Non-linear resonant coupling of tsunami edge waves using stochastic earthquake source models

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SUMMARY
Non-linear resonant coupling of edge waves can occur with tsunamis generated by large-magnitude subduction zone earthquakes. Earthquake rupture zones that straddle beneath the coastline of continental margins are particularly efficient at generating tsunami edge waves. Using a stochastic model for earthquake slip, it is shown that a wide range of edge-wave modes and wavenumbers can be excited, depending on the variability of slip. If two modes are present that satisfy resonance conditions, then a third mode can gradually increase in amplitude over time, even if the earthquake did not originally excite that edge-wave mode. These three edge waves form a resonant triad that can cause unexpected variations in tsunami amplitude long after the first arrival. An $M \sim 9$, 1100 km-long continental subduction zone earthquake is considered as a test case. For the least-variable slip examined involving a Gaussian random variable, the dominant resonant triad includes a high-amplitude fundamental mode wave with wavenumber associated with the along-strike dimension of rupture. The two other waves that make up this triad include subharmonic waves, one of fundamental mode and the other of mode 2 or 3. For the most variable slip examined involving a Cauchy-distributed random variable, the dominant triads involve higher wavenumbers and modes because subevents, rather than the overall rupture dimension, control the excitation of edge waves. Calculation of the resonant period for energy transfer determines which cases resonant coupling may be instrumentally observed. For low-mode triads, the maximum transfer of energy occurs approximately 20–30 wave periods after the first arrival and thus may be observed prior to the tsunami coda being completely attenuated. Therefore, under certain circumstances the necessary ingredients for resonant coupling of tsunami edge waves exist, indicating that resonant triads may be observable and implicated in late, large-amplitude tsunami arrivals.

Key words: Non-linear differential equations; Probability distributions; Tsunamis; Earthquake source observations.

INTRODUCTION
Edge waves are waves trapped by refraction that propagate parallel to the shoreline and occur in discrete modes with maximum amplitude occurring at the coast. In the 1950s, Greenspan (1956) and Munk et al. (1956) showed that large-scale edge waves can be generated by atmospheric pressure disturbances propagating along continental margins. Later, it was demonstrated that tsunamis are particularly effective at generating large-scale edge waves (Abe & Ishii 1987; Golovachev et al. 1992), both from local (Carrier 1995; Fujima et al. 2000) and far-field sources (Bricker et al. 2007; Horrillo et al. 2008). Especially along oblique propagation paths in the near field and in the far field, the maximum amplitude associated with a tsunami is often ascribed to the arrival of edge waves, occurring later than direct arrivals and thus, quite hazardous (Geist 2009).

In a previous study, Geist (2013) examined near-field edge waves generated by subduction zone earthquakes, using slip distributions specified by crack and stochastic rupture models and linear wave theory. One of the primary findings from that study is that edge-wave energy is dependent on the downdip location of the rupture zone relative to the shoreline. In addition, introduction of stochastic models for slip within the rupture zone result in significant variation of edge-wave amplitudes, as they do for broadside, non-trapped tsunami arrivals (Geist 2002). Essentially, the stochastic models result in subevents in terms of coseismic displacement and tsunami generation. Linear wave theory indicates that these subevents can constructively interfere, both for non-trapped and trapped waves, resulting in regions of high tsunami amplitude and runup along propagation paths oblique to the rupture area.

Previous investigations of edge waves using weakly non-linear theory has primarily focused on (1) possible modulation...
instabilities and (2) mode interaction or resonant coupling. For the former, Akylas (1983) demonstrates by way of the non-linear Schrödinger equations that edge waves can be subject to unstable, large-scale modulation perturbations, forming wave-envelope solitons, although laboratory experiments by Yeh (1985) indicate that such solitons develop asymmetrically and eventually breakdown. In addition, Dubinina et al. (2004) indicate that, for the same wave steepness, higher edge-wave modes exhibit less of the effects from non-linear modulation instability. In terms of mode interactions (2), Kenyon (1970) describes how the general energy transfer equations developed by Hasselmann (1966) are applied for edge waves and demonstrates the existence of coupling among three edge-wave modes termed a triad. Kirby et al. (1998) and Dubinina et al. (2006) further develop the theory of edge-wave triads, described in the next section. Pelinovsky et al. (2010) review these and other mechanisms that potentially cause the occurrence of freak edge waves. Observationally, Hsieh & Mysak (1980) associated the three dominant peaks in frequency for continental shelf waves (akin to edge waves) observed by Cutchin & Smith (1973) along the Oregon coast to resonant coupling.

In this study, the theory of non-linear resonant coupling among edge-wave modes is extended to tsunami-induced edge waves. The primary focus is on edge waves caused by continental subduction zone earthquakes (i.e. as an initial-value problem), although offshore landslide sources are also known to generate edge waves (Lynett & Liu 2005). As with Geist (2013), stochastic earthquake slip models are used for this study, although a broader range of models, in terms of slip variability, is considered here. In addition, a longer rupture is used as a test case in this study, similar to the rupture dimension of the M ~ 9 1700 Cascadia earthquake (Satake et al. 2003). The longer rupture length compared to Geist (2013) allows for a broader range of edge-wave modes and wavenumbers that are potentially generated by the stochastic model. This in turn allows the determination of the range of resonance periods for energy to transfer among modes for tsunamis generated by a large subduction zone earthquake. Although not a specific focus of this study, edge waves are also generated at far-field sites in combination with other effects such as shelf and bay resonance (Rabinovich 1997), such that tsunami waves are active over a long period of time that is characterized by an exponentially decaying coda with an e-folding time of 13–45 hr (Rabinovich et al. 2011). Islands too can trap edge waves with their own characteristics (Tinti & Vannini 1995). The overall objective of this study, however, is to determine whether resonant coupling among near-field edge waves is possible for large-magnitude, subduction zone earthquakes along continental margins.

**BACKGROUND**

Edge waves propagate in the alongshore direction, decay exponentially in the offshore direction, and occur in discrete modes. For constant slope ($\beta$), the amplitude equation for edge waves is separated between a cross-shore term $\eta(x)$ and an oscillatory term in time ($t$) and in the alongshore direction ($y$):

$$\xi(x, y, t) = \eta(x) e^{i(ky-\omega t)},$$

where $\omega$ is frequency and $k$ is the wavenumber in the alongshore ($y$) direction. $\omega$ and $k$ can have either positive or negative values, depending on the direction of propagation. The cross-shore wave profile is given by

$$\eta(x) = e^{-|k|z} L_n(2|k|x),$$

where $L_n$ is the Laguerre polynomial of order $n$. The foundation of resonant coupling described in this study is the branched dispersion relation for edge waves. In contrast, the simple dispersion relation for shallow-water, non-trapped tsunami waves will not allow for resonant coupling. The dispersion relation associated with edge waves for a constant slope is given by

$$\omega_n^2 = g k (2n+1) \tan \beta, n = 0, 1, 2, \ldots$$

in the shallow-water regime and

$$\omega_n^2 = g k \sin(2n+1) \beta, n = 0, 1, 2, \ldots$$

without the long-wave assumption (Ursell 1952). For shallow slopes and low mode numbers, the two dispersion relations are approximately equal (Leblond & Mysak 1978).

Edge waves caused by tsunami sources, most notably continental subduction zone earthquakes, can be treated as an initial-value problem in the near field. Assuming a periodic component of amplitude parallel to shore and in time, the complete wavefield can be given by Carrier (1995) and Fujima et al. (2000)

$$\xi(x, y, t) = \int_{-\infty}^{\infty} \sum_{n = 0}^{\infty} a_n(|k|) e^{-|k|z} \times L_n(2|k|x) e^{-i\omega_n t} e^{i\omega_n t} dk,$$

with coefficients (Carrier 1995)

$$a_n(|k|) = |k| \int_{0}^{\infty} c_l(x) e^{-x|k|} L_n(2|x|) dx,$$

where $c_l(x)$ are the Fourier coefficients of the source at $t = 0$.

**Stochastic slip**

For earthquake-generated tsunamis, coseismic displacement of the seafloor determines the initial wavefield. Coseismic displacement occurs as an elastic response of the earth to slip within the fault rupture zone of an earthquake. There is relatively little uncertainty in how the elastic earth behaves to static slip values, although realistic earthquake structure does have an influence (Geist & Yoshioka 1996; Geist 1999). In contrast, there is high aleatoric uncertainty in how a subduction megathrust slips during an earthquake. Various stochastic slip models constrained by earthquake observations have been proposed that can be used as input to elastic deformation models in calculating coseismic displacement of the seafloor. In general, stochastic slip $D(x)$ within a rupture zone is given by Liu-Zeng et al. (2005) and Lavallée et al. (2006)

$$D(x) = D_0 F^{-1} [R(k) D(k)],$$

with random variable $R$, constant $D_0$, inverse Fourier transform $F^{-1}$ (radial wavenumber space) and slip spectrum given by (after Herrero & Bernard 1994)

$$D(k) \propto \Delta \sigma \times \frac{L}{G k^v}.$$  

The slip spectrum above is scaled by physical parameters: $\Delta \sigma$ is the mean stress drop of the earthquake, $G$ is the shear modulus and $L$ is the fault dimension. $v$ is the spectral decay exponent, which canonically is 2 in two dimension (Andrews 1980), but can also be considered a variable parameter (Tsai 1997; Lavallée et al. 2006).

Originally, the random variable $R(x)$ (with Fourier transform $R(k)$ indicated in eq. 7) was assumed to follow a Gaussian distribution (i.e. white noise). However, Lavallée & Archuleta (2003) noted that slip inversions exhibit significantly more variability than provided
by the Gaussian model. They propose a more general model in which \( R(x) \) follows a Lévy \( \alpha \)-stable distribution family. There are several forms of the characteristic function for this distribution. The one used in this study is given by

\[
\varphi(k) = \begin{cases} 
\exp \left[ i\lambda k - c |k| \right] & \text{if } \alpha = 1 \\
\exp \left[ i\lambda k - c\alpha |k| \right] & \text{if } \alpha \neq 1
\end{cases} \tag{9}
\]

where \( \alpha \) is a stability parameter (\( 0 < \alpha \leq 2 \)), \( \theta \) a skewness parameter (\( -1 \leq \theta \leq 1 \)), \( \lambda \) a location parameter (real number) and \( c \) a scale parameter (\( c > 0 \)). Gaussian (\( \alpha = 2 \)) and Cauchy (\( \alpha = 1; \theta = 0 \)) distributions are special cases of the Lévy \( \alpha \)-stable distribution.

The stability parameter \( \alpha \) controls the asymptotic behaviour of the distribution with lower values resulting in heavier tails and hence, more variability in slip. Slip inversions typically result in values of \( \alpha \) between 1 and 2 (Lavallée et al. 2006; Lavallée et al. 2011); that is, heavy-tailed distributions compared to a Gaussian distribution.

Geist (2013) uses a stochastic model specified by a \( v = 2 \) spectral decay exponent and Cauchy random variables to examine near-field edge waves. As with this previous study, analytic expressions by Okada (1985) are used to calculate coseismic deformation for static deformation of a homogeneous elastic half-space. The static stochastic slip model, in comparison to slip associated with a crack model (i.e. constant stress drop), introduces substantial variability in maximum edge-wave amplitudes (Geist 2013). In particular, the stochastic model generates higher edge waves. As indicated below, for longer subduction earthquake ruptures than considered by Geist (2013), the Lévy \( \alpha \)-stable distributions (\( \alpha < 2 \)) produce an even larger range of edge-wave modes, whereas Gaussian random variables tend to concentrate energy at the fundamental mode.

**Resonant coupling**

For waves with multibranch dispersive relations in general, non-linear resonant coupling has been investigated for many decades, including notable papers by Bretheron (1964) who derived the temporal characteristics of energy exchange within a wave triad and by Hasselmann (1966) who more generally developed the criteria for wave interactions and the equations for energy transfer. Resonance occurs within a triad if (1) wavenumber and frequency resonance conditions are met and (2) the coupling (interaction) coefficients are non-zero. The first condition involves simple sum and difference equations for frequencies and wavenumbers of the three waves. The coupling coefficients indicated for the second condition are part of the dynamic equations for energy transfer and are specific to individual triads. For edge waves, the coupling coefficients depend on the cross-shore wave profile, indicated in eq. (2). In addition to triads, clusters of three-wave interactions are possible (Kartashova 2011).

More recently, Kirby et al. (1998) and Dubina et al. (2006) provide a focused investigation of resonant coupling among edge waves. Kirby et al. (1998) provide equations for the coupling coefficients specific to edge waves. In addition, using a pseudospectral model based on the non-linear shallow-water wave equations, they numerically verify that resonant coupling occurs for the lowest order triad involving counter-propagating waves. Dubina et al. (2006) provide wavenumbers and coupling coefficients for 34 triads through mode 4 for a constant bathymetric slope (Table 1). They demonstrate that although some triads involving unidirectional-propagating edge waves result in coupling coefficients equal to zero, there are other cases where unidirectional triads result in non-zero coupling coefficients and non-linear resonance.

**METHOD**

The combination of stochastic slip models and calculation of coseismic displacement is used to determine the distribution of edge-wave energy among different modes (discrete) and wavenumbers (continuous) (eq. 6). It is assumed that attenuation from turbulence in the form of bottom friction and wave breaking is insignificant. It also assumed that there is no dissipation from wave–wave interaction.

The first condition for resonance among edge-wave modes involves addition and subtraction relationships for wavenumber and frequency:

\[
\pm k_i \pm k_m = k_n \pm \alpha \omega_i' \pm \omega_m^* = \omega_n^* \tag{10}
\]

Here and in the equations that follow, the subscript is a wave index number (1)–(3) and the superscript represents the edge-wave mode number. Resonant coupling may involve both unidirectional- and counter-propagating edge waves as demonstrated by Dubina et al. (2006). For tsunamis, counter-propagating edge waves in the near field can be envisioned as being generated by backscattering from coastline irregularities. Forward-scattered trapped waves may be preferentially attenuated in comparison to backscattered waves and both contain more power than leaky modes (i.e. modes that are incompletely refracted with some energy leaking to the deep ocean) (Fuller & Mysak 1977). Using the shallow-water form of the dispersion relation (eq. 3), the above equations can be written in Diophantine form (Kartashova 2011) and solved using numerical methods (Smart 1998). Dubina et al. (2006) provide the resonance wavenumbers for triads using the four lowest edge-wave modes. Values for the wavenumbers in each resonant triad listed in Table 1 reproduce those results in exact form.

To understand the dynamics of non-linear resonance, multiple timescales are introduced (Mei et al. 2005):

\[
t \to t + \epsilon t = t + T, \tag{11}
\]

where \( \epsilon \) is the weak non-linearity small parameter. The interaction equation that governs the ‘slow’ time evolution of edge-wave amplitude of \( O(\epsilon) \) is given by Kirby et al. (1998)

\[
\frac{\partial A_n^*}{\partial T} = i \sum_{l=1}^T \sum_{m} \sum_{p} \sum_{q} \left\{ \begin{array}{c} T_{\text{lin}}^{np} A_l^m A_n^* \delta(l + m - n) \\
\times \delta \left( \omega_l^p - \omega_m^p - \omega_n^* \right) \\
+ T_{\text{lin}}^{np} A_l^m A_n^* \delta(m - l - n) \delta \left( \omega_l^p - \omega_m^p - \omega_n^* \right) \\
+ T_{\text{lin}}^{np} A_l^m A_n^* \delta(m - l - n) \delta \left( \omega_m^* - \omega_l^p - \omega_n^* \right) \end{array} \right\} \tag{12}
\]

where \( \delta \) is Dirac’s delta function, \( T \) and \( T_p \) are the interaction coefficients for the sum and difference interactions, respectively and \( A_n^* \) is the slowly varying amplitude for wave \( n \) and mode \( r \) (eq. 6). The amplitude solution for the linear system \( \zeta^{(1)} \) is given by

\[
\zeta^{(1)} = \sum_{n} \sum_{r} \left\{ \frac{1}{2} A_n^r(T) F_n^{(r)}(\epsilon) e^{i(\omega_n^r - \omega_n^*) \epsilon} + \text{c.c.} \right\}, \tag{13}
\]

where \( F_n^{(r)}(\epsilon) \) are the eigenmodes given by eq. (2) and c.c. stands for complex conjugate. The complete amplitude solution including non-linear terms (\( \zeta \)) is given by the asymptotic series expansion

\[
\zeta = \zeta^{(1)} + \epsilon \zeta^{(2)} + \epsilon^2 \ldots \tag{14}
\]
Table 1. Wavenumbers and coupling coefficients of triad combinations through mode 4 indicated by Dubinina et al. (2006; their table 2). Modified by including exact expressions for wavenumber and results of dominant triad search for each stochastic source model (last column—number in parentheses indicates number of found instances from triad search out of 20 runs). Rows shaded where coupling coefficients are zero (i.e. no resonance).

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Wavenumber</th>
<th>Coefficientsa</th>
<th>Stochastic Modelb</th>
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<td>n₁ n₂ n₃</td>
<td>k₁/k₃</td>
<td>μ₁/μ</td>
<td>μ₂/μ</td>
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aμ = k₁√2k₂tanβ.
bG-Gaussian; s-Lévy α stable; c-Cauchy.

These coefficients are typically calculated for a constant slope (Dubinina et al. 2006), although coefficients have also been derived for a step-like slope (Dubinina et al. 2008) or an exponential slope in the case of shelf waves (Hsieh & Mysak 1980). Coefficients for combinations of the first four modes, assuming a constant slope, are given in Table 1 (Dubinina et al. 2006). Triads that have coupling coefficients equal to zero and therefore do not result in resonant coupling are shaded in Table 1.

In the case of a single triad and under the conditions of weak non-linearity and simple bathymetry, closed-form expressions have been derived for the evolution of non-linear resonant coupling. Considering only time dependencies, eq. (12) reduces to the following

\[ F_{\text{p}} = \omega F_0 \left[ 8 \omega_0^2 \int_0^\infty (F_n^r)^{1/2} dx \right]^{-1} \]

\[ \int_0^\infty \left[ 2 (\omega F)^r + \omega F_0^r F_n^r \right] dx \]

\[ + \left[ 2 (\omega F)^r - \omega F_0^r k_1^r \right] \left[ \mp k_0^r - \omega a_0^r (k_0^r)^2 \right] \left[ F_0^r F_n^r \right] \]

\[ \right] \]
slow amplitude equations:
\[
\begin{align*}
\frac{dA_1}{dT} &= i\mu_1 A_2^* A_3 \\
\frac{dA_2}{dT} &= i\mu_2 A_1^* A_3 \\
\frac{dA_3}{dT} &= i\mu_3 A_1 A_2
\end{align*}
\]  

(16)

where \(\mu_i\) are the simplified coupling coefficients. In order to solve these equations, the Manley–Rowe invariants based on conservation of energy are also needed (Kartashova 2011):
\[
\begin{align*}
I_{12} &= |A_1|^2 - |A_2|^2 \\
I_{13} &= |A_1|^2 + |A_3|^2 \\
I_{23} &= |A_2|^2 + |A_3|^2
\end{align*}
\]

(17)

Two of the three invariants are independent. Let \(A = a \exp(i\theta)\), where \(a\) and \(\theta\) are real. For constant phase where energy transfer is maximized (\(\theta = \pi/2\)) these equations reduce to
\[
\begin{align*}
\frac{da_1}{dT} &= \mu_1 a_2 a_3 \\
\frac{da_2}{dT} &= \mu_2 a_1 a_3 \\
\frac{da_3}{dT} &= -\mu_3 a_1 a_2
\end{align*}
\]

(18)

The effect of dynamic phase is discussed by Bustamante & Kartashova (2009).

For \(a_1(0) > 0, a_2(0) > 0\) and \(a_3(0) = 0\), it can be shown (Armstrong et al. 1962) that solutions to the ordinary differential eq. (18) are given by Jacobi elliptic functions (Kaup et al. 1979; Hsieh & Mysak 1980; Kartashova 2011)
\[
\begin{align*}
a_1 &= a_1(0) \text{dn}(\sigma T|m) \\
a_2 &= a_2(0) \text{cn}(\sigma T|m) \\
a_3 &= a_2(0)(\mu_3/\mu_2)^{1/2}\text{sn}(\sigma T|m)
\end{align*}
\]

(19)

where, using the notation of Abramowitz & Stegun (1972),
\[
\sigma = a_1(0)(\mu_2/\mu_3)^{1/2} \quad \text{and} \quad m = \mu_1 a_2(0)^2 / [\mu_2 a_1(0)^2]
\]

(20)

For \(0 \leq m \leq 1\), the period of energy transfer is given by the period of the elliptic function \(\text{dn}(\sigma T|m)\) (Abramowitz & Stegun 1972; Hsieh & Mysak 1980; Kartashova 2011):
\[
T_r = \frac{K(m)}{a_1(0)(\mu_2/\mu_3)^{1/2}}
\]

(21)

where \(K(m)\) is the complete elliptic integral of the first kind. At \(\frac{1}{2} T_r\), energy exchanged to the third wave is at its maximum, whereas from \(\frac{1}{2} T_r\) to \(T_r\), energy is transferred back to the other two waves. \(T_r\) is dependent on the initial distribution of amplitudes among the different modes, as well as bathymetric slope and wavenumber in its dimensional form.

**RESULTS**

As a test case, a long subduction zone rupture with fixed dimensions and position relative to the shoreline is considered. A dip of 18° and rupture dimensions of 1100 km along strike (L) and 100 km (W) along dip are similar to those for the 1700 Cascadia earthquake as estimated by Satake et al. (2003). The offset between the downdip rupture extent and the coastline (\(x_s\) in Fig. 1) is -40 km. As with previous studies, a planar bathymetric slope is assumed.

Results are presented for three stochastic slip models using different random-variable (R) distributions with parameters as follows (eq. 9): Gaussian distribution (standard normal: i.e. \(\alpha = 2\)), the general Lévy \(\alpha\)-stable distribution (\(\alpha = 1.5, \theta = -0.3, \lambda = 0, c = 0.3\)) and Cauchy distribution (\(\alpha = 1, \theta = 0, \lambda = 0.1, c = 0.3\)) with the latter two presenting more variable slip relative to the Gaussian case. Probability density functions for the three slip models are shown in Fig. 2. The parameters used for the Lévy \(\alpha\)-stable distribution include a slight skewness in the negative direction and a heavier tail (i.e. \(\alpha = 1.5\)) than the Gaussian distribution (\(\alpha = 2\)) (cf. Lavallée et al. 2011). The Cauchy distribution (\(\alpha = 1\)) has an even heavier tail (Fig. 2b), resulting in more variation in slip. Twenty slip realizations are computed for each distribution. In contrast to the Gaussian distribution, the statistical moments of the Cauchy and Lévy \(\alpha\)-stable distribution are undefined. Therefore, the slip model in these cases
cannot be directly related to a constant scalar seismic moment for a prospective earthquake \(M_0 = G \bar{D} A\) where \(\bar{D}\) is mean slip and \(A = LW\) is rupture area). As such, results are normalized with respect to the maximum amplitude of each realization.

**Triad search**

An example of how spectral amplitudes are distributed for modes 0 through 8 is shown in Fig 3(a) using the Gaussian stochastic slip model. In this and subsequent figures, to obtain the physical wavenumber the discrete wavenumber \(k\) is scaled by \(2\pi/(N_y \Delta_y)\), where \(N_y\) is the number of points in the discrete Fourier transform and \(\Delta_y\) is the along-shore sampling interval. Peaks in amplitude are indicated by the red dots. Three steps are used to identify the dominant triads that have the most potential for resonance. First, the wavenumber and mode with the highest amplitude is selected (wave 1—solid arrow in Fig. 3a). Second, a set of modes and wavenumbers for wave 2 that satisfy the resonance conditions are found from Table 1. That is, possible wavenumbers for wave 2 are calculated from values in the \(k_2/k_1\) column for the mode of wave 1, excluding those with zero coupling coefficients shown by the shaded regions in Table 1. The wavenumber with the highest amplitude of this set is chosen as wave 2. Third, the mode and wavenumber of wave 3 is then given by the resonance conditions as listed in Table 1 (wavenumber from \(k_3/k_1\) column). Importantly, the third wave does not need to have been excited by the source—it can initially have zero amplitude.

For example, for all of the 20 Gaussian realizations tested, the wavenumber with the highest amplitude is associated with the along-strike dimension of the rupture zone (marked by the solid arrow in Fig. 3a), termed here as the ‘characteristic wavenumber’ \(k_\text{c} = 5\) in non-dimensional units for the case study \(k_\text{c} = N_y \Delta_y/(2L)\). Side lobes are evident at regular intervals off this peak, owing to source finiteness, similar to the seismic source spectrum determined from seismograms in the frequency domain (Ben-Menahem 1961; Aki & Richards 1980). For the first step of the dominant triad search, wave 1 occurs at the fundamental mode (mode 0) and the characteristic wavenumber. Wave 2 with the highest amplitude of those specified in Table 1 is associated with a wavenumber approximately half of the characteristic wavenumber (subharmonic). For this case, the three waves are of mode 0, 0 and 2 respectively, termed a 002 triad. The 003 triad is equivalent to the 002 triad according to the discretization of the problem (third and fifth rows of Table 1: \(k_2/k_1 = 4/9\) or 9/16 round to the same discrete wavenumber in the case study). That is, the mode of wave 3 is either 2 or 3, with approximately the same wavenumber. These are termed the characteristic triads and are shown in Fig. 4 by the heavy black triangles. The triad search yields the same results for all 20 of the Gaussian realizations.

![Figure 2](https://academic.oup.com/gji/article-abstract/204/2/878/593777)
Figure 3. Spectral amplitudes associated with modes 0–8, for sample, realizations from each stochastic source model: (a) Gaussian; (b) Lévy \( \alpha \)-stable; (c) Cauchy. Solid black arrow indicates peak amplitude over all modes and is designated as wave 1 for the dominant triad. Wave 2 (open arrow) is the highest amplitude of those constrained by Table 1 and wave 3 (open arrow) satisfies the resonance conditions, but does not have to be initially excited by the source. Wavenumber \( k \) given in model units (scale by \( 2\pi/(N_y \Delta y) \) to obtain physical units). Amplitude \( A \) normalized with respect to maximum amplitude in each realization.
Of the 20 realizations using the Lévy \( \alpha \)-stable distribution, the peak wave 1 amplitude of four of the realizations deviates from that associated with the fundamental mode and characteristic wavenumber. One of the non-characteristic triads is shown in Fig. 3(b), where the solid arrow indicates the peak amplitude (wave 1) and the open arrows indicate waves 2 and 3 resulting from the dominant triad search. All four non-characteristic triads determined from the triad search are shown by the blue triangles in Fig. 4. For each of the four dominant triads, waves 1 and 2 occur at the fundamental mode, but at higher wavenumbers than the characteristic wavenumber. The other 16 realizations are associated with the characteristic triads.

For the Cauchy distribution, the peak amplitude for all realizations is associated with high wavenumbers and a range of modes. An example of a non-characteristic triad is shown in Fig. 3(c). The dominant triads associated with the Cauchy distribution are shown by the red triangles in Fig. 4. For 11 triads, waves 1 and 2 occur at the fundamental mode. For the other 9 triads, either both waves 1 and 2 are associated with higher modes. In one case, a 234 triad is dominant. Otherwise, modes higher than mode 3 are not included in the dominant triads. The wide range of triads associated with the Cauchy stochastic model in comparison to the Gaussian and Lévy \( \alpha \)-stable modes is consistent with the heavier tail distribution as indexed by the stability parameter \( \alpha \). Subevents are more distinct with Cauchy model and are responsible for exciting high-wavenumber edge waves, rather than the low-wavenumber edge waves associated with the along-strike rupture dimension excited by the other stochastic source models.

**Examples of triad dynamics**

The first example of triad dynamics uses characteristic triads 002 and 003 (heavy black triangles in Fig. 4). Shown in Fig. 5 is the variation with time of the slow real amplitude variable \( \alpha \), using eqs (19) and (20). Amplitude of the third wave gradually increases at the expense mainly of the first wave, although the second wave has to be present for the interaction to occur. The resonant period \( T_i \) for the 002 triad is approximately 42 wave periods (non-dimensionalized with respect to the mode 0 wave), such that \( \alpha_3 \) reaches a maximum at 21 wave periods. The dynamics of triad 003 (Fig. 5b) is similar to that of the 002 triad with a slightly longer resonant period.

Recalling the equation for resonant period (eq. 21), the sensitivity of \( T_i \) is examined relative to the initial amplitudes of waves 1 and 2. For the case shown in Fig. 6, the coupling coefficients are specific to the 002 triad (Fig. 5a). There is a complex dependence of \( T_i \) on the amplitude of wave 2 (Fig. 6a) in which there is a singularity. A narrow range of initial amplitudes in the edge-wave spectrum, therefore, may result in very long resonant periods. In contrast, the initial amplitude of wave 1 has little effect on \( T_i \) (Fig. 6b).

To demonstrate the effect that higher modes and wavenumbers have on the triad dynamics, results from a 223 dominant triad using the Cauchy stochastic model (indicated by heavy red triangle in Fig. 4) are shown (Fig. 7). The corresponding amplitudes are shown in Fig. 3(c). In this case, the amplitude of wave 3 eventually exceeds that of wave 1, although the resonant period is much longer. There is little change in the amplitude of wave 2, although again this wave has to be present for resonance to occur.
Figure 4. Plot of triads resulting from search method described in the text for each of the stochastic source models. Heavy black triangles: characteristic triads (002 and 003). Blue triangles: non-characteristic triads from the Lévy $\alpha$-stable model (Fig. 3b—heavy blue triangle). Red triangles: non-characteristic triads from the Cauchy model (Fig. 3c—heavy red triangle). Wavenumber ($k$) given in model units (scale by $2\pi/(N_0\Delta y)$ to obtain physical units).

Secondary peaks at higher mode numbers and wavenumbers than the dominant triads (Gaussian and Lévy $\alpha$-stable models) can also result in resonant coupling. Shown in Fig. 8(a) are amplitudes associated with a realization of the Lévy $\alpha$-stable stochastic models (different than the one shown in Fig. 3b). The green arrows show the result of the triad search, using a secondary mode 3 peak as wave 1 (312 triad). For comparison, mode 3 amplitudes associated with the Gaussian model are typically smaller than for the heavier tail distributions, resulting overall smaller amplitudes for the other waves in the triad (Fig. 8b). For the Gaussian model, the highest amplitude for wave 2 among those listed in Table 1 is at a relatively low wavenumber (323 triad), such that wave 2 is at a higher amplitude than wave 1. For the Lévy $\alpha$-stable model, resonant coupling results in wave 3 reaching a higher amplitude than the initial amplitude of wave 1 (Fig. 9a). The overall amplitudes for the Gaussian model using secondary peaks are lower (Fig. 9b). In both cases, there is little change in the amplitude of wave 2.

The sensitivity of $T_i$ to initial amplitudes is shown in Fig. 10 with coupling coefficients specific to the Lévy $\alpha$-stable 312 triad (Fig. 8a). As with this previous sensitivity study, values of $T_i$ can reach large values because of the singularity associated with the initial amplitude of wave 2 (Fig. 10a). Other than the singularity occurring at smaller initial values of $a_2$, the functional forms are similar to the results presented in Fig. 6.

Given that there are any number of triads possible in an edge-wave field generated by a stochastic source, these triads will interact in more complex ways than presented here. Analysis of triad clusters is described extensively in Kartashova (2011). For the Gaussian and most of the Lévy $\alpha$-stable source models examined in this study ($\alpha = 1.5$), however, the characteristic triads will dominate nonlinear resonant coupling.

**DISCUSSION**

The above results indicate that the form of the stochastic model has an important role in determining which triads are activated during tsunami edge-wave propagation. The source model based on a Gaussian random variable results in the least-variable slip (eq. 7). The largest edge-wave amplitude is associated with the characteristic wavenumber and fundamental mode (wave 1). The corresponding triads (termed the characteristic triads) involve a subharmonic, fundamental mode wave (wave 2) and a mode 2 or 3 wave (wave 3). Distribution parameters that were used for the Lévy $\alpha$-stable source model (namely, $\alpha = 1.5$) correspond to a heavier tail than the Gaussian distribution, and hence, more variable slip. In several of the cases (20 per cent of those tested), higher wavenumber triads appear to be dominant, whereas in the other cases the characteristic triad still dominates. For the Cauchy source model ($\alpha = 1$), the associated distribution has the heaviest tail and most variable slip. In none of the cases tested for this model are the characteristic triads dominant. Most of the dominant triads have high wavenumbers and modes greater than the fundamental mode and are associated with very long resonant periods.

In addition to the distribution parameters associated with the stochastic source models, the spectral decay exponent $\nu$ may also have an effect on the distribution of edge-wave modes and wavenumbers (cf. Geist 2002 for non-trapped tsunami waves).
Results are also somewhat dependent on the specified geometry. Previous studies (Kajiura 1972; Fujima et al. 2000; Geist 2013), indicate that a more offshore position for rupture results in a shift of energy to higher edge-wave modes. However, there is also a concomitant decrease in the excitation of edge waves. Other parameters such as fault dip and along-strike rupture length may also have an effect on the results.

The analysis presented in this paper assumes no energy loss from turbulence (bottom friction, wave breaking) and scattering from coastline irregularities. Attenuation of progressive edge waves from turbulence and scattering is likely not very significant, owing to estimated e-folding distances of hundreds of kilometres for low-order waves (Fuller & Mysak 1977). Voronovich et al. (2008) examine the effect of bottom friction on non-linear modulation instability for general ocean waves, but it is unclear what effect bottom friction has on non-linear coupling of edge waves. Bottom friction likely has more of an attenuating effect on the higher edge-wave modes (cf. Hsieh & Mysak 1980). For the most energetic modes, however, edge waves can be observed at epicentral distances of more than 2000 km, as in the case of the 1952 Kamchatka tsunami (Ishii & Abe 1980).

These attenuation effects are partially offset by regional resonance (e.g. continental shelf) and scattering and reflections during transoceanic propagation, resulting in a continual ‘pumping’ of energy into the coast over long periods of time. Van Dorn (1984, 1987) determined decay constants that appeared to be similar for each ocean basin, ascribing the cause to normal mode oscillations across the basins. Rabinovich et al. (2011), however, refined this analysis using data from the 2004 Indian Ocean tsunami and found a substantial variation in decay times (13–45 hr) that depend on a number of factors, including island versus mainland recording sites and traveltime. The e-folding times for sites near the source are slightly less than far-field receiver locations.

Typical periods for low-mode edge waves associated with an M9 earthquake range from tens of minutes to over one hour (Ishii & Abe 1980; Galletta & Vittori 2004; Bricker et al. 2007). Therefore, although the transfer of energy time (i.e. $\frac{1}{2} T_r$) for low-mode triads is predominantly less than observed e-folding decay times (Fig. 6), it remains to be demonstrated whether this resonant effect can be observed from instrumental records as it has for continental shelf waves (Hsieh & Mysak 1980). Bricker et al. (2007) were able to detect the first three edge-wave modes of the 2006 Kuril tsunami in Hawaii from ADCP measurements and indicated that the arrival of edge waves were responsible for the highest surges from this event. Moreover, they were able to infer that counter-propagating edge waves were present for this event. Even under simple initial conditions, Dubinina et al. (2006, 2008) show that the composite wavefield from a triad is highly irregular in space and time. It is possible that edge-wave dynamics are implicated in late large-amplitude tsunami arrivals (e.g. Borrero et al. 2013), although identifying the origin of a particular phase is especially difficult.
Figure 8. Amplitudes associated with modes 0–8 for sample realizations from the (a) Lévy α-stable and (b) Gaussian stochastic source model. Open green arrows indicate amplitudes used for triad dynamics (Fig. 9), where wave 1 is a mode 3 secondary peak. Wave 2 is the highest amplitude of those constrained by Table 1 and wave 3 satisfies the resonance conditions, but does not have to be initially excited by the source. Wavenumber (k) given in model units (scale by $2\pi/(N_y \Delta_y)$ to obtain physical units). Amplitude (A) normalized with respect to maximum amplitude in each realization.
Future studies can be envisioned that further refine observations of non-linear interaction among tsunami edge-wave modes as well as extend the theoretical description of this phenomenon. It is demonstrated in this study that highly variable slip can distribute tsunami edge waves over a significant range of wavenumbers and modes, potentially activating a number of different triads. Further work needs to be performed to constrain the parameters of stochastic source models from earthquake observations to better determine the range of slip variability. In addition, examining the kinematic and dynamic effects of triad clusters (Kartashova 2011) may yield more accurate information on the characteristics of energy transfer among tsunami edge-wave modes, especially the dominant resonance period. Finally, the focus of this study has primarily been on near-field edge waves. It is equally important to determine the characteristics of non-linear edge-wave resonance at far-field locations as tsunami energy is continually pumped into the coastal region from the deep ocean (Rabinovich et al. 2011).

**CONCLUSIONS**

Conditions exist for resonant coupling among tsunami edge-wave modes generated from a large-magnitude subduction zone earthquake. Complexity of earthquake rupture results in edge-wave amplitudes that are distributed among a wide range of modes and wavenumbers. This distribution implies that a number of different wave triads are possible that result in resonant coupling. Although some triad combinations involve unidirectional-propagating edge-wave modes, many of these triads involve counter-propagating edge waves that can be generated by backscattering from coastline irregularities. Counter-propagating edge waves were observed, for example, for the 2006 Kuril earthquake in Hawaii (Bricker et al. 2007). Therefore, the necessary ingredients for resonant coupling of tsunami edge waves appear to be present.

In this study, which resonant triads are activated is directly dependent on the parameters used for the stochastic source model. In particular, the stability parameter $\alpha$ associated with the Lévy $\alpha$-stable distribution inversely controls the variability of slip, and hence, the range of edge-wave modes and wavenumbers that are excited. Commonly, a Gaussian distribution ($\alpha = 2$) random variable is used in stochastic source models, resulting in less variable slip such that only the characteristic triads are excited. These triads are defined by the along-strike rupture dimension and fundamental mode. On the other end of the spectrum, a Cauchy distribution ($\alpha = 1$) results in highly variable slip where tsunami generation occurs from distinct subevents. For this model, a wide number of possible triads are excited, owing to the variability in subevent size and position rather than by being controlled by the overall along-strike rupture dimension as in the case of the Gaussian model. In addition to probability distribution parameters, other parameters such as the spectral decay exponent of slip and geometric parameters likely influence resonant coupling of tsunami edge waves.

A critical aspect in determining whether resonant coupling may be observed is the time in which energy is transferred from one mode of the triad to another (one half of the resonant period $T_r$). Calculations of $T_r$ from the stochastic source model ranges from tens (low-mode triads) to hundreds (high-mode triads) of wave periods (Figs 6 and 10). Tsunami wave amplitudes persist at a tide gauge station for $e$-folding times of tens of hours and low-mode progressive edge waves have been observed to propagate long distances ($e$-folding distances of hundreds of kilometres). Therefore, given careful analysis, it may be possible to detect resonant coupling from instrumental observations at or near the shore, for at least the low-mode triads.
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REFERENCES


