SAINT-VENANT TORSION OF ANISOTROPIC SHAFTS:
THEORETICAL FRAMEWORKS, EXTREMAL BOUNDS AND
AFFINE TRANSFORMATIONS

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Summary

We study the Saint-Venant torsion of anisotropic shafts. Theoretical frameworks for torsion
of anisotropic composite shafts are derived in terms of warping function, conjugate function
as well as stress potential, parallel to the existing frameworks for torsion of isotropic shafts.
We prove an extremal property for the torsional rigidity of anisotropic composite shafts. For
homogeneous shafts, an affine coordinate transformation is introduced in the formulation, which
demonstrates how the cross-sectional shape of the shaft is deformed (stretching and rotation)
under the mapping, and how the warping field and the torsional rigidity of an anisotropic
shaft are correlated to those of an isotropic one. We find that a certain class of anisotropic
elliptical shafts, simply- or multiply-connected, will not warp under an applied torque. Of all
homogeneous shafts with a given cross-sectional area and the same shear rigidity matrix, the
torsional rigidity, associated with zero warping displacement, can be proven as extremal upper
bounds. Finally, families of anisotropic shafts that are equivalent to isotropic ones, including
elliptical and hollow elliptical shafts, and cylindrical shafts with specific cross-sections of
parallelogram and triangle shape, are characterized.

1. Introduction

The problem of cylindrical shafts under a twisting torque has long been a subject of solid mechanics
since the advent of Saint-Venant’s conjecture (1). Under the deformation mode, the torsional
rigidity can be directly linked to the axial displacement of the cross-section, referred to as the
warping function. Apart from a scale factor of the shear rigidity, torsional rigidity strongly depends
on the geometric shape of the shaft. Classical monographs on mathematical theory of elasticity,
such as (2 to 4), contain various expositions of mathematical frameworks. In the literature, one
branch of studies is dedicated to finding solutions of boundary-value problems. Another branch
is aimed at exploring general properties of torsional behaviour of cylindrical shafts. Rigorous
bounds on the estimate of torsional rigidity are typical examples of the latter kind (5 to 8). Recently,
substantial advances have been made for torsion of composite shafts, constituted by two or more
different materials. For example, in a series of works (9 to 11), we showed that an assemblage of
multicoated cylinders is an exactly solvable microgeometry in torsion. Other developments include
a number of theorems on variational bounds of the torsional rigidities for shafts containing coated
cylinders (12), and variational bounds on the torsional rigidity of shafts containing imperfectly
bonded cylinders (13).

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The above developments are mainly focused on cylindrical shafts with isotropic constituents. In contrast, there is a lack of a unified theory on the mathematical frameworks of anisotropic shafts under torsion. The physical motivation of this study is apparent, as some natural materials are intrinsically anisotropic. On a different perspective, the anisotropy of the shafts may result from spatially preferential compositions of constituent cylinders at a lower-dimensional scale. Within the literature, the book of Lekhnitskii (14) seems to be the only treatise that contains a more detailed exposition of anisotropic torsion. But its formulation is mainly focused on the framework of generalized torsion, that is, a torque will induce bending deformation. Sokolnikoff (3, §51) and Horgan and Miller (15) proposed a simple concept which briefly outlined a procedure showing how the torsion problem for an orthotropic shaft can be reduced to that of an isotropic one.

In this work we aim to present a unified exposition on the mathematical theory of Saint-Venant torsion of an anisotropic shaft. The cylindrical shaft is anisotropic, but the planes normal to the axis of the cylinder must coincide with the planes of elastic symmetry, so that a simple torsion will not induce curvature (bending effects) (14). Monoclinic and orthotropic materials with symmetry planes normal to the shaft direction are typical examples of the considerations. Mathematical frameworks for the field quantities are derived in terms of warping function, conjugate function as well as stress potential, entirely parallel to those of torsion of isotropic shafts. These include the governing equations, boundary conditions, interfacial continuity relations, extremal variational principles based on minimum of potential energy and minimum of complementary energy, and the torsional rigidity. With the derived variational principles, we prove a new extremal property on the torsional rigidity of anisotropic shafts in section 3. In particular, we show that, for a given cross-sectional area, the upper bound of the torsional rigidity is realized for a certain class of elliptical shafts. This finding can be viewed as the anisotropic counterpart of the well-known Saint-Venant conjecture for isotropic shafts: ‘Of all simply-connected isotropic cross-sections with given area, a circle has the maximum torsional rigidity’ (5). In section 4, we introduce an affine coordinate transformation in a spirit similar to that of Sokolnikoff (3), but more general in formulation. We demonstrate how the cross-sectional shape of the shaft is stretched and rotated under an affine mapping, and how to correlate the warping field and the torsional rigidity of an anisotropic shaft to those of an isotropic one. We show that the transformation matrix admits a number of variants, all linked by the Cholesky factorization (16). We demonstrate, without solving any field equation, that the torsional rigidity of an anisotropic elliptical shaft admits a simple, explicit form. Of particular interest is the situation in which the transformed cross-sections become a circle or concentric circles. We have demonstrated that a certain class of anisotropic elliptical shafts, simply- or multiply-connected, will not warp under an applied torque. In fact, for homogeneous anisotropic shafts with a given cross-sectional area, the corresponding torsional rigidities can be proven as extremal upper bounds. Finally, families of anisotropic shafts that are equivalent to isotropic ones are explored in section 5. Examples of elliptical and hollow elliptical shafts, and shafts with specific cross-sections of a parallelogram and a triangle are studied. In all these cases the torsion solutions can be readily found from the isotropic ones.

2. Anisotropic torsion

2.1 Mathematical formalisms

We consider the Saint-Venant torsion of an anisotropic shaft with cross-section Ω of arbitrary shape. Let us introduce a Cartesian coordinate system $X = x^I + y^J + z^k$ positioned at one end of the shaft,
and also let the $z$-axis be the axial direction along the shaft. The displacement field of the shaft under Saint-Venant torsion can be characterized by $u = \vartheta z \times X + \vartheta \varphi(x, y)k$, or explicitly written as

$$u_x = -\vartheta y z, \quad u_y = \vartheta x z, \quad u_z = \vartheta \varphi(x, y),$$

(2.1)

where $\vartheta$ is the angle of twist per unit length of the shaft and $\varphi$ is the axial displacement of the cross-section, commonly referred to as the warping displacement (function). The warping function typically varies from point to point in the cross-section, but is virtually independent of the axial coordinate for cross-sections sufficiently far away from the ends (17). We consider that the shaft is rectilinearly anisotropic, possessing at least one symmetry plane normal to the shaft direction (the $z$-axis). As such, a twisting torque will not induce curvature (bending effects) (14), and vice versa. Monoclinic and orthotropic cylinders with symmetry planes normal to the shaft direction are typical examples of this kind. The non-zero strains $\varepsilon$ and stresses $\sigma$ corresponding to (2.1) are

$$\sigma = \mu \varepsilon \quad \text{with} \quad \varepsilon = \vartheta (\nabla \varphi + x_\perp),$$

(2.2)

where $x_\perp \equiv -y i + x j$ is the plane position vector $x = xi + y j$ rotated anticlockwise by 90 degrees,

$$x_\perp = R_\perp x \quad \text{with} \quad R_\perp \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(2.3)

and

$$\sigma = \begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 2 \varepsilon_{xz} \\ 2 \varepsilon_{yz} \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_{55} & \mu_{45} \\ \mu_{45} & \mu_{44} \end{pmatrix}.$$  

(4.4)

The equilibrium condition, $\nabla \cdot \sigma = 0$, implies that $\varphi$ should satisfy

$$\nabla \cdot (\mu \nabla \varphi) = 0 \quad \text{for} \quad x \in \Omega.$$  

(2.5)

Further, the traction-free condition on the lateral surface $\partial \Omega$ of the cross-section is

$$\mu (\nabla \varphi + x_\perp) \cdot n |_{\partial \Omega} = 0,$$

(2.6)

with $n$ being the unit outward normal of the boundary $\partial \Omega$.

Alternatively, one may introduce a stress potential function $\Phi$,

$$\sigma = -\vartheta \mu R_\perp \nabla \Phi,$$

(2.7)

so that the stress tensor $\sigma$ automatically fulfills the equilibrium condition. Here $\mu$ could be any arbitrary constant, but for later algebraic convenience we shall take its value as

$$\mu \equiv \sqrt{\det \mu}.$$  

(2.8)

Now setting the equivalence between (2.2) and (2.7), in reference to (2.5), we are led to

$$\nabla \cdot (\mu \nabla \Phi) = -2\mu \quad \text{for} \quad x \in \Omega.$$  

(2.9)
On $\partial \Omega$, the traction free condition is
\[
\sigma \cdot n|_{\partial \Omega} = 0 \quad \Rightarrow \quad \mu R_{\perp} \nabla \Phi \cdot n|_{\partial \Omega} = 0 \quad \Rightarrow \quad \nabla \Phi \cdot s|_{\partial \Omega} = 0,
\]
(2.10)
where $s = R_{\perp} n$ is the unit tangent vector to the boundary $\partial \Omega$. This suggests that
\[
\Phi|_{\partial \Omega} = 0,
\]
(2.11)
where we have set the integration constant to be zero. We mention, however, that if $\Omega$ is a multiply-connected domain containing more than one closed contour, then the remaining integration constants cannot be specified arbitrarily (4). In summary, equations (2.5) and (2.6) constitute the governing system for the warping function $\varphi$, while (2.9) and (2.11) are the framework for the stress potential $\Phi$.

To proceed, let us now consider the general solution of (2.5). Without loss in generality we assume a solution of the form
\[
\varphi(x, y) = w(z) \quad \text{with} \quad z = x + py,
\]
(2.12)
where $w$, referred to as a complex potential in the sequel, is an arbitrary twice-differentiable function and $p$ is some constant to be determined. Substitution of (2.12) into (2.5), via direct differentiations, gives
\[
\mu^{55} p^2 + 2 \mu^{45} p + \mu^{44} = 0,
\]
which provides two roots for $p$. Due to the positive definiteness of $\mu$, one of them is
\[
p = (-\mu^{45} + i \mu)/\mu^{44} \equiv p' + ip'',
\]
(2.13)
and the other is its complex conjugate $\bar{p}$. Since $\varphi$ is a real-valued function, one may let
\[
\varphi = \left( w(z) + \overline{w(z)} \right)/2 = \text{Re} w(z).
\]
(2.14)
The corresponding stresses, obtained from (2.14) and (2.2), are
\[
\sigma_{zx} = \partial \text{Re} [(\mu^{55} + p \mu^{45}) w'(z)] + \partial (\mu^{45} x - \mu^{55} y),
\]
\[
\sigma_{zy} = \partial \text{Re} [(\mu^{45} + p \mu^{44}) w'(z)] + \partial (\mu^{44} x - \mu^{45} y).
\]
(2.15)
Further, from (2.13) it is known (18) that
\[
\mu^{55} + p \mu^{45} = -ip \mu, \quad \mu^{45} + p \mu^{44} = i \mu.
\]
(2.16)
Accordingly, (2.15) can be recast as
\[
\sigma_{zx} = \partial \mu \text{Im} [pw'(z)] + \partial (\mu^{45} x - \mu^{55} y),
\]
\[
\sigma_{zy} = -\partial \mu \text{Im} [w'(z)] + \partial (\mu^{44} x - \mu^{45} y).
\]
(2.17)
Now in comparing (2.7) and (2.17), we notice that the stress function $\Phi$ can be represented by $w(z)$ in the form
\[
\Phi = \text{Im} w(z) - (\mu^{44} x^2 - 2 \mu^{45} xy + \mu^{55} y^2)/(2 \mu).
\]
(2.18)
Note that
\[
\frac{1}{2\mu} \left( \mu_{44} x^2 - 2\mu_{45}xy + \mu_{55} y^2 \right) = \frac{\mu}{2} \mu^{-1} x \cdot x = \frac{1}{2\mu} \mu x_\perp \cdot x_\perp = \frac{\mu_{44}}{2\mu} \overline{z}z. \tag{2.19}
\]

For convenience, we shall introduce a real function \(\psi\) defined as \(\psi \equiv \text{Im} \, w(z)\), analogous to that of harmonic conjugate function to \(\varphi\), so that one has
\[
\varphi(x, y) + i\psi(x, y) = w(x + py), \quad \Phi(x, y) = \psi(x, y) - \frac{\mu_{44}}{2\mu} \bar{z}z, \tag{2.20}
\]
in reference to (2.18) and (2.19). Now upon the substitution of (2.20) into (2.9), we find
\[
\nabla \cdot (\mu \nabla \psi) = 0 \quad \text{for} \; x \in \Omega. \tag{2.21}
\]

In terms of \(\psi\), the traction free condition on \(\partial\Omega\), from (2.11) and (2.20), becomes
\[
\psi|_{\partial\Omega} = \frac{\mu_{44}}{2\mu} \bar{z}z|_{\partial\Omega}. \tag{2.22}
\]

In addition to \(\varphi\) and \(\Phi\), (2.21) with boundary condition (2.22) constitutes another framework to characterize the torsional behaviour of anisotropic shafts.

When the shaft is isotropic, namely \(\mu = \mu I\), \(I\) being a \((2 \times 2)\) identity matrix, it can be readily verified that the above frameworks for \(\varphi\), \(\psi\) and \(\Phi\) all reduce to those for torsion of isotropic shafts (3).

### 2.2 Interface conditions

When the shaft is constituted of two or more different materials, then the warping displacement \(\varphi\) and the shear traction \(\sigma_{ni}\) need to be continuous across the interface \(J\) under the perfect bonding condition. Here \(n\) is the unit normal to \(J\). For example, at the interface \(J\) between material 1 and material 2, we need to impose
\[
\varphi_1 - \varphi_2|_J = 0, \quad \mu_1 (\nabla \varphi_1 + x_\perp) \cdot n|_J = \mu_2 (\nabla \varphi_2 + x_\perp) \cdot n|_J. \tag{2.23}
\]

In terms of the complex potential \(w(z)\), these conditions can be recast in the forms
\[
\text{Re} \, w_1 - \text{Re} \, w_2|_J = 0, \quad \mu_1 \text{Im} \, w_1 - \mu_2 \text{Im} \, w_2|_J = \frac{1}{2} \left( \mu_{44}^{(1)} z_1 \bar{z}_1 - \mu_{44}^{(2)} z_2 \bar{z}_2 \right)|_J,
\]
where \(z_1 \equiv x + p_1 y\) and \(z_2 \equiv x + p_2 y\). Alternatively, one can write the continuity conditions in terms of \(\Phi\). Specifically, for the traction continuity condition, using (2.7) we have
\[
[\sigma \cdot n] = [-\partial \mu \mathbf{R}_\perp \nabla \Phi \cdot n] = [\partial \mu \nabla \Phi \cdot \mathbf{R}_\perp \mathbf{n}] = \partial [\mu \nabla \Phi \cdot s] = \partial [\mu d\Phi/ds] = 0. \tag{2.24}
\]
This implies
\[
\mu_1 \Phi_1|_J = \mu_2 \Phi_2|_J. \tag{2.25}
\]
Note that the square brackets in (2.24) denote the jump of the quantity inside. For continuity of warping displacement, we can start from (2.2) and (2.7) in the following steps:

\[-\mu R_{\perp} \nabla \phi_1 = \mu (\nabla \phi + x_{\perp})\]

\[\Rightarrow -\mu \mu^{-1} R_{\perp} \nabla \phi = \nabla \phi + x_{\perp} \Rightarrow -\frac{\mu}{\det \mu} R_{\perp} \mu R_{\perp}^T R_{\perp} \nabla \phi = \nabla \phi + x_{\perp}\]

\[\Rightarrow -\frac{1}{\mu} R_{\perp} \mu \nabla \phi \cdot s = (\nabla \phi + x_{\perp}) \cdot s \Rightarrow -\frac{1}{\mu} \mu \nabla \phi \cdot n = (\nabla \phi + x_{\perp}) \cdot s. \quad (2.26)\]

Since $\phi$ is continuous across $J$, one has

\[\frac{1}{\mu_1} \mu_1 \nabla \phi_1 \cdot n \bigg|_J = \frac{1}{\mu_2} \mu_2 \nabla \phi_2 \cdot n \bigg|_J. \quad (2.27)\]

2.3 Torsional rigidity

We now derive the torsional rigidity for the composite shaft. Let us start from the definition for the torsional rigidity $T$ in the steps

\[T = \frac{1}{\theta} \int \int_{\Omega} (x_\sigma_{y\zeta} - y_\sigma_{x\zeta}) \, dx \, dy = \sum_r \int \int_{\Omega_r} \mu_r (\nabla \phi + x_{\perp}) \cdot x_{\perp} \, dx \, dy\]

\[= \sum_r \int \int_{\Omega_r} \mu_r x_{\perp} \cdot x_{\perp} \, dx \, dy + \sum_r \int \int_{\Omega_r} \mu_r \nabla \phi \cdot x_{\perp} \, dx \, dy\]

\[= \sum_r P_r - \sum_r D_r(\phi). \quad (2.28)\]

where the subscript $r$ is summed over the number of different phases. From (2.19), it is readily seen that $P_r$ is always positive as $\mu_r$ is positive definite. Next, making use of (2.23), (2.6), (2.5) and the divergence theorem, we can evaluate the second integral of (2.28) as

\[\sum_r D_r(\phi) = -\sum_r \int \int_{\Omega_r} \mu_r \nabla \phi \cdot x_{\perp} \, dx \, dy\]

\[= -\sum_r \int \int_{\Omega_r} \left( \mu_{44} \frac{\partial (x \phi)}{\partial y} + \mu_{45} \frac{\partial (x \phi)}{\partial x} - \mu_{45} \frac{\partial (y \phi)}{\partial y} - \mu_{55} \frac{\partial (y \phi)}{\partial x} \right) \, dx \, dy\]

\[= -\int_{\partial \Omega} \left( -\mu_{55} y + \mu_{45} x \right) n_1 + \left( -\mu_{45} y + \mu_{44} x \right) n_2 \, \phi \, ds\]

\[= -\int_{\partial \Omega} (\mu x) \cdot n_1 \, \phi \, ds = \int_{\partial \Omega} (\mu \nabla \phi) \cdot n \, \phi \, ds = \int \int_{\Omega} (\mu \nabla \phi) \cdot \nabla \phi \, dx \, dy.\]

Thus, the torsional rigidity of an anisotropic shaft is as follows:

\[T = \sum_r \int \int_{\Omega_r} \mu_{44} x_{\perp} \cdot x_{\perp} \, dx \, dy - \sum_r \int \int_{\Omega_r} (\mu \nabla \phi) \cdot \nabla \phi \, dx \, dy. \quad (2.29)\]

We mention that when the shear rigidity $\mu$ is isotropic, then $D_r(\phi)$ is known as the Dirichlet integral (19). The expression (2.29) indicates that upper and lower bounds for $\sum_r D_r(\phi)$ will, respectively, yield lower and upper bounds for $T$. 

\[\text{Downloaded from https://academic.oup.com/qjmam/article-abstract/58/2/269/1939736 by guest on 05 March 2019}\]
Alternatively, using (2.25) and the divergence theorem, we can derive $T$ in terms of $\Phi$ as

$$
T = - \sum_r \iiint_{\Omega_r} \mu_r \left( \frac{x}{\partial} \Phi + \frac{y}{\partial} \Phi \right) dx dy = - \sum_r \iiint_{\Omega_r} \mu_r \left( \nabla f \cdot \nabla \Phi \right) dx dy \\
= \sum_r 2 \iiint_{\Omega_r} \mu_r \Phi_r dx dy.
$$

(2.30)

Here the scalar function $f = x \cdot x/2$. Further, making use of (2.9) in (2.30), we are led to

$$
T = - \sum_r \iiint_{\Omega_r} \mu_r \Phi \left( \nabla \cdot \nabla \Phi \right) dx dy \\
= - \oint_{\partial \Omega_r} \mu \nabla \psi \cdot \nabla \Phi \cdot \nabla dx dy + \sum_r \iiint_{\Omega_r} \mu \nabla \Phi \cdot \nabla \Phi dx dy.
$$

(2.31)

The latter expression also reflects the fact that the torsional rigidity is always positive.

The torsional rigidity can also be written in terms of $\psi$. To do this, using (2.20) and (2.19), we can recast (2.31) as

$$
T = \sum_r \iiint_{\Omega_r} (\mu \nabla \Phi \cdot \nabla \psi - \mu \nabla \psi \cdot \nabla \Phi) dx dy \\
= \sum_r \iiint_{\Omega_r} (\mu \nabla \psi \cdot \nabla \psi + \mu^2 \mu^{-1} x \cdot x - \mu x \cdot \nabla \psi - \mu \nabla \psi \cdot \mu^{-1} x) dx dy.
$$

(2.32)

To proceed, first it is evident that

$$
\sum_r \iiint_{\Omega_r} \mu x \cdot \nabla \psi dx dy = \sum_r \iiint_{\Omega_r} \left( \mu \mu \nabla \psi \cdot \mu^{-1} x \right) dx dy.
$$

(2.33)

Next, we note that

$$
\sum_r \iiint_{\Omega_r} (\mu \nabla \psi \cdot \mu^{-1} x) dx dy = \sum_r \iiint_{\Omega_r} \mu \nabla \psi \cdot \nabla \left( \frac{\mu x}{2 \mu} \right) dx dy \\
= \oint_{\partial \Omega_r} \mu \nabla \psi \cdot \nabla \psi dx dy.
$$

(2.34)

In the above derivations, we have employed the connections (2.21), (2.22) and (2.19) and the divergence theorem. Thus, substituting (2.33) and (2.34) into (2.32), we obtain

$$
T = \sum_r \iiint_{\Omega_r} (\mu \nabla \psi \cdot \nabla \psi) dx dy = \sum_r P_r - \sum_r D_r(\psi).
$$

(2.35)

Lastly, we mention that when the constituents of the shaft are isotropic, our derived expressions for the torsional rigidity, (2.29), (2.30), (2.31) and (2.35) exactly recover the known formulae in the literature (20).
2.4 Variational principles

It is known that the torsional rigidity can be related to minimum energy principles. The first variational principle is based on constructing complementary energy in terms of virtual stress potentials \( \tilde{\Phi} \) that vanish on the lateral boundary of the shaft and are square integrable with square integrable gradients. Without including detailed formulations, the minimum principle of complementary energy can be expressed in the form

\[
T = -2 \min_{\tilde{\Phi}} \left\{ \frac{1}{2} \int\int_{\Omega} (\mu(x) \nabla \tilde{\Phi}) \cdot \nabla \tilde{\Phi} \, dx \, dy - 2 \int\int_{\Omega} \mu(x) \tilde{\Phi} \, dx \, dy \right\}.
\]

(2.36)

The second variational principle is given in terms of virtual warping functions \( \tilde{\phi} \) that are square integrable with square integrable gradients. Specifically, it has the form

\[
T = \min_{\tilde{\phi}} \left\{ \int\int_{\Omega} \mu(x) (\nabla \tilde{\phi} + x_{\perp}) \cdot (\nabla \tilde{\phi} + x_{\perp}) \, dx \, dy \right\}.
\]

(2.37)

These two variational forms will be useful in constructing bounds on the torsional rigidity of composite anisotropic shafts. We mention that when the constituent properties of the shaft are isotropic, the two variational forms reduce to the known expressions (12).

3. An extremal property of the torsional rigidity

Since \( D_r(\psi) \) and \( D_r(\psi) \) are non-negative, from (2.29) and (2.35) we conclude that the torsional rigidity of a composite anisotropic shaft is bounded above:

\[
T \leq \sum_r \int\int_{\Omega_r} \left( \mu_{44}^{(r)} x^2 - 2 \mu_{45}^{(r)} xy + \mu_{55}^{(r)} y^2 \right) \, dx \, dy.
\]

(3.1)

Equality holds only when \( D_r(\psi) = D_r(\psi) = 0 \). This will occur if and only if the warping displacement is a constant. Chen (21) recently proved that a certain class of homogeneous elliptical shafts will not warp under torsion. Thus the upper bound on the right-hand side of (3.1) can actually be attained by certain anisotropic elliptical shafts. We elaborate this more in sections 5.1 and 5.2.

4. Affine transformation

We now confine our attention to homogeneous shafts. In terms of warping displacement, the field is governed by (2.5) together with the Neumann type of boundary condition (2.6). We now introduce an affine coordinate transformation (22)

\[
x' = Ax,
\]

(4.1)

where \( x' = x'i + y'j \) and the transformation matrix \( A \) does not depend on \( x \). Using the chain rule of differentiation, we have \( \nabla \phi = A^T \nabla \psi \), in which \( \nabla \) is the differential operator \( \nabla = (\partial/\partial x', \partial/\partial y') \). Under this transformation, (2.5) becomes

\[
(A^T \nabla') \cdot (\mu A^T \nabla \phi) = \nabla' (A\mu A^T \nabla \phi) = 0.
\]

(4.2)
Now if one chooses \( A \) such that \( A\mu A^T \) becomes a multiple of the unit matrix \( I \),

\[
A\mu A^T = \zeta I \quad \text{or} \quad \mu = \zeta (A^T A)^{-1},
\]

then the governing equation (2.5) will be transformed into Laplace’s equation \( \nabla^2 \varphi = 0 \), similar to that of torsion of an isotropic shaft. Here \( \zeta \) is a scalar constant that can be selected arbitrarily. By taking the determinant of both sides of (4.3), we see that \( \zeta \) must be positive as \( \mu \) is positive definite.

Next, let us consider the traction free condition (2.6) under the transformation (4.1). To proceed, we first describe the boundary \( \partial \Omega \) of the cross-section \( \Omega \) by \( f (x) = 0 \). As such, the unit normal to \( \partial \Omega \) can be expressed as \( n = \nabla f (x)/|\nabla f | \). The boundary of the transformed cross-section \( \partial \Omega' \) is now changed to \( f (A^{-1} x) = 0 \) with \( n' = \nabla f / |\nabla f | \). The boundary condition (2.6) can be recast in the following steps:

\[
\begin{align*}
\mu \nabla \varphi \cdot \nabla f |_{\partial \Omega'} &= -\mu R_{\perp} x \cdot \nabla f |_{\partial \Omega'} \\
\Rightarrow \mu A^T \nabla \varphi \cdot A^T \nabla f |_{\partial \Omega'} &= -\mu R_{\perp} A^{-1} x' \cdot A^T \nabla f |_{\partial \Omega'} \\
\Rightarrow \zeta \nabla \varphi \cdot \nabla f |_{\partial \Omega'} &= -\zeta (A^T)^{-1} R_{\perp} A^{-1} x' \cdot \nabla f |_{\partial \Omega'} \\
\Rightarrow \zeta \nabla \varphi \cdot \nabla f |_{\partial \Omega'} &= -\zeta \det(A^{-1}) R_{\perp} x' \cdot \nabla f |_{\partial \Omega'} \\
\Rightarrow (\det A) \nabla \varphi \cdot n |_{\partial \Omega'} &= -x'_\perp \cdot n' |_{\partial \Omega'}.
\end{align*}
\]

In these derivations we have used (4.3) and the identity \( (A^T)^{-1} R_{\perp} A^{-1} = \det(A^{-1}) R_{\perp} \). Thus if one sets

\[
\varphi' = (\det A) \varphi,
\]

it turns out that the traction free condition (2.6) has exactly the same form as that appearing in torsion of an isotropic shaft with boundary \( \partial \Omega' \). Hence the solution of the torsion of an anisotropic shaft with cross-section \( \Omega \) is now transformed into that for an isotropic shaft with a new cross-section \( \Omega' \).

Now we derive the torsional rigidity of an anisotropic shaft in terms of that of an isotropic one. The torsional rigidity of the anisotropic shaft, defined in (2.28), can be written as

\[
T = \iint_{\Omega} x'_\perp \cdot \mu (\nabla \varphi + x'_\perp) \, dx \, dy.
\]

To proceed, the integrand of (4.5), via the affine transformation, can be expressed as follows:

\[
\begin{align*}
x'_\perp \cdot \mu (\nabla \varphi + x'_\perp) &= R_{\perp} A^{-1} x' \cdot \zeta (A^T A)^{-1} (A^T \nabla \varphi + RA^{-1} x') \\
&= x' \cdot \zeta (A^T)^{-1} R_{\perp} A^{-1} \nabla \varphi + x' \cdot \zeta (A^T)^{-1} R_{\perp} A^{-1} (A^T)^{-1} R_{\perp} A^{-1} x' \\
&= \zeta \det(A^{-1}) x' \cdot R_{\perp} A^{-1} \nabla \varphi + \zeta (\det A)^{-2} x'_\perp \cdot x'_\perp \\
&= \zeta (\det A)^{-2} (x'_\perp \cdot \nabla \varphi' + x'_\perp \cdot x'_\perp).
\end{align*}
\]

Further, since

\[
\frac{\partial (x, y)}{\partial (x', y')} \, dx' \, dy' = (\det A^{-1}) \, dx \, dy,
\]

we have

\[
\begin{align*}
dx \, dy &= \frac{\partial (x, y)}{\partial (x', y')} \, dx' \, dy' = (\det A^{-1}) \, dx' \, dy'.
\end{align*}
\]
a direct substitution of (4.6) and (4.7) into (4.5), using (4.4), gives

\[ T = \frac{\zeta}{(\det A)^3} \int_{\Omega'} \left( \nabla' \psi' + x'_\perp \right) \cdot x'_\perp \, dx' \, dy' \equiv \frac{\zeta}{(\det A)^3} T_{\Omega'}; \] (4.8)

\( T_{\Omega'} \) is the torsional rigidity of an isotropic shaft (with unit shear rigidity) with cross-section \( \Omega' \).

Back to (4.3), we now explore the possible expressions for the transformation matrix \( A \). To do this, we first rewrite (4.3) in the form

\[ A^T A = \zeta \mu^{-1}. \] (4.9)

According to the QR factorization (16, p. 112), any matrix \( A \) can be decomposed as an upper triangular matrix \( \tilde{A} \) multiplied by an orthogonal matrix \( Q \):

\[ A = Q \tilde{A}. \] (4.10)

With this relation, (4.9) can be recast as

\[ \tilde{A}^T \tilde{A} = \zeta \mu^{-1}. \] (4.11)

Further, by the Cholesky factorization (16, p. 114), there always exists a unique upper triangular \((n \times n)\) matrix \( \tilde{A} \) that fulfills (4.11). We thus conclude that, for any given positive definite shear rigidity matrix \( \mu \), one can uniquely determine the matrix \( \tilde{A} \) for any selected positive value \( \zeta \). The general expression of \( A \) can be readily obtained from (4.10) by pre-multiplying an orthogonal matrix \( Q \). We remark that any \( (2 \times 2) \) orthogonal matrix can always be expressed as

\[ Q = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad \text{or} \quad Q = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}, \] (4.12)

in which \( \theta \) is a real parameter.

A simple illustration is the characteristic trajectory outlined in (2.12), in which the two new independent variables, \( x' \) and \( y' \) with \( z = x' + iy' \), are defined by the formulae \( x' = x + p'y \) and \( y' = p'y \). In this case we have

\[ A = \begin{pmatrix} 1 & p' \\ 0 & p'' \end{pmatrix} \] (4.13)

when \( \theta \) in (4.12) is set to be zero. A direct substitution of (4.13) into (4.9) will give

\[ \zeta = \mu^2/\mu_{44} \quad \text{and} \quad \det A = \mu/\mu_{44}. \] (4.14)

Another simple choice of \( A \) that fulfills (4.9) is \( A = \mu^{-\frac{1}{2}} \). In this case, one has \( \zeta = 1 \) and \( \det A = 1/\mu \).

5. Examples of anisotropic shafts that are equivalent to isotropic shafts

For torsion of isotropic shafts, closed-form expressions for warping fields and torsional rigidity are known for a few cross-sectional shapes, such as circles, ellipses, and equilateral triangles (3, 4). For anisotropic shafts under torsion, apart from the geometric shape of \( \Omega \), the corresponding field
quantities also depend on the components of the shear rigidity matrix $\mathbf{\mu}$. Here we are not concerned with solving any field equation. Instead, we raise the following question. Given an anisotropic shaft with a shear rigidity matrix $\mathbf{\mu}$, can we identify permissible shapes of $\Omega$, relevant to $\mathbf{\mu}$, so that after the affine transformation they will carry onto a circle, an ellipse or an equilateral triangle? If anisotropic shafts of such kinds can be characterized, then the corresponding field solutions and torsional rigidity of the anisotropic shafts readily follow from the existing solutions for isotropic shafts through the formulae of (4.4) and (4.8) without any further derivations.

5.1 Elliptical shafts

Let us now consider an anisotropic elliptical shaft with the shear rigidity matrix given in (2.4). The elliptical cross-section $\Omega$ can be characterized by the quadratic form

$$x \cdot \mathbf{L} x = 1,$$

(5.1)

where the geometric matrix

$$\mathbf{L} = \begin{pmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \end{pmatrix}$$

(5.2)

is necessarily positive definite for an elliptical $\Omega$. Upon an introduction of the affine transformation (4.1), it will carry the domain $\Omega$ onto a new cross-section $\Omega'$,

$$x' \cdot \mathbf{L}' x' = 1,$$

(5.3)

in the $x'$ coordinates, where

$$\mathbf{L}' = (\mathbf{A}^{-1})^T \mathbf{L} \mathbf{A}^{-1}.$$  

(5.4)

Simple algebra shows that tr $\mathbf{L}' > 0$ and det $\mathbf{L}' > 0$, which means that $\mathbf{L}'$ is still positive definite. This implies that the new cross-section $\Omega'$ remains an ellipse, but deformed (stretched and rotated) with $\Omega$.

Without loss of any generality, we can assume that the elliptical cross-section $\Omega$ is coaxial with the coordinates $x$, so that $\mathbf{L}$ has the form

$$\mathbf{L} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}^{-1},$$

(5.5)

where $a$ and $b$ are the semi-axis lengths of the ellipse. We now wish to derive the torsional rigidity of the anisotropic elliptical shaft. Since the torsional rigidity of an isotropic elliptical shaft is a simple function of the semi-axis lengths of $\Omega$ (see, for instance (3, p. 122)), to derive the torsional rigidity of the anisotropic elliptical shaft it is necessary to find the new semi-axis lengths of $\Omega'$. For doing this, we can start from the quadratic form of $\Omega'$, (5.4), with $\mathbf{L}$ given in (5.5), by transforming the coordinates $x'$ into the principal axes of $\Omega'$, designated by $\xi = (\xi, \eta)$ (Fig. 1(a)). The new semi-axis lengths of $\Omega'$ can then be written in terms of the eigenvalues of (5.4). With some algebra, the torsional rigidity of the anisotropic elliptical shaft, via (4.8), can be expressed in a simple, explicit form:

$$T_{\text{ellipse}} = \frac{\pi a^3 b^3 \mu_2}{a^2 \mu_{44} + b^2 \mu_{55}}.$$  

(5.6)
When the shaft is isotropic, namely $\mu_{44} = \mu_{55} = \mu$ and $\mu_{45} = 0$, one can easily verify that (5.6) recovers the known torsional rigidity of isotropic elliptical shafts.

Of particular interest is the situation when the transformed cross-section $\Omega'$ becomes a circle. Since an isotropic circular shaft will not warp under torsion, this would imply that a certain class of anisotropic elliptical shafts will not warp under an applied torque. From (5.3) and (5.4), we can see...
that if the transformed cross-section $\Omega'$ is a circle, it is necessary that $L'$ be a multiple of $I$, within a scalar factor. In reference to (4.9), we find that $L$ and $\mu$ are necessarily connected by

$$L\mu = kI,$$  \hspace{1cm} (5.7)

where $k$ is a scalar. This suggests that, for anisotropic elliptical shafts with no warping, the geometric matrix $L$ must be a scalar multiple of $\mu^{-1}$. For simplicity, let us suppose that the principal directions of $\Omega$ are coaxial with the coordinates $x$, that is, $L$ has the form (5.5). In this case, the condition (5.7) is equivalent to

$$b/a = \sqrt{\mu_{44}/\mu_{55}}, \quad \mu_{45} = 0. \hspace{1cm} (5.8)$$

The above connections exactly agree with a recently found constraint for zero warping of homogeneous elliptical shafts (21) through a different approach. Now using (5.8) in (5.6), the corresponding torsional rigidity is obtained as

$$T = \frac{\pi}{2} \mu_{44} a^3 = \frac{\pi}{2} \mu_{55} ab^3. \hspace{1cm} (5.9)$$

which also agrees with that in (21, (2.12)).

As proven in section 3, a minimum of $D(\phi)$ will lead to an upper bound for the torsional rigidity. We thus state the following theorem.

**Theorem 1.** Of all simply connected $\Omega$ with a given cross-sectional area and the same shear rigidities $\mu$, an elliptical shaft satisfying (5.7) has the maximum torsional rigidity. Equivalently, when the coordinates $x$ are coaxial with the principal directions of $\Omega$, then of all $\Omega$ an elliptical cross-section whose major and minor axis lengths satisfy the condition (5.8) has the maximum torsional rigidity.

This theorem can be viewed as the anisotropic counterpart of the celebrated Saint-Venant conjecture for isotropic shafts: ‘Of all simply-connected isotropic cross-sections with given area, a circle has the maximum torsional rigidity’ (5).

### 5.2 Hollow elliptical shafts

We now consider hollow shafts in which the cross-section $\Omega$ is bounded by two ellipses. As in section 5.1, the outer boundary $\partial \Omega$ of the hollow cross-section is described by (5.1) and (5.2). The inner boundary of $\Omega$, denoted by $\partial \Omega_c$ which is the boundary of the cavity core, is described by the equation

$$x \cdot L_c x = 1,$$  \hspace{1cm} (5.10)

with the geometric matrix

$$L_c = \begin{pmatrix} l_{c11} & l_{c12} \\ l_{c12} & l_{c22} \end{pmatrix} \hspace{1cm} (5.11)$$

being positive definite. As shown, the affine transformation will map the outer boundary $\partial \Omega$ onto a new boundary $\partial \Omega'$, characterized by (5.3) and (5.4). Likewise the inner boundary $\partial \Omega_c$ will transform into a new boundary $\partial \Omega'_c$, written as

$$x' \cdot L'_c x' = 1,$$  \hspace{1cm} (5.12)
with
\[ L'_c = (A^{-1})^T L_c A^{-1}. \] (5.13)

We consider two simple situations, in which the torsion solutions for the transformed hollow shafts are known to exist. One is that the transformed cross-section \( \Omega' \) is a hollow elliptical cross-section bounded by two similar ellipses; the other is that \( \Omega' \) is a hollow confocal ellipse. In the first case, we can write
\[ L' = \begin{pmatrix} a'^2 & 0 \\ 0 & b'^2 \end{pmatrix}^{-1}, \quad L'_c = \begin{pmatrix} k^2 a'^2 & 0 \\ 0 & k^2 b'^2 \end{pmatrix}^{-1}, \quad \text{with } 0 < k < 1. \] (5.14)

Here \( a' \) and \( b' \) are the semi-axis lengths of the transformed ellipse \( \Omega' \). Now use of (5.14) in (5.13) and (5.4) will give
\[ L = k^2 L_c, \] (5.15)

which suggests that \( \partial \Omega \) and \( \partial \Omega_c \) be coaxial. In other words, an affine transformation will carry similar ellipses onto similar ellipses. An illustration of the mapping is shown in Fig. 2(a). As the isotropic torsion solution for a hollow ellipse bounded by two similar ellipses is known (4, §115), an anisotropic hollow shaft with the two boundaries characterized by (5.15) can be readily found.

Of particular interest is the situation in which \( a' = b' \), namely
\[ (a')^2 L' = I, \quad k^2 (a')^2 L'_c = I. \] (5.16)

This implies that the transformed cross-section is a concentric circle and thus it will not warp under torsion. Now use of (5.16) in (5.4) and (5.13), in reference to (4.9), will give
\[ (a')^2 L = \zeta \mu^{-1}, \quad k^2 (a')^2 L_c = \zeta \mu^{-1}. \] (5.17)

For simplicity we consider that the two elliptical curves \( \partial \Omega \) and \( \partial \Omega_c \) are coaxial with the coordinates \( x \). Then the outer boundary \( \partial \Omega \) can be described as in (5.1) and (5.5) and the inner boundary \( \partial \Omega_c \) can be written as (5.10) with
\[ L_c = \begin{pmatrix} a_c^2 & 0 \\ 0 & b_c^2 \end{pmatrix}^{-1}, \] (5.18)
in which \( a_c \) and \( b_c \) are the semi-axis lengths of the inner ellipse. Using (5.5) and (5.18), the constraint (5.17) is equivalent to
\[ \frac{b}{a} = \frac{b_c}{a_c} = \sqrt{\frac{\mu_{44}}{\mu_{55}}}, \] (5.19)

which agrees with a recent finding (21, Corollary 2.2) for a hollow elliptical shaft without warping through a different approach. Based on this observation, we can make the following theorem.

**Theorem 2.** Of all multiply-connected cross-sections with a given cross-sectional area and non-zero shear rigidities \( \mu_{55} \) and \( \mu_{44} \), a hollow ellipse bounded by two similar elliptical boundaries, whose major and minor axis lengths fulfill the algebraic conditions (5.19), has the maximum torsional rigidity.
The isotropic counterpart of this theorem is that of all multiply-connected cross-sections the maximum torsional rigidity occurs when the cross-section is a concentric ring \((6, 20)\).

We next consider the other case in which the transformed cross-section \(\Omega'\) is a hollow confocal ellipse. In this case, we can write

\[
L' = \begin{pmatrix} a' & 0 \\ 0 & b'^2 \end{pmatrix}^{-1}, \quad L_c' = \begin{pmatrix} a'^2 & 0 \\ 0 & b'^2 \end{pmatrix}^{-1}.
\]

Since two elliptical boundaries are confocal, the following constraint exists \((22)\):

\[
(L')^{-1} = (L_c')^{-1} + hI,
\]

where \(h\) is a positive constant. From (5.21) and (5.13), this implies that \(L\) and \(L_c\) satisfy

\[
L^{-1} = L_c^{-1} + (h/\zeta)\mu.
\]

In the physical coordinate \(x\), (5.22) suggests that the two boundaries of the hollow ellipse \(\Omega\) need
not have the same principal axes. A schematic illustration of the hollow ellipse, which after the affine transformation will map onto a hollow confocal ellipse, is shown in Fig. 2(b). As the isotropic torsion solution for a confocal ellipse is known (23, pp. 407–409), the torsion solution of an anisotropic hollow shaft with the two elliptical boundaries characterized by (5.22) can be readily found.

5.3 Parallelogram

We now consider the case when the cross-section of the anisotropic shaft is a parallelogram. To see how a parallelogram is transformed under the transformation (4.13), we consider a point \( A \) in Fig. 1(b) relative to the origin \( O \) of the coordinate \( x \) whose position is given by \( x + iy = le^{i\alpha} \). Now the transformation (4.1) with \( A \) given in (4.13) will carry the vector \( \overrightarrow{OA} \) onto the point \( \overrightarrow{O'A} = x' + iy' \), where \( x' = l \cos \alpha + p'l \sin \alpha \), and \( y' = p''l \sin \alpha \). In terms of the polar form \( x' + iy' = l'e^{i\alpha'} \), we have

\[
\frac{l'}{l} = \mu \sqrt{\frac{\mu_{44}}{\mu_{45}}} \left( a^T \mu^{-1} a \right)^{\frac{1}{2}}, \quad \tan \alpha' = \frac{p'' \sin \alpha}{\cos \alpha + p' \sin \alpha},
\]

where \( a = (\cos \alpha, \sin \alpha) \) is the unit vector along \( \overrightarrow{OA} \). Equation (5.23) indicates that the stretching ratio \( l'/l \) and the rotation angle \( \alpha' - \alpha \) depend on the shear rigidities \( \mu \) as well as on the orientation of the vector \( \overrightarrow{OA} \). Here we discuss three simple situations: (a) the original vector \( \overrightarrow{OA} \) is along the horizontal axis, namely \( \alpha = 0 \). A simple observation of (5.23) shows that \( l' = l \) and \( \alpha' = \alpha = 0 \). This means that the horizontal line will not stretch nor rotate during the transformation (4.13); (b) the vector \( \overrightarrow{OA} \) is along the vertical direction, \( \alpha = \pi/2 \). In this case, (5.23) gives

\[
\frac{l'}{l} = \mu \sqrt{\frac{\mu_{55}}{\mu_{44}}} \quad \text{and} \quad \tan \alpha' = \frac{-p'' \sin \alpha}{\cos \alpha + p' \sin \alpha}.
\]

Unless \( \mu_{45} = 0 \), a vertical line in the \( x \) coordinates will stretch as well as rotate under the transformation. (c) for the third situation, we ask what is the orientation of \( \overrightarrow{OA} \) so that, after the transformation, the line becomes vertical, namely \( \alpha' = \pi/2 \)? Equation (5.23) tells us that the orientation of the vector \( \overrightarrow{OA} \) needs to be

\[
\cot \alpha = \mu_{45}/\mu_{44}
\]

and, in such circumstances, the stretch ratio follows as

\[
\frac{l'}{l} = \frac{\mu}{\sqrt{\mu_{44}^2 + \mu_{45}^2}}.
\]

These simple illustrations show that the transformation (4.13) will carry a rectangle into a parallelogram if \( \mu_{45} \neq 0 \). If closed-forms solutions for torsion of isotropic parallelograms were known, the torsion solutions of anisotropic rectangular shafts can be directly found from solutions of isotropic shafts. In case (c), it suggests that it is possible to transform a certain class of parallelograms onto rectangles, and thus the torsion of anisotropic shafts of this particular kind can be readily found from those of torsion solutions for isotropic rectangles without solving any field equations. The torsional rigidity of anisotropic shafts, with cross-sections satisfying (5.25),
can be obtained from that of an isotropic rectangular shaft via a multiplication factor (4.8). The resulting torsional rigidity is

\[ T_{\text{parallelogram}} = \mu_{44}^2 a^3 b - \mu_{44}^4 \frac{64a^4}{\mu} \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^5} \tanh \left( \frac{(2n + 1)\pi b\mu}{2a\mu_{44}} \right). \]  

(5.27)

Let us summarize the solvable parallelograms in the following theorem.

**THEOREM 3.** We consider anisotropic shafts in which the cross-sections are parallelograms in shape, Fig. 1(b). We find that if the angle \( \alpha \) between the two sides fulfills the condition (5.25), then the affine transformation (4.1) with (4.13) will carry parallelograms onto rectangles. Note that the base length remains unchanged \( a' = a \), while \( l' \) is given by (5.25). The associated torsional rigidity follows from the formula (5.27), through the corresponding solutions for torsion of isotropic rectangles.

### 5.4 Triangle

For torsion of isotropic shafts, an equilateral triangle is another exactly solvable cross-sectional shape. The idea is to explore permissible cross-sectional shapes of the anisotropic shafts, so that, after the affine transformation, the transformed cross-section becomes an equilateral triangle. Thereby, the torsion solutions of anisotropic shafts can be directly linked to those of isotropic shafts with equilateral triangles.

Without loss of any generality, we assume that the base line (with length \( a \)) of the triangle is along the \( x \)-axis of the coordinate \( x \). As in previous subsection, we ask what is the orientation \( \alpha \) and length \( l_1 \) of \( \overrightarrow{OA} \), so that, after the affine transformation, the orientation of the line becomes \( \pi/3 \) and its magnitude is equal to the base length \( a \). Now letting \( \alpha' = \pi/3 \) in (5.23), we find

\[ \cot \alpha = \frac{\sqrt{3}\mu_{45} + \mu}{\sqrt{3}\mu_{44}}, \quad l_1 = \frac{a}{2\mu} \sqrt{\mu_{44}(3\mu_{44} + \mu_{55}) + 2\mu_{45}(\mu_{45} + \sqrt{3}\mu)}. \]  

(5.28)

Equivalently, we may consider the other end of the base line (point \( B \)). To ensure that the triangle \( OAB \) will map onto an equilateral triangle, it can be shown that the orientation and the length of \( \overrightarrow{BA} \) need to be

\[ \cot \beta = \frac{\sqrt{3}\mu_{45} - \mu}{\sqrt{3}\mu_{44}}, \quad l_2 = \frac{a}{2\mu} \sqrt{\mu_{44}(3\mu_{44} + \mu_{55}) + 2\mu_{45}(\mu_{45} - \sqrt{3}\mu)}. \]  

(5.29)

The torsional rigidity of anisotropic triangular shafts, which fulfill (5.28) or (5.29), is equivalent to that of isotropic shafts with equilaterally triangular cross-sections. The torsional rigidity of the anisotropic triangular shaft can then be derived from that of the corresponding isotropic shaft as

\[ T_{\text{triangle}} = \frac{\sqrt{3}}{40} \frac{\mu_{44}^2 a^4}{\mu}. \]  

(5.30)

### 6. Concluding remarks

To summarize, we have derived the mathematical frameworks for Saint-Venant torsion of anisotropic shafts. Parallel to those for isotropic shafts, a number of new frameworks and new
exact theorems in anisotropic counterpart have been constructed. For an isotropic shaft, its torsional behaviour depends on the cross-sectional shape of the shaft as well as on its microgeometry. But for an anisotropic shaft, the field quantities and its torsional rigidity may depend on three more degrees of freedom (shear rigidities). These additional shear rigidities have added to the mathematical complexity of the frameworks and, at the same time, enriched the versatility of the physical behaviour. With the recent advances of general theorems and variational bounds on the torsional rigidity of composite shafts, the present study will serve as a theoretical basis for further studies on relevant subjects.

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