STABILITY OF AN IMPLODING SPHERICAL
SHOCK WAVE IN A VAN DER WAALS GAS II

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Summary

The emission of light from a sonoluminescing bubble may depend on whether shock waves launched each acoustic cycle by the imploding surface of the bubble focus on to a sufficiently small volume at the bubble centre. This in turn may depend on whether the shock maintains its stability as it travels inwards. With this application in mind, the linear stability of an imploding spherical shock was studied in Part I, using a van der Waals equation of state for the gas. Conditions for instability were determined, but the subsequent fate of the perturbations of the bubble surface was unknown. Would the instabilities grow and persist at finite amplitude or would they disappear during implosion? The answers to such questions are sought here by integrating the gas dynamics equations using the finite-difference essentially non-oscillatory method of Shu and Osher. The shock is initiated by a nearly spherical ‘piston’ and its subsequent evolution, including its finite-amplitude deviations from sphericity, is determined. Two types of behaviour are found depending on the parameter \( \tilde{b} = b \rho_0 \), where \( b \) is the van der Waals excluded volume and \( \rho_0 \) is the initial uncompressed density of gas ahead of the shock. When \( \tilde{b} \) is sufficiently large, an initially smooth shock front remains smooth as it focuses and, although it is impossible to continue the integrations up to the moment of implosion, it appears that it will focus on a small volume at the centre of the bubble. This is in sharp contrast to what happens at smaller values of \( \tilde{b} \) for which the initial distortion of the shock front, if sufficiently large, becomes and remains polygonal shaped. This is consistent with experimental results for cylindrical imploding shocks as well as with earlier theoretical investigations of imploding cylindrical and spherical shocks in an ideal gas (\( b = 0 \)) that used the Chisnell, Chester and Witham (CCW) approximation or the geometrical shock approximation of Whitham. Plausibly, the polygonal distortions reduce the volume on to which the imploding shock in a sonoluminescing bubble focuses.

1. Introduction

1.1 Motivation

We examine the stability of imploding spherical shock waves in a van der Waals gas. Our motivation is primarily sonoluminescence but the stability of spherically converging shocks is also important in other contexts, most notably for inertial confinement fusion (1). Sonoluminescence is the phenomenon where a bubble of gas, trapped in a liquid, undergoes rapid pulsations due to a periodically varying sound field and emits light.

A leading explanation of the light emission is the so-called shock wave theory of sonoluminescence (2, 3); see (4, 5) for recent discussions. This model proposes the existence of a shock wave

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launched by the supersonic collapse of the bubble surface that concentrates energy in a small region near the centre of the bubble as it implodes. This hot spot is heated to temperatures high enough to ionize the gas and emit light through a process such as bremsstrahlung \((6, 7)\), from the collisions of free electrons and ions in the plasma.

The success of the shock wave theory might ultimately depend on how successful shocks are at depositing energy near the centre of the bubble. This in turn may depend on how well the shocks preserve their spherical shape during implosion. For this reason, shock stability is a significant issue for the shock wave theory of sonoluminescence; one that has already been addressed in Part I of this series \((8)\).

Imploding shock waves that remain spherical as they collapse were first obtained for an ideal gas by Guderley \((9)\), who found the asymptotic form of the solution immediately before and after the moment \(t = 0\) of implosion. This solution has a similarity form, the radius \(R_S\) of the shock being proportional to \((-t)^{\alpha}\) before implosion and \(t^{\alpha}\) after implosion, where \(\alpha(>0)\) depends on the ratio \(\gamma\) of specific heats. The densities reached in sonoluminescing bubbles is so high that Guderley’s results had to be generalized to more realistic equations of state, the simplest being a simplified van der Waals equation of state that recognizes that the excluded molecular volume \(b\) is finite; see \((4, 6, 7)\).

Similarity solutions of Guderley form were again found, with \(\alpha\) a function of both \(\gamma\) and \(\tilde{b} = b\rho_0\), where \(\rho_0\) is the initial density.

The similarity solutions provided the basic time-dependent state for the linear stability analysis of implosions undertaken in \((8)\). This found that the ‘normal modes’ of perturbation are proportional to \((-t)^{\alpha(\beta+1)}\). It was shown that, for all sufficiently small \(\tilde{b}\), some normal modes grow relative to the basic state as the shock moves inwards, that is, \(\beta < 0\). Such states were therefore termed ‘unstable’. In contrast, for the ‘stable’ states of larger \(\tilde{b}\), all normal modes decay relative to the basic state, that is, \(\beta > 0\). The transition from instability occurs at a value of \(\tilde{b}\) that depends on \(\gamma\) and on the form of the perturbation of the shock surface. The question remained, ‘What is the fate of the instabilities?’ Do they grow in amplitude until the moment of implosion, or do they saturate and re-stabilize?

Answers to these key questions can be sought by experimental and theoretical methods, and in fact both have demonstrated the existence of instabilities; further details are given in sections 5 and 6 below. Nearly all investigators have focused on gases that are ideal \((\tilde{b} = 0)\) or effectively ideal \((\tilde{b} \ll 1)\). Only in \((8)\) have larger \(\tilde{b}\) been studied, although these are the cases that are believed to be relevant to bubble collapse. It is beyond the capabilities of current sonoluminescent experimentation to discern an imploding shock within a bubble, much less to determine its structure. Several theoretical approaches suggest themselves, but it appears from \((8)\) that the so-called CCW approximation \((10\text{ to }12)\) is less successful for finite \(\tilde{b}\) than for the ideal gas. A referee suggested to us that weakly nonlinear stability theory might be useful in answering the key questions, particularly in cases where \(\tilde{b}\) is close to unity, when the amplitude of the perturbation might be expected to saturate. In this paper, however, we seek answers from direct numerical integrations. Even though it is impossible to continue such integrations to the instant of implosion, one can seek tell-tale signs of instability in the form of structural changes in the shock front that are still growing at the time when the integrations have to be terminated. If we find these (as we do), we may call the shock ‘unstable’. If, however, the initial distortion of the shock front dies out, or at least does not grow, we may claim that the shock is ‘stable’.

We follow the evolution of a distorted spherical shock wave by solving the conservation laws of gas dynamics. The states ahead of and behind the shock are initially uniform, being separated by a non-spherical diaphragm or ‘piston’ that is removed at the start of the integration. Our
study therefore investigates stability of the sonoluminescing bubble before the final, asymptotic, Guderley-type regime arises; unlike 8, it does not study the stability of the similarity solutions. We use spherical coordinates \((r, \theta, \phi)\) and a spherical computational grid in order to avoid spurious perturbations to spherical shock surfaces. To make the computations feasible with the available computer resources, we assume axial symmetry throughout. Although the coordinate singularity at \(r = 0\) makes it impossible for us to trace the evolution of the shock right up to the moment of implosion, it is clearly desirable to follow it for as long as possible. This is accomplished by contracting the computational grid radially to keep pace with the shock as it moves inwards. This is obviously not the same as using Lagrangian coordinates and is superior since it maintains numerical resolution better as implosion is approached. The relevant equations in the moving frame are derived in section 2. To solve the resulting hyperbolic system of conservation laws, we use the finite-difference essentially non-oscillatory (ENO) method of Shu and Osher (13), which is a third-order total variation diminishing (TVD) scheme. A sketch of our numerical implementation will be found in section 3. Results from our numerical experiments are presented in section 4, and are discussed in section 5. The paper concludes with a brief summary (section 6).

We should emphasize that the material presented here is slightly unusual in two respects. First, it shows how deviations in the equation of state from the perfect gas law affect the finite-amplitude evolution of the shock. Secondly, its findings are derived from the full gas dynamic equations; it makes no use of the CCW approximation or Whitham’s geometric approximation (14). In this respect, our work parallels that of Takayama and Watanabe (15), who solved the full gas dynamic equations for cylindrical implosions of a perfect gas, using a different total variation diminishing (TVD) numerical scheme.

2. Moving reference frames

2.1 Change of frames in general case

In this section, we derive the conservation laws of gas dynamics in a moving frame in spherical coordinates. Our approach, although presumably equivalent to that of Vinokur (16), is more convenient for the present application. We denote the ‘laboratory’ reference frame by \(\mathcal{F}^*\) and the contracting frame by \(\mathcal{F}\). After the governing equations have been derived, they are used to generate the results of section 4, but are returned to \(\mathcal{F}^*\) before the discussion in section 5.

Let \(\mathbf{x}^*\) be the position of a fluid element at time \(t^*\) in frame \(\mathcal{F}^*\). Refer this to a point \(\mathbf{x}\) at time \(t\) in frame \(\mathcal{F}\) through a mapping

\[
\mathbf{x}^* = \mathbf{x}(\mathbf{x}, t), \quad t^* = t.
\]

The mapping is one-to-one, the inverse being

\[
\mathbf{x} = \mathbf{x}(\mathbf{x}^*, t^*), \quad t = t^*.
\]

The use of both \(t\) and \(t^*\) makes it obvious whether time derivatives are carried out at constant \(\mathbf{x}\) or at constant \(\mathbf{x}^*\). Thus, for example,

\[
u'_i = \frac{\partial x_i}{\partial t^*}
\]

is clearly a time derivative at constant \(\mathbf{x}^*\); we may regard \(\mathbf{u}'\) as the velocity of frame \(\mathcal{F}^*\) relative to frame \(\mathcal{F}\); it varies with position and time.
Let $v^*(x, t) = v^*(x^*(x, t), t)$ be the actual fluid velocity relative to $F^*$; expressed in $F$, this is
\[ v_i = \frac{\partial x_i}{\partial x_j^*} v_j^*. \] (2.4)

Here, $\partial x_i/\partial x_j^*$ represents the distortion created by the transformation, including the concomitant ‘rotation’ of axes. We define $u$ by
\[ u = v + u'. \] (2.5)

This is the fluid velocity relative to $F$, which quite naturally is the fluid velocity relative to $F^*$ plus the velocity of $F^*$ relative to $F$.

We have
\[ v \cdot \nabla = \frac{\partial x_i}{\partial x_j^*} \frac{\partial}{\partial x_i^*} v_j^* = v_j^* \frac{\partial}{\partial x_j} = v^* \cdot \nabla^*. \] (2.6)

Clearly
\[ \frac{D}{Dt^*} = \frac{\partial}{\partial t^*} + v^* \cdot \nabla^* = \left( \frac{\partial}{\partial t} + u' \cdot \nabla \right) + v \cdot \nabla = \frac{\partial}{\partial t} + u \cdot \nabla = \frac{D}{Dt}, \] (2.7)

that is, the material derivative is preserved between the two frames.

We want to define the state variables in the un-starred frame in such a way that when expressing the conservation laws in $F$, the resulting equations will be as close to their original conservation forms as possible. Define
\[ \rho = J \rho^*, \] (2.8)

where
\[ J = \text{det} \left( \frac{\partial x_i^*}{\partial x_j} \right) = \frac{1}{3!} \epsilon_{ijk} \epsilon_{lmn} \frac{\partial x_i^*}{\partial x_l^*} \frac{\partial x_j^*}{\partial x_m^*} \frac{\partial x_k^*}{\partial x_n^*} \] (2.9)
is the Jacobian of the transformation. Then
\[ \frac{D\rho}{Dt} + \rho \nabla \cdot u = J \left[ \frac{D\rho^*}{Dt^*} + \rho^* \nabla^* \cdot v^* \right] = 0. \] (2.10)

The equation of continuity in $F^*$ therefore transforms to
\[ \frac{D\rho}{Dt} + \rho \nabla \cdot u = 0. \] (2.11)

The fact that this has the same form as the continuity equation in $F^*$ motivated the choice (2.8). Other similarly convenient choices are
\[ S = S^*, \quad \epsilon = \epsilon^*, \quad T = T^*, \quad P = J P^* \] (2.12)
for the specific entropy, specific internal energy, temperature and pressure, respectively.

We ignore heat conduction, so that by the first law of thermodynamics $D\epsilon^*/Dt^* = -P^* Dp^{*-1}/Dt^*$. By (2.7) and (2.12), this implies that
\[ \rho \frac{D\epsilon}{Dt} = -P \left( \nabla \cdot u + \frac{j}{j} \right). \] (2.13)

We shall use this to obtain the equation expressing conservation of total energy in the frame $F$. 

2.2 Radial contraction

Now, we specialize to the case of a radially contracting frame $F$. Let $\Lambda(t)$ be the time-dependent geometric contraction factor:

$$x_i^* = \Lambda(t)x_i,$$

(2.14)

where $\dot{\Lambda} < 0$. From the definition of the mapping $x^* = x^*(x, t)$, it follows that

$$\frac{\partial x_i^*}{\partial x_j} = \Lambda \delta_{ij}, \quad \frac{\partial x_i}{\partial x_j^*} = \frac{1}{\Lambda} \delta_{ij}, \quad J = \Lambda^3. \tag{2.15}$$

Also, we have

$$v_i^* = \Lambda u_i + \dot{\Lambda} x_i, \quad u'_i = -(\dot{\Lambda}/\Lambda) x_i$$

(2.16)

and

$$\rho = \Lambda^3 \rho^* \quad \text{and} \quad P = \Lambda^3 P^*. \tag{2.17}$$

After similar reductions, we find that conservation of momentum,

$$\rho Dv_i^* \frac{Dt}{D^*} = -\frac{\partial P^*}{\partial x_i^*}, \tag{2.18}$$

becomes

$$\frac{\partial}{\partial t} (\Lambda^2 \rho u_i) + \frac{\partial}{\partial x_j} (\Lambda^2 \rho u_i u_j) - \frac{\dot{\Lambda}}{\Lambda} (\Lambda^2 \rho u'_i) = -\frac{\partial P}{\partial x_i}. \tag{2.19}$$

From equation (2.13) and $J = \Lambda^3$, we get

$$\rho \frac{D\epsilon}{Dt} = -P \left( \nabla \cdot u + \frac{3}{2} \frac{\dot{\Lambda}}{\Lambda} \right) = -P \nabla \cdot (u - u'). \tag{2.20}$$

Equation (2.19) can be written as

$$\frac{Du_i}{Dt} = \frac{1}{\Lambda^2} \left[ -2\Lambda \dot{\Lambda} u_i - \Lambda \ddot{\Lambda} x_i - \frac{1}{\rho} \frac{\partial P}{\partial x_i} \right]. \tag{2.21}$$

Also

$$\frac{Du'_i}{Dt} = \left( \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j} \right) \left( -\frac{\dot{\Lambda}}{\Lambda} x_i \right) = -\frac{\ddot{\Lambda} \Lambda - \dot{\Lambda}^2}{\Lambda^2} x_i - \frac{\dot{\Lambda}}{\Lambda} u'_i,$$ \tag{2.22}

so that

$$\frac{D}{Dt} (u - u') = -\frac{\dot{\Lambda}}{\Lambda} (u - u') - \frac{1}{\Lambda^2 \rho} \nabla P. \tag{2.23}$$

It therefore follows that

$$\rho \frac{D}{Dt} \left[ \frac{1}{2} \Lambda^2 (u - u')^2 \right] = -(u - u') \cdot \nabla P. \tag{2.24}$$

A combination of (2.11), (2.20) and (2.24) gives

$$\frac{\partial}{\partial t} \left[ \rho \left( \epsilon + \frac{1}{2} \Lambda^2 (u - u')^2 \right) \right] + \nabla \cdot \left[ \rho u \left( \epsilon + \frac{1}{2} \Lambda^2 (u - u')^2 \right) + P (u - u') \right] = 0. \tag{2.25}$$
Writing the total energy per unit volume as
\[ E = \rho \left\{ \epsilon + \frac{1}{2} \Lambda^2 (u - u')^2 \right\}, \] (2.26)
we can express conservation of energy as
\[ \frac{\partial E}{\partial t} + \nabla \cdot \left[ E u + P (u - u') \right] = 0. \] (2.27)

2.3 Governing equations in spherical polar coordinates

Because we are interested in the distortions of spherical shock waves, we now express the conservation laws (2.11), (2.19) and (2.27) for \( \mathcal{F} \) using spherical coordinates \((r, \theta, \phi)\). We restrict attention to cases in which the deviations from spherical symmetry are axisymmetric with respect to the axis 0z of the spherical coordinate system. It is easily shown that a state that is initially axisymmetric remains axisymmetric for all times \( t \). After discarding \( u_\phi \) and all derivatives with respect to \( \phi \), the conservation laws become

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r} (\rho u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{1}{r} (2 \rho u_r + \cot \theta \rho u_\theta) = 0, \] (2.28)

\[ \frac{\partial (\Lambda^2 \rho u_r)}{\partial t} + \frac{\partial}{\partial r} (\Lambda^2 \rho u_r^2 + P) + \frac{1}{r} \frac{\partial}{\partial \theta} (\Lambda^2 \rho u_r u_\theta)
+ \frac{1}{r} (2 \Lambda^2 \rho u_r^2 + \cot \theta \Lambda^2 \rho u_r u_\theta - \Lambda^2 \rho u_\theta^2) - \frac{\ddot{\Lambda} \Lambda^2 \rho u_r'}{\Lambda} = 0, \] (2.29)

\[ \frac{\partial (\Lambda^2 \rho u_\theta)}{\partial t} = \frac{\partial}{\partial r} (\Lambda^2 \rho u_r u_\theta) + \frac{1}{r} \frac{\partial}{\partial \theta} (\Lambda^2 \rho u_\theta^2 + P)
+ \frac{1}{r} (3 \Lambda^2 \rho u_r u_\theta + \cot \theta \Lambda^2 \rho u_\theta^2) = 0, \] (2.30)

\[ \frac{\partial E}{\partial t} + \frac{\partial}{\partial r} [Eu_r + P (u_r - u'_r)] + \frac{1}{r} \frac{\partial}{\partial \theta} [(E + P) u_\theta]
+ \frac{1}{r} [2 Eu_r + 2 P (u_r - u'_r) + \cot \theta (E + P) u_\theta] = 0. \] (2.31)

Since the frame contraction is radially inwards, \( u'_\theta = 0 \); (2.26) becomes
\[ E = \rho \left\{ \epsilon + \frac{1}{2} \Lambda^2 [(u_r - u'_r)^2 + u_\theta^2] \right\}. \] (2.32)

2.4 Van der Waals model

Because of the extremely high pressures and densities reached during implosion, the ideal gas law is an inadequate model; a simplified van der Waals equation of state is much more satisfactory (see, for example, (7, 8)). For this model
\[ \epsilon^* = \frac{(V^* - b) P^*}{\gamma - 1}, \quad P^* = (\gamma - 1) \epsilon^*, \quad c_s^2 = \left( \frac{\partial P^*}{\partial \rho^*} \right)_s = \frac{\gamma V^* s^2 P^*}{V^* - b}, \] (2.33)
where \( V^* (=1/\rho^*) \) is the specific volume, \( b \) is the van der Waals excluded volume, \( \gamma \) (\( =\) constant) is the ratio of specific heats and \( c^* \) is the (adiabatic) speed of sound; (2.33)\(_{2,3}\) follow from (2.33)\(_1\) by standard thermodynamic relations. By applying (2.17), it is seen that

\[
\begin{align*}
\epsilon &= (\Lambda^3 - b\rho)\rho, \\
P &= \frac{(\gamma - 1)\Lambda^3 \epsilon}{\Lambda^3 - b\rho}, \\
c^2 &= \left( \frac{\partial P}{\partial \rho} \right)_S = \frac{\gamma \Lambda^3 P}{\rho(\Lambda^3 - b\rho)}. \tag{2.34}
\end{align*}
\]

From the definition of the total energy (2.32), together with (2.34), we find the equation of state in frame \( F \) as

\[
P = \frac{(\gamma - 1)\Lambda^3}{\Lambda^3 - b\rho} \left\{ E - \frac{1}{2} \Lambda^2 \rho \left[ (u_r - u'_r)^2 + u_\theta^2 \right] \right\}. \tag{2.35}
\]

3. Numerical method

3.1 Formulation

We write (2.28), (2.29), (2.30) and (2.31) as

\[
U_t + [F(U)]_r + \frac{1}{r} [G(U)]_\theta + H(U) = 0, \tag{3.1}
\]

where

\[
U = \begin{pmatrix}
\rho \\
\Lambda^2 \rho u_r \\
\Lambda^2 \rho u_\theta \\
E
\end{pmatrix}, \quad F(U) = \begin{pmatrix}
\rho u_r \\
\Lambda^2 \rho u_r^2 + P \\
\Lambda^2 \rho u_r u_\theta \\
E u_r + P(u_r - u'_r)
\end{pmatrix}, \tag{3.2}
\]

\[
G(U) = \begin{pmatrix}
\rho u_\theta \\
\Lambda^2 \rho u_r u_\theta \\
\Lambda^2 \rho u_\theta^2 + P \\
(E + P) u_\theta
\end{pmatrix}, \tag{3.3}
\]

and

\[
H(U) = \begin{pmatrix}
r^{-1}(2\rho u_r + \cot \theta \rho u_\theta) \\
r^{-1}(2\Lambda^2 \rho u_r^2 + \cot \theta \Lambda^2 \rho u_r u_\theta - \Lambda^2 \rho u_\theta^2 - (\tilde{\Lambda}/\Lambda)(\Lambda^2 \rho u'_r)) \\
r^{-1}(3\Lambda^2 \rho u_r u_\theta + \cot \theta \Lambda^2 \rho u_\theta^2) \\
r^{-1}(2E u_r + 2P(u_r - u'_r) + \cot \theta (E + P) u_\theta)
\end{pmatrix}. \tag{3.4}
\]

Equation (3.1) expresses the gas dynamic laws in \( F \) in conservative form. More precisely, because spherical coordinates are used so that \( H(U) \) is non-zero, (3.1) is in weak conservative form.

We use the third-order finite-difference ENO method of Shu and Osher with a third-order Runge–Kutta ordinary differential equation (ODE) solver to solve the hyperbolic system (3.1). This is a high-accuracy, shock-capturing numerical method for hyperbolic systems of conservation laws. It is a semi-discrete method (also called the method of lines), that is, spatial derivatives are differenced independently of time differencing. Our implementation is exactly as described in (13).
The ENO discretization is applied independently to the convection terms \([F(U)]_r\) and \(r^{-1}[G(U)]_\theta\). The \(H(U)\) term, which involves no differentiation, is simply evaluated at point values. In the discretization procedure for the convection terms, the eigenvalues and a system of orthonormal left and right eigenvectors of the Jacobian matrices \(\partial F/\partial U\) and \(\partial G/\partial U\) are required. We evaluate the eigensystems midway between grid points, using Roe’s average of states (17).

Once we have a numerical approximation to the terms \([F(U)]_r\), \([G(U)]_\theta\) and \(H(U)\), we can write the system of equations (3.1) abstractly as a system of ODEs:

\[
U_r = f(t, U). \quad (3.5)
\]

For the time integration of this system, the TVD third-order Runge–Kutta method of Shu and Osher is used:

\[
\begin{align*}
U^* &= U^n + \Delta t f(U^n), \\
U^{**} &= U^n + \Delta t \left[ \frac{1}{6} f(U^n) + \frac{1}{2} f(U^*) \right], \\
U^{n+1} &= U^n + \Delta t \left[ \frac{1}{6} f(U^n) + \frac{1}{2} f(U^*) + \frac{2}{3} f(U^{**}) \right].
\end{align*}
\]

Our computational domain is a half-disc, corresponding to a meridional cross-section of our (axisymmetric) shock. Grid points are uniformly distributed in both the \(r\)- and \(\theta\)-directions, as required by the ENO method. The physical domain consists of \(N\) grid points, \(i = 3, \ldots, N + 2\), in the \(r\)-direction and \(M + 2\) grid points, \(j = 2, \ldots, M + 3\), in the \(\theta\)-direction. There is no grid point at the origin, the first radial grid point being at \(r_3 = \frac{1}{2} \Delta r\). The grid spacings are therefore

\[
\Delta r = \frac{R_{\text{max}}}{N - \frac{5}{2}}, \quad \Delta \theta = \frac{\pi}{M + 1}. \quad (3.6)
\]

In addition, similarly spaced ‘ghost points’ are situated at \(i = 0\) to \(2\), \(i = (N + 3)\) to \((N + 5)\), \(j = 0\) to \(2\) and \(j = (M + 4)\) to \((M + 6)\) to ensure that the boundary and symmetry conditions can be properly satisfied. Full details can be found in (18).

3.2 Test problem: plane shock wave

Although a plane shock has a simple solution that would normally be obtained by a one-dimensional integration, it is also an axisymmetric solution that can be solved on our spherical grid. It then provides a stringent test of our procedures. It must be stressed that although the one-dimensional integration can be cast in strong conservation form, the axisymmetric integration cannot, since it necessarily involves curvature terms, such as \(H(U)\) in our formulation. The possibility exists that conservation is violated. This violation might, for instance, manifest itself as an unphysical source of velocity, causing the shock to propagate with different speeds at different locations on the shock front, so destroying its planar form.

As a test of our method, we therefore used our spherical code to solve for a plane shock wave propagating in the positive \(z\)-direction in a polytropic ideal gas. Initially

\[
U(z, 0) = \begin{cases} 
U_1 & \text{for } z > z_0, \\
U_2 & \text{for } z < z_0,
\end{cases} \quad (3.7)
\]
where
\[ \rho_1 = 1, \quad P_1 = 1, \quad v_1 = 0 \] (3.8)
and
\[ \rho_2 = 5.97, \quad P_2 = 1001, \quad v_2 = 28.85. \] (3.9)
Here, \( v \) is velocity in the \( z \)-direction. The initial conditions are such that at the discontinuity, the Rankine–Hugoniot jump conditions are satisfied (see (14, pp. 172–174) for various forms of the shock conditions); thus, the discontinuity propagates as a shock wave along the \( z \)-axis with a constant speed \( u_z \). The shock strength is
\[ S = \frac{P_2 - P_1}{P_1} = 1000 \] (3.10)
so that the Mach number, for the assumed \( \gamma = \frac{7}{5} \), is
\[ M = \left( 1 + \frac{\gamma + 1}{2\gamma} S \right)^{1/2} = 29.29. \] (3.11)
The speed of sound in region ‘1’ is \( c_1 = 1.18 \) and the shock speed is \( u_z = Mc_1 + v_1 = 34.66 \).

On the spherical boundary, \( r = 2 \), we impose the correct solution
\[ U(z, 0) = \begin{cases} U_1 & \text{for } z > z_0 + u_z t, \\ U_2 & \text{for } z < z_0 + u_z t. \end{cases} \] (3.12)
As already noted in section 1.1, it is not possible in spherical coordinates to follow the evolution of a shock moving through \( r = 0 \), and we therefore initiate the shock at \( z_0 = 0.5 \) so that it moves away from the origin. The integration terminates at \( t = 1/u_z \) so that the final shock location is \( z = 1.5 \). Results are shown in Fig. 1. It is evident that the shock maintains its planar structure well.

The flatness of the \( \rho \), \( P \) and \( v \) ‘tables’ describing the state behind the shock is particularly striking.

4. Numerical experiments
4.1 The initial state
To investigate the stability of imploding spherical shock waves, we examined the evolution of shocks with various initially perturbed shapes. The shocks were launched by a perturbed spherical ‘piston’, which is a diaphragm of perturbed (non-spherical) shape that separates the gas at \( r = R_s + \epsilon f(\theta) \) into two domains with different initial states:
\[ U(r, \theta, 0) = \begin{cases} U_{\text{in}} & \text{for } r < R_s + \epsilon f(\theta), \\ U_{\text{out}} & \text{for } r > R_s + \epsilon f(\theta). \end{cases} \] (4.1)
The gas is initially at rest everywhere and the pressure in the outer domain is \( 10^4 \) times the pressure in the inner domain. The ratio of specific heats \( \gamma \) is taken to be \( \frac{7}{5} \). At \( t = 0 \), the diaphragm is removed and the shock is launched. All results are given in dimensionless units.
Fig. 1 Plane shock test results: density, pressure and flow speed in dimensionless units shown in the $x > 0$ part of the $(x, z)$-cross-section, where $x$ is the distance from the symmetry axis; left-hand column, initial state and right-hand column, final state.

4.2 Spherical harmonic deformations of a spherical shock

Axisymmetric deformations of the sphere represented by spherical harmonics are produced by taking

$$f(\theta) = L_n(\cos \theta), \quad (4.2)$$

where $L_n$ is a Legendre polynomial of degree $n$. Figure 2 shows the deformations of a sphere represented by the three spherical harmonics $L_2$, $L_3$ and $L_4$. (The $n = 1$ case is uninteresting since it corresponds to bodily translation of the spherical shock along the $z$-axis.)

For each of the three types of deformation, we conducted experiments with various combinations of the following parameter values.

- The van der Waals excluded volume: $b = 0, 0.05, 0.25, 0.4$ and $0.8$.
- The uniform initial density (for $b \neq 0$): $\rho = \rho_{\text{max}}/20$ and $2\rho_{\text{max}}/5$, where $\rho_{\text{max}} = 1/b$ is the maximum density of the van der Waals gas.
- The initial deformation amplitude: $\epsilon = R_s/8$ and $R_s/40$ corresponding to ‘big’ and ‘small’ deformations, respectively.

4.3 Definitions and results of experiments

Results from eight numerical experiments are shown. They are representative of the various types of evolution of the imploding shock that were observed. As in the linear stability analysis of the spherical shock wave in (8), the results are best understood in terms of the dimensionless...
Fig. 2 Deformations of the sphere represented by the Legendre polynomials $L_2$, $L_3$ and $L_4$ shown in (a), (b) and (c), respectively.

Fig. 3 Experiment 1: evolution of the shock front after a large initial $L_3$-deformation ($\epsilon = R_s/8$) for $\tilde{b} = 0.4$ parameter

$$\tilde{b} = b\rho_0,$$

where $\rho_0$ is the initial uncompressed density. All experiments were carried out on an $N \times M = 400 \times 100$ grid, with a constant time step $\Delta t = 2 \times 10^{-7}$. The grid was contracted at a constant rate $\Lambda$, suitably adjusted for each experiment; the value of $\Lambda$ is given.

Experiment 1: $L_3$ deformation, $\tilde{b} = 0.4$. In this experiment, $b = 0.05$, $\rho = 8$ everywhere initially and $\epsilon = R_s/8$, where $R_s = \frac{2}{3}$, that is, it is a ‘big’ deformation; $\Lambda = -70$. The shock evolves as shown in Fig. 3. Contour plots of density and pressure at various times are shown in Figs 4 and 5. The shock profiles shown in Fig. 3, and in all similar figures that follow, were obtained by choosing the shock position along each direction from the origin to be the closest grid point to the origin.
Fig. 4 Experiment 1: evolution of the density after a large initial $L_3$-deformation for $\tilde{b} = 0.4$

Fig. 5 Experiment 1: evolution of the pressure after a large initial $L_3$-deformation for $\tilde{b} = 0.4$
at which the pressure jumps radially by a fixed factor, which we chose to be 10. The time \( t \) after shock initiation is specified for each profile by the key in the top right-hand corner of the figure. Each profile (except the one that shows the initial shape of the ‘piston’) lies within the profile for the previous time.

**Experiment 2:** \( L_3 \) deformation, \( \tilde{b} = 0.05 \). This resembles experiment 1, except that initially \( \rho = 1 \) everywhere; \( \Lambda = -140 \). Note that for \( b = 0.05 \), \( \rho_{\text{max}} = 20 \), so that the initial density is significantly further from the van der Waals hardcore limit than in experiment 1. Results of this experiment are shown in Figs 6–8.

**Experiment 3:** \( L_4 \) deformation, \( \tilde{b} = 0.4 \). In this experiment, \( \rho = 1 \) everywhere initially and \( \epsilon = R_s/8 \), where \( R_s = \frac{2}{3} \); \( \Lambda = -205 \). The shock front evolution is shown in Fig. 9.

**Experiment 4:** \( L_4 \) deformation, \( b = 0 \). This resembles experiment 3, except that the gas is ideal; \( \Lambda = -133 \). The shock front evolution is shown in Fig. 10.

The final four experiments concerned \( L_2 \) deformations: the evolution of shock waves launched by a spheroidally shaped ‘piston’. The spheroid has an equatorial radius of \( a \) and a polar radius of \( d \), so that it is oblate if \( a > d \) but prolate if \( a < d \). The inner and outer domains are initially separated at \( r = 1/\sqrt{(\cos^2 \theta/d^2 + \sin^2 \theta/a^2)} \).

**Experiment 5:** Oblate deformation for \( \tilde{b} = 0.05 \). In this experiment, \( \rho = 1 \) everywhere initially and \( a/d = 9/7 \); \( \Lambda = -140 \). The shock front evolution is shown in Fig. 11.

**Experiment 6:** Prolate deformation, \( \tilde{b} = 0.05 \). This resembles experiment 5, except that \( a/d = 7/9 \); \( \Lambda = -140 \). The shock front evolution is shown in Fig. 12.

Finally, we repeat Experiments 5 and 6 with a different \( b \).

**Experiment 7:** Oblate deformation for \( \tilde{b} = 0.4 \). This experiment is otherwise similar to experiment 5 and \( a/d = 9/7 \). The shock front evolution is shown in Fig. 13.

**Experiment 8:** Prolate deformation for \( \tilde{b} = 0.4 \). This experiment is otherwise similar to experiment 6 and \( a/d = 7/9 \). The shock front evolution is shown in Fig. 14.

### 5. Analysis and discussion

Converging cylindrical and spherical shock waves have been the subject of many investigations. Whitham (12) studied converging cylindrical shocks using his theory of geometric shock dynamics, which is an approximate method that determines the motion of the leading shock front explicitly, without determining the flow behind the shock. He showed that, as small disturbances to the cylindrical shock surface grow, discontinuities in Mach number and slope along the leading shock front, referred to as ‘shock-shocks’, would develop. At each discontinuity, or ‘corner’ along the shock front, a shock, also called a ‘Mach tail’, propagates along the main shock front.

Schwendeman and Whitham (19) used Whitham’s theory of geometrical shock dynamics to study cylindrical shocks that initially have regular polygonal-shaped cross-sections. They showed that these polygonal-shaped forms repeat with successive contractions in scale. They found that there is a strong tendency for distorted circular forms to assume polygonal shapes.

In (20), the behaviour of converging shocks in three dimensions was also considered. It was found that smooth, three-dimensional distortions of converging cylindrical and spherical shocks tend to form nearly planar Mach stems similar to those observed in the two-dimensional case.
Fig. 6 Experiment 2: evolution of the density after a large initial $L_3$-deformation for $\tilde{b} = 0.05$

Fig. 7 Experiment 2: evolution of the pressure after a large initial $L_3$-deformation for $\tilde{b} = 0.05$
The stability of converging cylindrical and spherical shock waves in an ideal gas was investigated by Gardner et al. (21), using the so-called ‘CCW approximation’ (10 to 12). This study concentrated on the behaviour of converging cylindrical and spherical shocks subject to initial distortions (possibly large) in the form of isolated bulges. An initial bulge flattened and ultimately formed
Fig. 10 Experiment 4: evolution of shock front in an ideal gas after large initial $L_4$-deformation

shock-shocks and planar Mach stem-like shocks in agreement with the results of Schwendeman and Whitham.

The tendency for converging cylindrical shocks to develop planar sides and polygonal shapes was observed experimentally by Tayakayma et al. (22) and theoretically also by Watanabe and
Fig. 12 Experiment 6: evolution of shock front after large initial prolate deformation; $\tilde{b} = 0.05$

Fig. 13 Experiment 7: evolution of shock front after large initial oblate deformation; $\tilde{b} = 0.4$

Takayama (15). In the experiments reported by Apazidis et al. (23), converging polygonal shocks were produced by reflecting a cylindrical outgoing shock from a smoothly distorted circular boundary. Some of their results are reproduced in Fig. 15, where shock-shocks are seen behind every corner of the shock front.
Fig. 14 Experiment 8: evolution of shock front after large initial prolate deformation; $\tilde{b} = 0.4$.

Fig. 15 Converging shock wave in an argon-filled chamber, together with the wave system behind the shock. Segments $A$, $B$ and $D$ mark three separate shocks joined at a triple point. Segment $C$ indicates a contact discontinuity emerging from the triple point and $E$ shows a system of outgoing waves that interact with the oncoming shock. [From (23), with the permission of the authors and publisher (Springer)]
In our numerical experiments, we observed two types of behaviour. In the first type, illustrated by the results of experiments 1, 3, 7 and 8, the shock surface remains smooth as it implodes. Regions that are initially concave forward, that is, have a smaller radius of curvature, accelerate faster and overtake neighbouring regions on the front. At this point, the geometry is reversed and the regions that now lag behind begin to accelerate more rapidly. For the shapes studied here, the effect is a repeated oscillation in the shape of the shock as it implodes. According to Fig. 3, at the final computed time in experiment 1, the second half of the oscillation, during which the shock reverts to its initial shape, appears to be only beginning, whereas in experiment 3, the initial shock shape is reproduced at \( t = 0.0039 \) and by the final computed time one and a half oscillations have been completed (see Fig. 9).

In the second type of behaviour, exemplified by experiments 2, 4, 5 and 6, the initially smooth shock surfaces generate polygonal-shaped cross-sections. Regions where the shock front is initially concave forward develop straight segments as they accelerate past neighbouring portions of the front. As these regions are overtaken by the straight segments and start to accelerate faster because of their smaller radius of curvature, they in turn evolve into straight segments too. Eventually, the cross-section of the shock front consists entirely of straight segments. Further evolution of the shock front sees adjacent sides of the polygonal-shaped shock converge, absorbing a straight segment as they do so. This process is clearly seen in Figs 6 and 10.

The results of our study suggest that in the range \( 0 \leq \tilde{b} \leq 0.25 \), the front of a perturbed imploding strong spherical shock evolves into a polygonal shape for a sufficiently large initial perturbation of any type. In our axisymmetric setting, polygonal cross-sections correspond to a shock surface consisting of conical slices. In the non-axisymmetric case, the shock surface would more probably consist of planar patches. The structure of the shock front is similar to those found in the investigations of ideal gases mentioned above, with ‘shock-shocks’ connecting Mach stem-like straight segments and ‘Mach tails’ developing behind each corner of the shock cross-section. These features are clearly visible in the contour plots of experiment 2; see Figs 7 and 8. The Mach tails, which are shocks in the flow behind the main shock front, are a source of dissipation of shock energy. We regard the spherical shock to be (mildly) unstable in this regime.

For \( \tilde{b} \geq 0.4 \), the shock front remains smooth for all types of perturbation during the time that our computation was able to follow the implosion. There is no evidence of any instability. For the symmetric initial forms studied here, the shock shape undergoes oscillations, as regions that are concave forward accelerate past neighbouring portions and are overtaken in their turn. This is a stable focusing mechanism that would probably also prevent a more general initial distortion of the front from deviating far from spherical form as the shock implodes. This suggests that the converging spherical shock is stable in this regime.

6. Summary

We have carried out a numerical investigation of the nonlinear stability of an imploding strong spherical shock wave in a van der Waals gas. A perturbed spherical piston launched an imploding shock wave, whose evolution was studied by numerically solving the conservation equations of gas dynamics in a moving frame. By solving the conservation laws directly, we were able to simulate the full dynamics of the converging shock, including the flow behind the main shock front. Previous studies (see references quoted in section 5), employing Whitham’s approximate theory of geometrical shock dynamics or the CCW approximation, were not able to do this.
We have observed two types of behaviour of the converging shock depending on the value of the parameter $\tilde{b} = b\rho_0$, where $b$ is the van der Waals excluded volume and $\rho_0$ is the initial uncompressed density. In the range $0 \leq \tilde{b} \leq 0.25$, the shock front evolves, in all the cases that we examined, into a polygonal shape, provided that the initial distortion of the shock front is sufficiently large. In our axisymmetric setting, polygonal cross-sections correspond to a shock surface consisting of conical slices. In the general, non-axisymmetric case, the shock surface would more probably consist of planar patches. The structure of the shock front is similar to those found in other investigations of converging cylindrical and spherical shocks in ideal gases. ‘Mach tails’, or shocks emanating from each corner of the shock front, are a source of dissipation of the shock energy. The spherical shock may be regarded as mildly unstable in this parameter regime.

For $\tilde{b} \geq 0.4$, the shock evolves smoothly, with no appearance of instability of any kind. Because of its smaller radius of curvature, a portion of the shock front that is concave forward accelerates past neighbouring sections but later, as the curvature of the lagging sections grows and their acceleration increases, they in turn overtake and resume their leading position. This oscillation continues until implosion and prevents the shock front from departing far from a spherical shape. Although these results were established only for a few special initial distortions of the shock front, the mechanism appears to be robust and likely to maintain a shock, initially distorted in a more general way, close to spherical form during implosion. We regard this as evidence that the shock is stable in this parameter range.

Our results suggest that stability of the imploding spherical shock is enhanced as $b$ is increased. This is consistent with the linear stability results of Wu and Roberts (8). Even for small $b$, where some of the shock energy is dissipated by reflecting shock waves, the shock maintains a front mainly consisting of flat patches as it implodes. In the application to single-bubble sonoluminescence, our results imply that the imploding shock wave may well be able to deposit a large amount of energy near the centre of the bubble, as proposed by the shock wave theory. It should be noted, however, that the results of our computations, which follow the shock from its inception by a piston, might not apply to the very late stages of the implosion, when the shock is known to approach a similarity form (8). To examine the stability of the imploding shock near the final stage of collapse, a non-linear stability analysis of the similarity solution itself should be performed but this is beyond the scope of this paper.

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References


