sandstone. Furthermore, the inverted $\rho_{HCPM}$ seems to reach a threshold starting from $P_{eff} \sim 30$ MPa for both rocks. Assumption 1 is thus verified in the case of both Berea and Fontainebleau sandstones. Note that only the first Fontainebleau sandstone subset will be used for comparison with Berea sandstone data in the following.

### 6.1.2 Effect of mineral composition

By comparing Figs 3 and 4, it appears that (i) $\rho_{MM} \neq \rho_{HCPM}$; and (ii) $\rho_{MM} > \rho_{HCPM}$ for Berea sandstone. In principle, this deviation originates from assumptions 2 or 5 in Model 3. Therefore, $\Delta \rho = \rho_{MM} - \rho_{HCPM}$ can be used as a measure of the relevance of these assumptions. The evolution of $\Delta \rho = \rho_{MM} - \rho_{HCPM}$ with effective pressure for Berea and Fontainebleau sandstones (subset 1) is shown in Fig. 9. In general, the two models ($\rho_{HCPM}$ from Model 2 or $\rho_{MM}$ from Model 3) yield more similar results of $\Delta \rho \rightarrow 0$ as pressure increases. Although $\Delta \rho$ seems to reach a plateau at $P_{eff} \sim 30$ MPa, it does not cancel for any of these sandstones. Note however that the stress sensitivity of $\Delta \rho$ is much more significant for Berea sandstone than for Fontainebleau sandstone. Since the latter is

![Figure 9](https://example.com/figure9.png)

**Figure 9.** Difference in inverted microcracks densities using: (i) known matrix properties; or (ii) the assumption of a mono-mineral matrix. Data from Lin et al. (2011), Gomez et al. (2010) and Mavko & Vanorio (2010).
made of almost pure quartz, then the asymptotic value $\Delta \rho \sim 0.2$ for Fontainebleau sandstone can reasonably be attributable to assumption 5, namely the Mori-Tanaka upscaling scheme used to account for the presence of spherical pores making up the bulk porosity. In contrast, the asymptotic $\Delta \rho \sim 0.5$ obtained for Berea sandstone is a priori attributable to both assumptions 5 and 2 (i.e. pure mono-mineral phase). Assuming similar bulk porosity for Fontainebleau and Berea sandstones, assumption 5 can be discarded by correcting $\Delta \rho$ obtained for Berea sandstones by the value of $\Delta \rho \sim 0.2$ found for Fontainebleau sandstones. Such a corrected value of $\Delta \rho$ for Berea sandstone still exhibits (i) an added offset (i.e. distance between both asymptotes) and (ii) a slight dependency on pressure ($\Delta \rho \in [0.35, 0.2]$), which in turn are reasonably attributable to assumption 2 alone.

The density differences $\Delta \rho$ extracted from either $P$-wave velocity or TC data are also compared in Fig. 9. The two types of physical properties yield similar trends for $\Delta \rho$, although at low pressure, $\Delta \rho$ extracted from $P$-wave velocity data is notably larger than that extracted from TC data. Such difference virtually disappears as pressure increases. The difference at low pressure is either due to (i) an inherent difference between the two sets of samples tested independently by Lin et al. (2011) and Mavko & Vanorio (2010); or (ii) to a more fundamental cause related to the accuracy of the models used here; or (iii) to an inherently greater sensitivity of EWV to microcracks as compared to TC. Unfortunately, there is no mean, with the data and models currently available, to discriminate between these plausible causes.

6.3 Effect of burial depth
As described in the literature, TC displays significant dependency on burial depth (e.g. Clauser 2006), which is consistent with laboratory studies showing its pressure dependency (Lin et al. 2011). Various studies reported in the literature investigated the effects of both temperature and pressure on rock’s TC. In these studies, the pressure dependency of TC is usually accounted for through an empirical pressure-dependent porosity $\phi_p$ (e.g. Zimmerman 1989; Abdulagatova et al. 2009). In contrast, the approach developed in this paper introduces the effect of pressure through pressure-dependent microvariables based on explicit models of the macroscopic EWV and TC associated with the rock’s microstructure (i.e. microcracks).

Temperature also appears as an important factor of deviation in TC predictions (e.g. Birch & Clark 1940; Clauser & Huenges 1995; Revil 2000). This dependence of TC on temperature has been widely studied either experimentally (e.g. Abdulagatova et al. 2009), or through empirical modelling of temperature corrections (e.g. Clauser 2006). Yet, only minor temperature effects on TC exist in the temperature range relevant for shallow crustal rocks and geothermal applications ($T \leq 150 ^\circ C$). For instance, Abdulagatova et al. (2009) have shown experimentally a decrease of about $\Delta TC/\lambda < 10$ per cent in the range $T \in [20, 150] ^\circ C$ in many sandstones. EWV can also be affected by temperature for temperatures $T \leq 150 ^\circ C$. Several authors (e.g. Jones & Nur 1983; Winkler & Murphy 1995) have shown a decrease of about $\Delta V_p/V_p \sim 5\ldots10$ per cent in the range $T \in [20, 150] ^\circ C$ for both dry and water-saturated sandstones.

The present approach, by successfully predicting TC from EWV measured under similar pressure conditions, seems promising. Furthermore, the same magnitude decrease from temperature is expected for both properties. It is thus believed that if the present approach is employed, pressure and temperature conditions would be intrinsically accounted for and relevant predictions for complete in situ (pressure and temperature) conditions would be achievable.

7 CONCLUSION
In this study, EWV and TC are shown to bear a very similar pressure dependency. Using EMT, and by analogy with modelling of effective elastic properties, a new TC model accounting for microcracks has been developed and validated. Based on this model and the corresponding effective elasticity model, a TC prediction workflow has been devised.

The validation of the TC model is based on experimental data found in the literature. Experimental measurements of EWV on well-known sandstone lithologies were used to predict TC. The predicted TC was then compared to direct TC measurements. The comparison of the predictions and the measurements shows reasonably good fit for clean sandstones (quartz content $\geq 80$ per cent).

A thorough analysis of the modelling assumptions (and limitations), and their impact on the prediction workflow has been performed. It turns out that the results obtained here for TC are comparable in terms of validity and applicability with classical effective elasticity models published in the literature. Yet, the limitations associated with the TC model do not seem to affect the applicability of the predictions workflow: predict TC from known EWV. It is believed that this good applicability is due to the fact that both of these properties exhibit similar dependency on microstructural variables such as volumetric density of microcracks $p$ and their aspect ratio $\epsilon$. It is also shown that the reliability of the TC predictions increases with effective pressure or burial depth, which has an even more positive impact on practical applications at the field scale.
ACKNOWLEDGEMENTS

The authors would like to thank Prof Weiren Lin and an anonymous reviewer for their instructive comments. This work has been supported by the CSIRO-ESRE development fund.

REFERENCES


APPENDIX A: LITERATURE DATA

Ideally, in order to validate the present approach, simultaneous measurements of TC and EWV under variable effective pressures on the same rock specimens in the laboratory would be required. Unfortunately, to the authors’ knowledge, no such complete data set exists. However, different authors have measured separately and on different specimens TC and EWV on the well-known Berea and Fontainebleau sandstones. To alleviate possible issues associated with the heterogeneity of a given rock lithology, particular attention is given to the mineralogical composition and microstructure of the specimens used from the various studies reported in the literature. Furthermore, TC and EWV measurements under atmospheric conditions on other clean sandstones are also used for model and workflow validation purposes. Finally, for the purpose of comparing two distinct properties such as EWV and TC, it resulted of importance to discuss the role of experimental uncertainties of the different measurements.

A1 Berea sandstone

Many authors (Christensen & Wang 1985; Sayers et al. 1990; Tao et al. 1995; Prasad & Manghnani 1997; Pagoulatos & Sondergeld 2004; Mavko & Vanorio 2010) measured EWV on dry and water-saturated Berea sandstone specimens under variable effective pressure. TC was also measured on dry (Woodside & Messmer 1961a; Lin et al. 2011) and water-saturated (Lin et al. 2011) specimens under similar pressure conditions.

Berea sandstone has been extensively used in the experimental rock mechanics, rock physics and petrophysics communities as an analogue of actual reservoir rocks. It is characterized by (i) a porosity between 17 and 22 per cent (Tao et al. 1995); (ii) a permeability between 1 and 1000 mD (Jones & Nur 1983; Tao et al. 1995; Pagoulatos & Sondergeld 2004); (iii) a quartz content between 70 and 95 per cent (Christensen & Wang 1985; Sayers et al. 1990; Baud et al. 2004; Pagoulatos & Sondergeld 2004) and (iv) a clay content that may occasionally reach 11 per cent. Average mineral composition from 19 Berea sandstone samples is reported in Table A1 (Christensen & Wang 1985; Sayers et al. 1990; Mohan et al. 1993; Baudracco & Aoubouazza 1995; Hart & Wang 1995; Baud et al. 2004; Pagoulatos & Sondergeld 2004).

As shown through microstructural observation (e.g. Prasad & Manghnani 1997), Berea sandstone is made of angular quartz grains located at three or more grain junctions (\(\sim150\ \mu m\)) and a pore network composed of (i) relatively equant pores; and (ii) intergranular thin discontinuities (flat geometry at two-grain junctions). The latter type of feature is known to be the major contributor to the observed stress sensitivity of various physical properties such as TC (Lin et al. 2011) and EWV (e.g. Seipold et al. 1998).

Despite variable permeability and mineral content of the tested specimens, porosity and magnitude of EWVs are correlated among the various samples tested, both under dry and water-saturated conditions. As shown by Tao et al. (1995), in the dry case at high confining pressure (\(\sim50\) MPa), lower porosity samples (\(\sim17\) per cent) yield higher P-wave velocity (\(\sim4.20\) km/s) while higher porosity samples (\(\sim22\) per cent) yield lower EWV (\(\sim3.79\) km/s). Similarly,


<table>
<thead>
<tr>
<th>Berea sandstone (19 samples)</th>
<th>Mean value (per cent)</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Porosity</td>
<td>19.7</td>
<td>0.7</td>
</tr>
<tr>
<td>Quartz</td>
<td>75.6</td>
<td>9.9</td>
</tr>
<tr>
<td>Carbonates</td>
<td>4.4</td>
<td>5.3</td>
</tr>
<tr>
<td>K-feldspar and Na-plagioclase</td>
<td>6.3</td>
<td>1.3</td>
</tr>
<tr>
<td>Clay minerals</td>
<td>11.1</td>
<td>4.8</td>
</tr>
<tr>
<td>Others</td>
<td>3.4</td>
<td>2.7</td>
</tr>
</tbody>
</table>
as shown by Lin et al. (2011), TC displays a significant sensitivity to effective pressure in both dry and water-saturated conditions. However, the stress sensitivity of these two properties are very similar among the tested specimens.

A2 Other clean sandstones

Although the full data set required to validate the present modelling approach seems to be only available for Berea sandstone, relevant insight into the relationship between TC and EWV can be gained from the analysis of incomplete data sets available for other quartz-rich sandstones, namely Fontainebleau (>98 per cent quartz), Berkeley (>95 per cent quartz) and Tensleep (98 per cent quartz) sandstones (Woodside & Messmer 1961a; Zamora et al. 1993; Gomez et al. 2010). For these sandstones, only TC measurements at room pressure or under low uniaxial stress conditions (i.e. Fontainebleau) are available. Because Fontainebleau sandstone is made of nearly pure quartz (grains and cement), it is used to assess the validity of the modelling approach assuming a mono-mineralic composition (solid phase made of a single mineral) when it is applied to non-pure sandstones.

A3 Experimental uncertainties

Uncertainties associated with sample dimensions (or transducers spatial positioning) and arrival-time picking (using the pulse transmission method) usually yield accuracies in the determination of P-wave velocity up to about 1–3 per cent. This uncertainty is expected to affect accordingly the inverted microstructural parameters $\rho$ and $\epsilon$, and therefore the TC predictions presented in Section 5. However, in view of the mathematical complexity of the mathematical models involved, propagating these uncertainties in the measurements of EWV to yield uncertainties in the predictions of TC is far from being straightforward.

Experimental uncertainties affect the measurement of TC as well, although estimates of this uncertainty is not always provided by the respective authors. Note also that the TC measurements reported in the literature use different experimental methods for which the uncertainties are probably different as well: (i) line source (e.g. Sass et al. 1984; Horai & Susaki 1989; Lin et al. 2011); (ii) optical scanner (e.g. Popov et al. 1999); (iii) divided bar (e.g. Birch & Clark 1940); (iv) probe (e.g. Woodside & Messmer 1961a).

Furthermore, measurement protocols for the pure minerals (used as input in the effective medium models) were found to be quite variable among authors (Clauser 2006): for pure quartz, some authors used quartz aggregates, while some others used single crystal specimens. The measurement methods are variable as well. For pure quartz, this yields values of spatially averaged TC spanning from $\lambda_{\text{dry}} = 7.6 - 7.7$ W m$^{-1}$ K$^{-1}$ (Horai 1971; Popov et al. 1999) to $\lambda_{\text{sat}} = 8.36$ W m$^{-1}$ K$^{-1}$ (Birch & Clark 1940). Values derived from single crystal quartz samples are the average of the three principal crystal directions. Values derived from aggregate samples intrinsically involve undesired effects associated with the microstructure of the aggregate (possibly porosity, displacement microcracks and impurities). Since the evaluation of these microstructural effects is the purpose of this paper, such measurements have been disregarded and only values of quartz TC obtained from single crystals have been used. More precisely, examinations on single-crystal samples, using the divided bar method are preferred and have been used in this paper as model inputs (Table A3). This method is preferred as it is a steady-state method (as opposed to a transient method), the result of which is independent from other properties of the tested material (e.g. heat capacity). Note that TC data for sandstone samples subjected to varying pressures used here have been reported by Lin et al. (2011), and they were obtained using the line source method.

### Table A2. TC for various quartz-rich sandstones found in the literature.

<table>
<thead>
<tr>
<th>Sandstone sample #</th>
<th>Quartz content (per cent)</th>
<th>Porosity (per cent)</th>
<th>Thermal conductivity ($\lambda_{\text{dry}} = 6.485/\lambda_{\text{sat}} = 7.406$)</th>
<th>Author</th>
</tr>
</thead>
<tbody>
<tr>
<td>Berkeley</td>
<td>98–99</td>
<td>3</td>
<td>$\lambda_{\text{dry}} = 6.485/\lambda_{\text{sat}} = 7.406$</td>
<td>Woodside &amp; Messmer (1961a)</td>
</tr>
<tr>
<td>Fontainebleau (F8)</td>
<td>98–99</td>
<td>4.3</td>
<td>$\lambda_{\text{dry}} = 6.485/\lambda_{\text{sat}} = 7.406$</td>
<td>Zamora et al. (1993)</td>
</tr>
<tr>
<td>Fontainebleau (F9)</td>
<td>98–99</td>
<td>9.7</td>
<td>$\lambda_{\text{dry}} = 4.485/\lambda_{\text{sat}} = 6.15$</td>
<td>Zamora et al. (1993)</td>
</tr>
<tr>
<td>St Peters</td>
<td>98–99</td>
<td>11</td>
<td>$\lambda_{\text{dry}} = 3.556/\lambda_{\text{sat}} = 6.360$</td>
<td>Woodside &amp; Messmer (1961a)</td>
</tr>
<tr>
<td>Tensleep</td>
<td>90–95</td>
<td>15.5</td>
<td>$\lambda_{\text{dry}} = 3.038/\lambda_{\text{sat}} = 5.862$</td>
<td>Woodside &amp; Messmer (1961a)</td>
</tr>
<tr>
<td>Fontainebleau (F5)</td>
<td>98–99</td>
<td>16.2</td>
<td>$\lambda_{\text{dry}} = 3.78/\lambda_{\text{sat}} = 5.43$</td>
<td>Zamora et al. (1993)</td>
</tr>
<tr>
<td>Berea_1</td>
<td>–</td>
<td>20.7</td>
<td>$\lambda_{\text{dry}} = 2.29/\lambda_{\text{sat}} = 4.89$</td>
<td>Lin et al. (2011)</td>
</tr>
<tr>
<td>Berea_2</td>
<td>88–89</td>
<td>22</td>
<td>$\lambda_{\text{dry}} = 2.389/\lambda_{\text{sat}} = 4.477$</td>
<td>Woodside &amp; Messmer (1961a)</td>
</tr>
</tbody>
</table>

### Table A3. TC of various minerals and fluids at room conditions (Clauser 2006). The mean TC $\lambda_{\text{mean}}$ denotes the spatial average for anisotropic minerals: $(2\lambda_{\text{per}} + \lambda_{\text{par}})/3$, where $\lambda_{\text{per}}$ denotes TC perpendicular to the crystal symmetry axis whereas $\lambda_{\text{par}}$ stands for that parallel to this axis. Measurement methods: LS, line source; OS, optical scanner.

<table>
<thead>
<tr>
<th>Mineral (clay mineral)</th>
<th>Thermal conductivity ($\lambda_{\text{mean}} = 8.36$)</th>
<th>Method</th>
<th>Author</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quartz</td>
<td>$\lambda_{\text{mean}} = 8.36$</td>
<td>LS</td>
<td>Birch &amp; Clark (1940)</td>
</tr>
<tr>
<td>K-feldspar</td>
<td>$\lambda_{\text{mean}} = 2.44$</td>
<td>LS</td>
<td>Sass et al. (1971)</td>
</tr>
<tr>
<td>Na-plagioclase</td>
<td>$\lambda_{\text{mean}} = 2.37$</td>
<td>OS</td>
<td>Popov et al. (1999)</td>
</tr>
<tr>
<td>Calcite</td>
<td>$\lambda_{\text{mean}} = 2.36$</td>
<td>LS</td>
<td>Birch &amp; Clark (1940)</td>
</tr>
<tr>
<td>Dolomite</td>
<td>$\lambda_{\text{mean}} = 5.97$</td>
<td>OS</td>
<td>Popov et al. (1999)</td>
</tr>
<tr>
<td>Chlorite (clay mineral)</td>
<td>$\lambda_{\text{par}} = 1.38, \lambda_{\text{per}} = 11.1$</td>
<td>OS</td>
<td>Popov et al. (1999)</td>
</tr>
<tr>
<td>Brine</td>
<td>$\lambda_{\text{mean}} = 0.628$</td>
<td>Probe</td>
<td>Woodside &amp; Messmer (1961a)</td>
</tr>
<tr>
<td>Gas</td>
<td>$\lambda_{\text{mean}} = 0.026$</td>
<td>Probe</td>
<td>Woodside &amp; Messmer (1961a)</td>
</tr>
</tbody>
</table>

### APPENDIX B: TC MODEL

The TC model developed here follows an approach originally suggested by Torquato (2001) and developed further by Giraud et al. (2007), which relies on the EMT. In particular, Giraud et al. (2007)
developed a model for clayrocks involving aligned spheroidal clay-mineral inclusions making up an anisotropic background medium in which spherical slit inclusions are embedded. Here, this approach is simplified to model spherical and randomly oriented spherical cavities filled with a fluid and embedded in an isotropic host medium. The spherical cavities simulate the stiff (stress-insensitive) porosity while the thin oblate spheroidal cavities simulate compliant (stress-sensitive) discontinuities such as microcracks (Fig. 2). This appendix reports the main derivation steps and important results associated with the effective TC of such a rock model. For sake of completeness and comparison purposes, the corresponding results associated with effective elasticity are also recalled.

### B1 Effective TC

The derivation of the TC models used in this paper is reported, starting from the fundamental Eshelby–Hill type of equations describing the effect of an ellipsoidal inclusion embedded in a solid host medium on the temperature field (Eshelby 1957; Hill 1963; Giraud et al. 2007). In the following, bold characters denote second-rank tensors and $\delta$ is the second-rank identity tensor. Following Giraud et al. (2007), the TC tensor of a solid (subscript m) embedding a single ellipsoidal inclusion (subscript I) made of a different material can be written in terms of the second-rank concentration tensor $A^m_I$:

$$\lambda_{eff} = \lambda_m + \phi_I (\lambda_I - \lambda_m). \quad < A^m_I > .$$

where $\lambda_m$ and $\lambda_I$ stand for the TC tensors of the background solid and of the inclusion material, respectively; $\phi_I$ is the total volume fraction of the inclusion; $< A^m_I >$ denotes the average of the concentration field in that single heterogeneity $A^m_I$, rotated and averaged over the representative volume element (RVE) containing that single heterogeneity:

$$< A^m_I > = \frac{1}{V} \int \int_V R_i^m \cdot A^m_I \cdot R^T_i dV,$$

where $R$ is an orthogonal tensor representing a rotation, and $R^T$ is its transpose (or inverse). Tensor $A^m_I$ is function of the symmetric interaction tensor $P^m_I$ defined by analogy with Hill’s tensor in elasticity as

$$A^m_I = [\delta + P^m_I (\lambda_I - \lambda_m)]^{-1} .$$

The interaction tensor of a spheroidal inclusion with symmetry axis along the unit vector $e_3$, embedded in an isotropic solid is a diagonal tensor with two independent components

$$P^m_I = \frac{Q_I}{\lambda_m} (e_i \otimes e_i + e_j \otimes e_j) + \frac{1 - 2Q_I}{\lambda_m} (e_3 \otimes e_3) ,$$

where $e_i (i \in \{x, y, z\})$ are the unit vectors of the local orthonormal frame attached to the inclusion and oriented along its principal directions; and $Q_I$ is a function of the inclusion’s geometry. More precisely, $Q_p$ for a spherical pore and $Q_c$ for microcracks (i.e. oblate spheroids) are

$$\begin{align*}
Q_p &= \frac{1}{\lambda_m} \\
Q_c &= \frac{\pi \sqrt{2}}{4} .
\end{align*}$$

where $\varepsilon$ is the aspect ratio of the spheroid defined as the ratio of the small to the large axes such that $\varepsilon < 1$ for an oblate spheroid. Therefore, B4 becomes

$$\begin{align*}
P_p^m &= \frac{1}{\lambda_m} \delta \\
P_c^m &= \frac{1}{\lambda_m} \left[ \frac{\varepsilon}{\pi} \left( e_3 \otimes e_3 + e_i \otimes e_i \right) + \left( 1 - \frac{\varepsilon}{\pi} \right) \left( e_3 \otimes e_3 \right) \right].
\end{align*}$$

for spheres and oblate spheroids, respectively. Combining B6 with B3 yields

$$\begin{align*}
A^m_p &= \left( 1 + \frac{\lambda_I}{\lambda_m} \right) \delta^{-1} \\
A^m_c &= \delta + \left( \frac{\lambda_I}{\lambda_m} \right) \left[ \frac{\varepsilon}{\pi} \left( e_3 \otimes e_3 + e_i \otimes e_i \right) + \left( 1 - \frac{\varepsilon}{\pi} \right) \left( e_3 \otimes e_3 \right) \right]^{-1} .
\end{align*}$$

Note that, up to this point, the detailed equations correspond to the main steps to be found in more details in Giraud et al. (2007) article. Introducing now, for uniformity with elasticity, the ratio of fluid to solid TC $\alpha_m = \frac{\lambda_m}{\lambda_f}$ yields

$$\begin{align*}
A^m_p &= \frac{3}{\pi \alpha_m} \delta \\
A^m_c &= \delta + \left( \frac{4\alpha_m}{4\alpha_m - 1} \right) \left( e_3 \otimes e_3 + e_i \otimes e_i \right) \\
&\quad + \left( \frac{\alpha_m - 2}{4\alpha_m - 1} \right) (e_i \otimes e_i) .
\end{align*}$$

In order to compute the average, isotropic contribution $< A^m_I >$ of a randomly oriented oblate spheroid in a 3-D volume with a global reference frame defined by the unit vectors $e_i (i \in \{X, Y, Z\})$, a combination of rotation tensors is introduced

$$R = R_z (\theta_1) \cdot R_y (\psi) \cdot R_z (\theta_2) ,$$

where $R_z$ and $R_y$ are rotations of angles $\theta_1$ and $\theta_2$ about the $z$- and $y$-axes, respectively, and $R_z$ is a rotation of angle $\psi$ about the $x$-axis. This rotation tensor allows any inclusion to be rotated arbitrarily with respect to the background solid. These rotations can be represented by orthogonal tensors such that B2 becomes

$$< A^m_I > = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} \left[ \lambda(\theta_1, \psi, \theta_2) A^m_I \cdot R^T (\theta_1, \psi, \theta_2) \right] \times \sin \psi \, d\theta_1 \, d\psi \, d\theta_1 ,$$

which leads to

$$\begin{align*}
< A^m_p > &= A^m_p = \frac{3}{\pi \alpha_m} \delta \\
< A^m_c > &= A^m_c = \frac{1}{\alpha_m} \left[ \frac{1}{\alpha_m (\varepsilon^2 + 1 - \frac{1}{\alpha_m})} + \frac{4}{4\alpha_m - 1} \right] \delta .
\end{align*}$$

As expected for many spheres or randomly oriented spheroids embedded in an isotropic solid, $< A^m_p >$ and $< A^m_c >$ are isotropic tensors (diagonal tensors with a single independent component). Since the TC tensor of the host solid $\lambda_m$ and that of the inclusion (i.e. fluid) $\lambda_I$ are isotropic, $\lambda_{eff}$ in eq. (B1) is also isotropic. Therefore, from now on, eq. (B1) is used in its scalar form to calculate the effective TC of an isotropic heterogeneous rock $\lambda_{eff}$. In the following, the equations associated with the three microstructural models used in the main body of the paper are derived. Several situations are detailed: dry and fluid-saturated cavities (spherical pores or thin spheroidal microcracks) and their effect on elastic moduli and TC.

### B2 Models 1: porous rock

#### B2.1 Effective elasticity

The effective elastic moduli of a solid embedding spherical pores have been reported by Fortin et al. (2007) using the Mori-Tanaka upscaling scheme. The fluid-saturated effective bulk and shear moduli $K^p_f$ and $G^p_f$ are functions of the moduli $K_m$, $G_m$ and Poisson’s...
ratio \( v_m \) of the host solid, the bulk modulus of the fluid \( K_t \), and the bulk porosity \( \phi_t \)
\[
\begin{align*}
\frac{K_m}{K_t} &= 1 + \frac{3}{2} \frac{\phi_t (1 - v_m)}{(1 - \phi_t (1 - 2v_m))} \left( \frac{\phi_t}{1} \right), \\
\frac{G_m}{G_t} &= 1 + \frac{15}{7} \frac{\phi_t (1 - v_m)}{(1 - \phi_t (1 - 5v_m))},
\end{align*}
\]  
(B12)
where \( \alpha_{el} = K_t/K_m \) stands for the ratio of fluid-to-solid bulk moduli, and \( \delta_{p} \) accounts for the fluid compressibility (e.g. water) and the pore geometry. In the case of a spherical pore
\[
\delta_{p} = \frac{2}{9} \left( \frac{E_m}{1 - v_m} \right) \left( 1 - \frac{\alpha_{el}}{\alpha_{el} K_m} \right),
\]  
(B13)
where \( E_m \) is the Young’s modulus of the host medium. The dry case is derived from the fluid-saturated one by taking the limit as \( \alpha_{el} \to 0 \) (or \( \delta_{p} \to \infty \)) such that
\[
\begin{align*}
\frac{K_m}{K_{\text{dry}}} &= 1 + \frac{3}{2} \frac{\phi_t (1 - v_m)}{(1 - \phi_t (1 - 2v_m))} \left( \frac{\phi_t}{1} \right), \\
\frac{G_m}{G_{\text{dry}}} &= 1 + \frac{15}{7} \frac{\phi_t (1 - v_m)}{(1 - \phi_t (1 - 5v_m))},
\end{align*}
\]  
(B14)
The bulk and shear moduli are related to the longitudinal and shear wave velocities \( V_L \) and \( V_S \) through
\[
V_L = \sqrt{\frac{\delta_{p} + \frac{4}{3} \frac{G_{\text{dry}}}{\phi_t}}{d_m (1 - \phi_t) + d_f \phi_t}}; \quad V_S = \sqrt{\frac{\delta_{p}}{d_m (1 - \phi_t) + d_f \phi_t}},
\]  
(B15)
where \( d_m \) is the average mass density of the minerals constituting the background solid, and \( d_f \) is the mass density of the saturating fluid (\( d_f \sim 0 \) in the dry case).

B2.2 Effective TC
The effective TC of a solid embedding spherical pores is derived from equations (B1)–(B11). The fluid-saturated TC \( K_{th} \) is a function of the TC of the host solid \( \lambda_{th} \), that of the fluid \( \lambda_{th} \), and the bulk porosity \( \phi_t \)
\[
\begin{align*}
\lambda_{th}^p &= \lambda_m \frac{2(1 + \alpha_{el}) - 2\phi_t (1 - \alpha_{th})}{(2 + \alpha_{el}) + \phi_t (1 - \alpha_{th})},
\end{align*}
\]  
(B16)
where \( \alpha_{el} = \lambda_{el}/\lambda_{th} \) stands for the ratio of fluid-to-solid thermal conductivities. Note that eq. (B16) is often referred to as the Eucken–Maxwell equation (e.g. Wang et al. 2006). It is widely used in the literature as the basis for various semi-empirical TC models for which deviations between predicted and measured thermal conductivities are accounted for by introducing additional empirical parameters (e.g. Wang & Yi 2004). Eq. (B16) can be rewritten in a form similar to that of the effective elastic moduli given by (B12) such that
\[
\begin{align*}
\frac{\lambda_m}{\lambda_{th}} &= 1 + \frac{3 \phi_t (1 - \alpha_{th})}{(2 + \alpha_{el}) - 2 \phi_t (1 - \alpha_{th})}.
\end{align*}
\]  
(B17)
The dry case is derived from the fluid-saturated one by taking the limit as \( \alpha_{el} \to 0 \) such that
\[
\begin{align*}
\frac{\lambda_m}{\lambda_{th}} &= 1 + \frac{3 \phi_t}{2(1 - \phi_t)}.
\end{align*}
\]  
(B18)

B3 Models 2: rock with microcracks

B3.1 Effective elasticity
The effective elastic moduli of a solid embedding oblate spheroidal cavities have been reported by Adelinet et al. (2011) using the NIA upscaling scheme. The fluid-saturated effective bulk and shear moduli \( K_{sat} \) and \( G_{sat} \) are functions of the moduli \( K_m \), \( G_m \) and Poisson’s ratio \( v_m \) of the host medium, the bulk modulus of the fluid \( K_t \), the volumetric density of microcracks \( \rho \) and their aspect ratio \( \varepsilon \)
\[
\begin{align*}
\frac{K_{sat}}{K_m} &= 1 + \frac{16}{9} \frac{(1 - v_m)}{(1 - 2v_m)} \left( \frac{\delta_{p}}{1} \right), \\
\frac{G_{sat}}{G_m} &= 1 + \frac{16}{9} \frac{(1 - v_m)}{(1 - 2v_m)} \left( \frac{\delta_{p}}{1} \right),
\end{align*}
\]  
(B19)
where \( \alpha_{el} = K_t/K_m \) stands for the ratio of fluid-to-solid bulk moduli, and \( \delta_{p} \) accounts for the fluid compressibility (e.g. water) and the microcracks geometry characterized by the aspect ratio \( \varepsilon \). In the case of an oblate spheroid
\[
\delta_{p} = \frac{\pi \varepsilon}{4} \left( \frac{E_m}{1 - v_m} \right) \left( 1 - \frac{\alpha_{el}}{\alpha_{el} K_m} \right),
\]  
(B20)
where \( E_m \) is the Young’s modulus of the host medium. The dry case is derived from the fluid-saturated one by taking the limit as \( \alpha_{el} \to 0 \) such that
\[
\begin{align*}
\frac{K_{sat}}{K_{dry}} &= 1 + \frac{16}{9} \frac{(1 - v_m)}{(1 - 2v_m)} \left( \frac{\delta_{p}}{1} \right), \\
\frac{G_{sat}}{G_{dry}} &= 1 + \frac{16}{9} \frac{(1 - v_m)}{(1 - 2v_m)} \left( \frac{\delta_{p}}{1} \right).
\end{align*}
\]  
(B21)
Note that in the dry case, \( K_{sat}^c \) and \( G_{sat}^c \) are independent of \( \varepsilon \).

B3.2 Effective TC
The effective TC of a solid embedding randomly oriented spheroidal cavities is derived from equations (B1)–(B11). The fluid-saturated TC \( K_{sat} \) is a function of the TC of the host solid \( \lambda_{sat} \), that of the fluid \( \lambda_{th} \), the volumetric density of microcracks \( \rho \) and their aspect ratio \( \varepsilon \)
\[
\lambda_{th} = 1 + \frac{16}{3} \frac{(4 + 3 \delta_{th} (4 - \pi \varepsilon))}{4 \delta_{th} + 3 \pi \varepsilon (1 + \delta_{th})} \left( \frac{1}{3 - 4 \pi \varepsilon \rho \delta_{th}} \right)^{-1},
\]  
(B22)
where \( \delta_{th} \) accounts for the TC of the fluid (\( \alpha_{th} = \lambda_{th}/\lambda_{th} \)) and the microcracks geometry characterized by the aspect ratio \( \varepsilon \) such that
\[
\delta_{th} = \frac{\pi \varepsilon}{4} \left( \frac{1 - \alpha_{th}}{\alpha_{th}} \right).
\]  
(B23)
The dry case is derived from the fluid-saturated one by taking the limit as \( \alpha_{th} \to 0 \) such that
\[
\frac{\lambda_{th}}{\lambda_{dry}} = 1 + \frac{8}{9} \rho \left( \frac{4 + 3 \pi \varepsilon}{4 - \pi \varepsilon} \right).
\]  
(B24)
This last result contrasts with the effective elastic moduli in this particular dry situation, although the microstructural models and the derivations are similar for the elastic and thermal properties. Whereas the TC is an explicit function of the aspect ratio \( \varepsilon \) (eq. B24), the elastic moduli are not (eq. B21). It is believed that this difference is rooted in the formal difference between Fourier’s law of thermal conduction (diffusion-type) and Hooke’s law of elastic deformation (propagation-type).

Furthermore, applying to eq. (B24) the limiting case of \( \varepsilon \to 0 \) (closed but not cemented microcracks), leads to
\[
\frac{\lambda_{th}}{\lambda_{dry}} = 1 + \frac{8}{9} \rho,
\]  
(B25)
which coincides with a result obtained by Bristow (1960) for the effective electrical conductivity of a microcracked composite material using the non-interaction upscaling scheme.
B4 Model 3: porous rock with microcracks

The model for rocks containing both stiff porosity (spheres) and compliant microcracks (thin oblate spheroids) is built in two steps: (i) stiff pores are first embedded in a homogeneous host solid using the Mori-Tanaka scheme (Model 1 above); then (ii) microcracks are subsequently embedded using the non-interaction scheme, with the porous rock from the previous step acting as the host medium. This method is called here pseudo-differential upscaling scheme as the interactions between pores and microcracks are accounted for in an approximate manner through this two-step procedure.

B4.1 Effective elasticity

Using eq. (B12) with a host solid made of a pure mineral (quartz), \( K_m = K_{MM} \) and \( G_m = G_{MM} \), leads to the elastic moduli of the water-saturated porous rock \( K_{sat} \) and \( G_{sat} \):

\[
\begin{align*}
\frac{K_{MM}}{K_{sat}} &= 1 + \frac{3\phi_h (1 - \alpha_{MM})}{2 (1 + \phi_h) (1 - 2\alpha_{MM})} \left( \frac{\phi_h}{1 + \phi_h} \right), \\
\frac{G_{MM}}{G_{sat}} &= 1 + \frac{3\phi_h (1 - \alpha_{MM})}{2 (1 + \phi_h) (1 - 2\alpha_{MM})} \left( \frac{\phi_h}{1 + \phi_h} \right),
\end{align*}
\] (B26)

where

\[
\begin{align*}
\delta_{M1} &= \frac{2}{9} \left( 1 - \alpha_{MM} \right) \left( \frac{\phi_h}{1 + \phi_h} \right), \\
\alpha_{M1} &= \frac{K_{MM}}{K_{sat}}.
\end{align*}
\] (B27)

Replacing \( K_m \) and \( G_m \) in eq. (B19) by the above \( K_{sat} \) and \( G_{sat} \) leads to the effective elastic moduli of the water-saturated rock containing pores and microcracks \( K_{eff}^p \) and \( G_{eff}^p \):

\[
\frac{K_{p}}{K_{eff}^p} = 1 + \frac{3\phi_h (1 - \alpha_{MM})}{2 (1 + \phi_h) (1 - 2\alpha_{MM})} \left( \frac{\phi_h}{1 + \phi_h} \right),
\]
\[
\frac{G_{p}}{G_{eff}^p} = 1 + \frac{3\phi_h (1 - \alpha_{MM})}{2 (1 + \phi_h) (1 - 2\alpha_{MM})} \left( \frac{\phi_h}{1 + \phi_h} \right),
\] (B28)

where

\[
\begin{align*}
\delta_{p1} &= \frac{3\phi_h (1 - \alpha_{MM})}{2 (1 + \phi_h) (1 - 2\alpha_{MM})} \left( \frac{\phi_h}{1 + \phi_h} \right), \\
\alpha_{p1} &= \frac{K_{sat}}{K_{eff}^p}, \\
\end{align*}
\] (B29)

and

\[
\begin{align*}
\delta_{p2} &= \frac{9\phi_h^2 G_{eff}^p}{3 K_{sat} + G_{sat}}, \\
\alpha_{p2} &= \frac{K_{eff}^p}{K_{sat}},
\end{align*}
\] (B30)

The dry case is derived from the fluid-saturated one by taking the limit \( K_f \to 0 \) in the above equations.

B4.2 Effective TC

Using eq. (B17) with a host solid made of a pure mineral (quartz), \( \lambda_m = \lambda_{MM} \), leads to the TC of the water-saturated porous rock \( \lambda_{sat}^f \):

\[
\frac{\lambda_{MM}}{\lambda_{sat}^f} = 1 + \frac{3\phi_h (1 - \alpha_{MM})}{2 (1 + \phi_h) (1 - 2\alpha_{MM})} \left( \frac{\phi_h}{1 + \phi_h} \right),
\] (B31)

where

\[
\alpha_{MM} = \frac{\lambda_{sat}^f}{\lambda_{MM}}.
\] (B32)

Replacing \( \lambda_m \) in eq. (B22) by the above \( \lambda_{sat}^f \) leads to the effective TC of the water-saturated rock containing pores and microcracks \( \lambda_{eff}^f \):

\[
\frac{\lambda_{MM}}{\lambda_{eff}^f} = 1 + \frac{3\phi_h (1 - \alpha_{MM})}{2 (1 + \phi_h) (1 - 2\alpha_{MM})} \left( \frac{\phi_h}{1 + \phi_h} \right),
\]
\[
\frac{\lambda_{MM}}{\lambda_{eff}^f} = 1 + \frac{3\phi_h (1 - \alpha_{MM})}{2 (1 + \phi_h) (1 - 2\alpha_{MM})} \left( \frac{\phi_h}{1 + \phi_h} \right),
\] (B31)

where

\[
\alpha_{eff}^f = \frac{\lambda_{eff}^f}{\lambda_{MM}}.
\] (B32)

Explicit expressions of \( \lambda_{eff}^f \), \( K_{eff}^p \), \( G_{eff}^p \), \( \lambda_{MM} \), and \( G_{MM} \) as a function of the solid and fluid properties for Model 3 are rather lengthy and cannot practically be reported here. Equations for all models have been implemented in the Mathematica\textsuperscript{TM} software package. This software package has also been used to generate the predictions and perform the inversions of experimental data.

As a direct output of the model implementation, Fig. B1 shows variations of the three effective properties \( \lambda_{eff}^f \), \( K_{eff}^p \) and \( G_{eff}^p \) as a function of (i) porosity; (ii) saturation (dry or water-saturated); (iii) microcracks density \( \rho \) and (iv) aspect ratio \( \epsilon \).

In the dry case (i.e. dashed lines), the effective properties \( \lambda_{eff}^f \), \( K_{eff}^p \) and \( G_{eff}^p \) are strong functions of \( \rho \). As shown from the derived equation B36, \( \lambda_{eff}^f \) is a weaker function of \( \epsilon \) (i.e. compare Figs B1a and b). On the other hand, \( K_{eff}^p \) (i.e. Figs B1c and d) and \( G_{eff}^p \) (i.e. Figs B1e and f) are not functions of \( \epsilon \). This is consistent with previous models (e.g. Shafiro & Kachanov 1997; Fortin et al. 2007).
In the water-saturated case (i.e. plain lines), effective properties exhibit a stronger dependency on $\varepsilon$, that is, (i) for a low value of $\varepsilon = 10^{-3}$, $\rho$ has almost no effect on $\lambda_{\text{sat}}^{\text{eff}}$, a weak effect on $K_{\text{sat}}^{\text{eff}}$ and a strong effect on $G_{\text{sat}}^{\text{eff}}$; (ii) for a higher value of $\varepsilon = 5 \times 10^{-2}$, the dependency of $\lambda_{\text{sat}}^{\text{eff}}$ and $K_{\text{sat}}^{\text{eff}}$ on $\rho$ increases, whereas $G_{\text{sat}}^{\text{eff}}$ is not significantly affected.

The elasticity results $K_{\text{sat}}^{\text{eff}}$, $G_{\text{sat}}^{\text{eff}}$, $K_{\text{dry}}^{\text{eff}}$ and $G_{\text{dry}}^{\text{eff}}$ are overall comparable to the corresponding results published by Fortin et al. (2007). Note however that the shear modulus (i.e. Figs B1e and f) exhibits slight differences, most probably related to the difference in the crack model used by these authors, that is, penny-shaped cracks as opposed to oblate spheroids here.