WAVE PROPAGATION THROUGH CASCADING SCREENS OF FINITE THICKNESS WITH PERIODIC DISTRIBUTION OF OPENINGS

by EDOARDO SCARPETTA and VINCENZO TIBULLO†

(Dipartimento di Ingegneria dell’Informazione e Matematica Applicata, University of Salerno, 84084 Fisciano (Salerno), Italy)

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Summary

We develop an analytical approach to study normal penetration of a scalar plane wave into an arbitrary number of parallel, equidistant screens of finite thickness; on each screen, there is a periodic distribution of rectangular openings. After reducing the problem to a system of integral equations and then to an algebraic linear system, suitable assumptions on the physical and geometrical parameters lead to explicit representations for the scattered wave field and the relevant parameters. Some figures are given to reflect the peculiar wave properties of the cascading structure under consideration.

1. Introduction

In some previous papers, the scalar problem of wave propagation through scattering structures made by periodic distributions of obstacles has been studied from a new analytical point of view (1 to 3). Of course, the geometrical characteristics of the structure are deeply involved in all scattering problems, whatever may be the method of approach. In (1) and (3), slit-type and rectangular scatterers have been considered, respectively, which are distributed in a one-dimensional (periodic) array, while in (2) a two-dimensional (2D) (doubly periodic) grating is treated in which an arbitrary number of arrays of slit-type scatterers are located one after the other, giving rise to the so-called `cascading’ structure.

As is known, this type of problem has a wide range of practical applications in many branches of engineering science, such as mechanics, acoustics and electromagnetism; we refer to the books of Krautkramers (4) and Jones (5) for a survey of the matters investigated in these fields.

Thus, in this paper, we aim to study a doubly periodic cascading structure, like in (2), in which the scatterers along each array are of rectangular form, like in (3). Of course, this is equivalent to consider as many (parallel and equidistant) screens of finite thickness having rectangular openings periodically—complementarily—distributed. The procedure will not be fully analytical (as it was in (1, 2)); nevertheless, by using some mild approximations that are quite typical in this context, we will derive explicit formulas with respect to frequency for the scattered wave field and the relevant parameters; however, to calculate certain coefficients, we will need to solve numerically—just once for each value given to the geometric parameters—a few linear integral equations independent of frequency (the same as in (3)).
The guidelines are as follows. Starting from natural boundary conditions and continuity assumptions, the problem will be first reduced to $2M$ integral equations based on the openings’ height, where $M$ is the number of screens (or arrays) and then to a $2M \times 2M$ linear system having fruitful properties from an algebraic standpoint. Solution of such a system by suitable recurrence formulas will thus lead to the sought representations of the main unknowns showing explicitly the dependence on frequency. Show and comment on the relevant properties of the structure, the results obtained will be finally reflected in some figures; a corresponding full-numerical investigation of the main equations will also provide an useful test of accuracy.

The approximations employed—and alluded to above—are based on the following assumptions: (i) the frequency of the (harmonic) incident wave implies ‘one-mode’ far-field propagation and (ii) the screens are sufficiently distant from each other. In this connection, we should note that a lot of papers in the literature which are devoted to scattering problems in analogous contexts are based on equivalent assumptions. Besides (2), the reader can be addressed to (6 to 10), along with the numerous references therein quoted, to become acquainted with other (somewhat different) treatments of such problems starting from those assumptions. However, we should note that, as far as we are concerned, such treatments sooner or later end by using direct numerical methods or approximations through suitable (Galerkin-type) expansions, and thus, the results they produce cannot be expressed by means of explicit formulas holding throughout a given range of frequency.

According to the specific interpretation given to the wave field and scattering structure, the proposed framework can be applied to study (linear) wave propagation in acoustics, elastodynamics or electromagnetism.

2. Mathematical formulation

Let us consider an unbounded 2D medium in which there is a cascading structure made by an arbitrary (but finite) number of parallel, equidistant screens having the same (nonzero) thickness. All such screens have the same periodic array of (identical) openings of rectangular form. We denote by $\delta = d - 2\ell$ the distance between adjacent screens, $2\ell$ their thickness, $2a$ the period on each (vertical) array and $2b$ the height of the openings (whose length clearly equals $2\ell$). See Fig.1 for a description of the geometry.
In the assumed harmonic regime, the time dependence implies the common factor $e^{-i \omega t}$ in all field variables, where $\omega = kc$ is the angular frequency, while $k$ and $c$ are the (transverse) wave number and wave speed of the involved material ($i^2 = -1$); this factor will be omitted throughout.

Thus, we consider a scalar plane wave of unit amplitude

$$\varphi_0(x, y) = e^{ikx}$$

(2.1)

which, arriving from $x = -\infty$, normally penetrates into the described structure, giving rise to scattered fields in the various parts of it. In this connection, denoted by $M (\geq 2)$ the number of screens, we will label the wave field $\varphi(x, y)$ by ‘left’ when $x < -\ell$; by ‘in(ternal)’ when $(m - 1)d - \ell < x < (m - 1)d + \ell$, $m = 1, \ldots, M$ (namely, inside the rectangular openings); by ‘cent(ral)’ when $(m - 1)d + \ell < x < md - \ell$, $m = 1, \ldots, M - 1$ (namely, in the regions between adjacent screens) and finally by ‘right’ when $x > (M - 1)d + \ell$.

Throughout the structure, the wave field is governed by the Helmholtz equation

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2)\varphi + k^2 \varphi = 0. \quad (2.2)$$

Due to the natural symmetry and periodicity along $y$, the problem can be restricted to the single strip $|y| < a$ with $M$ (equidistant) narrowings $|y| < b$ of length $2\ell$, where the following relations should hold:

$$\frac{\partial \varphi_{\text{left}}}{\partial y}(x, \pm a) = \frac{\partial \varphi_{\text{right}}}{\partial y}(x, \pm a) = \frac{\partial \varphi_{\text{cent}}}{\partial y}(x, \pm a) = \frac{\partial \varphi_{\text{in}}}{\partial y}(x, \pm b) = 0 \quad (2.3)$$

(for all pertinent values of $x$) according to a natural boundary condition (of stress-free type) on the strip’s edges and on the rectangles’ horizontal sides.

We start with Fourier-type representations of the wave field in the various regions along the strip, as follows (2, 3, 11):

$$\varphi_{\text{left}} = e^{ik(x+\ell)} + R e^{-ik(x+\ell)} + \sum_{n=1}^{\infty} A_n e^{q_n(x+\ell)} \cos a_n y, \quad x < -\ell, \ |y| < a; \quad (2.4a)$$

$$\varphi_{\text{in}}^m = \sum_{n=0}^{\infty} (C^m_n \cosh r_n (x - [(m - 1)d - \ell]) + D^m_n \cosh r_n (x - [(m - 1)d + \ell])) \cos b_n y, \quad m = 1, \ldots, M, \ (m - 1)d - \ell < x < (m - 1)d + \ell, \ |y| < b; \quad (2.4b)$$

$$\varphi_{\text{cent}}^m = \sum_{n=0}^{\infty} (E^m_n \cosh q_n (x - [(m - 1)d + \ell]) + F^m_n \cosh q_n [x - (md - \ell)]) \cos a_n y, \quad m = 1, \ldots, M - 1, \ (m - 1)d + \ell < x < md - \ell, \ |y| < a; \quad (2.4c)$$

$$\varphi_{\text{right}} = T e^{ik(x - [(M - 1)d + \ell])} + \sum_{n=1}^{\infty} B_n e^{-q_n(x - [(M - 1)d + \ell])} \cos a_n y, \quad x > (M - 1)d + \ell, \ |y| < a. \quad (2.4d)$$

Above, all capital letters denote unknown constants, $a_n = \pi n/a$, $b_n = \pi n/b$ and

$$q_n = \sqrt{a_n^2 - k^2}, \quad r_n = \sqrt{b_n^2 - k^2} \quad (2.5)$$
in order to satisfy the Helmholtz equation. Equations (2.3) also are satisfied. Somewhere, we prefer to let explicitly appear the terms in the summations with \(n\) equal zero (\(q_o = -ik\) by a radiation condition).

If we assume the wave number in the range

\[
(0 <) k < \pi /a,
\]

(2.6)

which implies \(q_n > 0\), \(\forall n \neq 0\), the coefficients of the zeroth-order modes \(R \equiv A_0\) and \(T \equiv B_0\) in (2.4a,d) can be fully interpreted as ‘reflection’ and ‘transmission parameters’, respectively, since with distance from the structure, all other (higher-order) modes in the summations are vanishing. This is just what defines the ‘one-mode regime’ for the far-field propagation (note that inequality (2.6) amounts to take the wavelength \(2\pi /k\) greater than vertical period \(2a\), which is quite usual when dealing with propagation in ‘waveguide’-like structures (11)).

As in our previous paper (2), besides upper bound (2.6) on frequency, we shall also assume that the screens are sufficiently distant from each other with respect to the width of the strip, namely

\[
\delta /a \gg 1.
\]

(2.7)

One can compare with (6 to 10) where similar assumptions are sometimes considered for other treatments of the problem. In particular, (2.7) is typically used to state the so-called ‘wide-spacing’ approximation (12) intended to imply that all nonhomogeneous (‘evanescent’) wave modes between adjacent screens can be \textit{a priori} disregarded; as it will be clear in the sequel, this is not our purpose. (Nevertheless, in Appendix B, the results obtained by a straightforward application of such an approximation will be discussed.)

Of course, a boundary condition of stress-free type should also be imposed on the vertical sides of the screens. In this connection, if we put

\[
\frac{\partial \phi_{\text{left}}}{\partial x} (-\ell, y) \equiv \begin{cases} 
  g_1(y), & |y| < b, \\
  0, & b < |y| < a,
\end{cases}
\]

(2.8a)

\[
\frac{\partial \phi_{\text{cent}}}{\partial x} ((m - 1)d + \ell, y) \equiv \begin{cases} 
  g_{2m}(y), & |y| < b, \\
  0, & b < |y| < a,
\end{cases}
\]

(2.8b)

\[
\frac{\partial \phi_{\text{cent}}}{\partial x} (md - \ell, y) \equiv \begin{cases} 
  g_{2m+1}(y), & |y| < b, \\
  0, & b < |y| < a,
\end{cases}
\]

(2.8c)

\[
\frac{\partial \phi_{\text{right}}}{\partial x} ((M - 1)d + \ell, y) \equiv \begin{cases} 
  g_{2M}(y), & |y| < b, \\
  0, & b < |y| < a,
\end{cases}
\]

(2.8d)

then, the continuity of the stress through the borderlines of the regions implies the following:

\[
\frac{\partial \phi_{\text{in}}}{\partial x} (-\ell, y) = g_1(y), \quad \frac{\partial \phi_{\text{in}}}{\partial x} ((m - 1)d + \ell, y) = g_{2m}(y),
\]

(2.9)

\[
\frac{\partial \phi_{\text{in}}}{\partial x} (md - \ell, y) = g_{2m+1}(y), \quad m = 1, \ldots, M - 1,
\]

\[
\frac{\partial \phi_{\text{in}}}{\partial x} ((M - 1)d + \ell, y) = g_{2M}(y), \quad |y| < b.
\]

Due to symmetry properties, such \(2M\) unknown functions \(g(y)\) are even functions in \(|y| < b\).
By using (2.4) and orthogonality of cosines, (2.8) and (2.9) can be integrated to give all unknown constants of Fourier series in terms of functions $g$ as follows:

$$R = 1 - \frac{1}{2ai_k}G_1, \quad T = \frac{1}{2ai_k}G_{2M}, \quad (2.10a)$$

$$C_o^m = -\frac{G_{2m}}{2bk \sin(2k\ell)}, \quad D_o^m = \frac{G_{2m-1}}{2bk \sin(2k\ell)} \quad (m = 1, \ldots, M), \quad (2.10b)$$

$$E_o^m = -\frac{G_{2m+1}}{2ak \sin(k\delta)}, \quad F_o^m = \frac{G_{2m}}{2ak \sin(k\delta)} \quad (m = 1, \ldots, M - 1),$$

as well as $(n \neq 0)$:

$$A_n = \frac{1}{aq_n}G_1^a(n), \quad B_n = -\frac{1}{aq_n}G_{2M}^a(n), \quad (2.11a)$$

$$C_n^m = \frac{G_{2m}^b(n)}{br_n \sinh(2r_n\ell)}, \quad D_n^m = -\frac{G_{2m-1}^b(n)}{br_n \sinh(2r_n\ell)} \quad (m = 1, \ldots, M), \quad (2.11b)$$

$$\sinh(q_n\delta)E_n^m = \frac{G_{2m+1}^a(n)}{aq_n}, \quad \sinh(q_n\delta)F_n^m = -\frac{G_{2m}^a(n)}{aq_n} \quad (m = 1, \ldots, M - 1), \quad (2.11c)$$

where

$$\begin{bmatrix} G_i^a(n) \\ G_i^b(n) \end{bmatrix} = \int_{-b}^b g_i(y) \cos \left( \frac{a_n y}{b_n y} \right) dy, \quad G_i \equiv G_i^{a,b}(n = 0) \quad (i = 1, \ldots, 2M). \quad (2.12)$$

On the other side, the continuity assumption for the wave field through the borderlines:

$$\varphi_\text{left} = \varphi_\text{in}^1 \text{ at } x = -\ell, \quad \varphi_\text{right}^M = \varphi_\text{right} \text{ at } x = (M - 1)d + \ell,$$

$$\varphi_\text{in}^m = \varphi_\text{cent}^m \text{ at } x = (m - 1)d + \ell, \quad \varphi_\text{cent}^m = \varphi_\text{cent}^{m+1} \text{ at } x = (m + 1)d - \ell, \quad m = 1, \ldots, M - 1, \quad (2.13)$$

implies the following equalities to hold in $|y| < b$:

$$1 + R + \sum_{n=1}^\infty A_n \cos a_n y = \sum_{n=0}^\infty (C_n^M + D_n^M \cosh 2r_n\ell) \cos b_n y, \quad (2.14a)$$

$$\sum_{n=0}^\infty (C_n^M \cosh 2r_n\ell + D_n^M) \cos b_n y = T + \sum_{n=1}^\infty B_n \cos a_n y, \quad (2.14b)$$

$$\sum_{n=0}^\in\infty (C_n^m \cosh 2r_n\ell + D_n^m) \cos b_n y = E_0^m + F_0^m \cos k\delta + \sum_{n=1}^\infty (E_n^m + F_n^m \cosh q_n\delta) \cos a_n y, \quad (2.14c)$$

$$E_0^m \cos k\delta + F_0^m + \sum_{n=1}^\infty (E_n^m \cosh q_n\delta + F_n^m) \cos a_n y = \sum_{n=0}^\in\infty (C_n^{m+1} + D_n^{m+1} \cosh 2r_n\ell) \cos b_n y \quad (m = 1, \ldots, M - 1). \quad (2.14d)$$
By the main assumptions (2.6) and (2.7), we can put \( q_n \delta \gg 1 \), so that

\[
\sinh q_n \delta \approx \cosh q_n \delta \gg 1 \quad (\text{for } n \neq 0)
\]

in (2.11c) and (2.14c,d) (in the worst case, which is for \( q_1 \), it is sufficient to take \( \delta/a = 2 \) to have (in the middle of the range (2.6)) \( q_1 \delta = \pi \delta/a \sqrt{1 - (ak/\pi)^2} \approx 5.4 \), and thus \( \sinh q_1 \delta \approx \cosh q_1 \delta \approx 115 \)). This enables us to neglect the terms \( E^m_n \) and \( F^m_n \) with respect to terms \( F^m_n \cosh q_n \delta \) and \( E^m_n \cosh q_n \delta \), respectively, in the brackets of (2.14c) and (2.14d). In Appendix B, a numerical investigation on the error implied will be performed.

By this approximation, inserting the values of all constants, taken from (2.10) and (2.11), into

\[
\int_{-b}^{b} K_1(t-y) g_1(t) \, dt + \int_{-b}^{b} K_2(t-y) g_2(t) \, dt
\]

\[
= -2 + \left( \frac{1}{2ai} \csc \frac{2k\ell}{2bk} \right) G_1 - \frac{1}{2bk} \csc \frac{2k\ell}{2bk} G_2;
\]

\[
(2.16a)
\]

\[
\int_{-b}^{b} K_1(t-y) g_{2m}(t) \, dt + \int_{-b}^{b} K_2(t-y) g_{2m-1}(t) \, dt
\]

\[
= - \frac{1}{2bk} \csc \frac{2k\ell}{2bk} G_{2m-1} + \left( \cot \frac{k\delta}{2ak} \csc \frac{2k\ell}{2bk} \right) G_{2m} - \frac{1}{2ak} \csc \frac{k\delta}{2bk} G_{2m+1},
\]

\[
m = 1, \ldots, M - 1;
\]

\[
(2.16b)
\]

\[
\int_{-b}^{b} K_1(t-y) g_{2m+1}(t) \, dt + \int_{-b}^{b} K_2(t-y) g_{2m+2}(t) \, dt
\]

\[
= - \frac{1}{2ak} \csc \frac{k\delta}{2ak} G_{2m} + \left( \cot \frac{k\delta}{2ak} \csc \frac{2k\ell}{2bk} \right) G_{2m+1} - \frac{1}{2bk} \csc \frac{2k\ell}{2bk} G_{2m+2},
\]

\[
m = 1, \ldots, M - 1;
\]

\[
(2.16c)
\]

\[
\int_{-b}^{b} K_1(t-y) g_{2M}(t) \, dt + \int_{-b}^{b} K_2(t-y) g_{2M-1}(t) \, dt
\]

\[
= - \frac{1}{2bk} \csc \frac{2k\ell}{2bk} G_{2M-1} + \left( \frac{1}{2ai} \csc \frac{2k\ell}{2bk} \right) G_{2M},
\]

\[
(2.16d)
\]

where we have put

\[
K_1(y) \equiv \sum_{n=1}^{\infty} \left( \frac{1}{aq_n} \cos a_n y + \frac{\coth 2r_n \ell}{br_n} \cos b_n y \right), \quad (2.17a)
\]

\[
K_2(y) \equiv \sum_{n=1}^{\infty} \frac{-1}{br_n \sinh 2r_n \ell} \cos b_n y. \quad (2.17b)
\]
It is worth noting now that, by suitably summing and subtracting two by two the above equations, we arrive at the following $2M$ integral equations ($M$ with the upper sign and $M$ with the lower sign):

\[
\int_{-b}^{b} K^\pm(t - y)[g_1(t) \pm g_2(t)] dt
\]

\[= -2 + \left( \frac{1}{2ai k} + \frac{\cos 2k \ell \mp 1}{2bk \sin 2k \ell} \right) G_1 \pm \left( \frac{\cot k \delta}{2ak} + \frac{\cos 2k \ell \mp 1}{2bk \sin 2k \ell} \right) G_2 \pm \frac{1}{2ak \sin k \delta} G_3; \]

(2.18a)

\[
\int_{-b}^{b} K^\pm(t - y)[g_{2m-1}(t) \pm g_{2m}(t)] dt
\]

\[= -2 - \frac{1}{2ak \sin k \delta} G_{2m-2} + \left( \frac{\cot k \delta}{2ak} + \frac{\cos 2k \ell \mp 1}{2bk \sin 2k \ell} \right) (G_{2m-1} \pm G_{2m}) \pm \frac{1}{2ak \sin k \delta} G_{2m+1}, \quad m = 2, \ldots, M - 1; \]

(2.18b)

\[
\int_{-b}^{b} K^\pm(t - y)[g_{2M-1}(t) \pm g_{2M}(t)] dt
\]

\[= -2 - \frac{1}{2ak \sin k \delta} G_{2M-2} + \left( \frac{\cot k \delta}{2ak} + \frac{\cos 2k \ell \mp 1}{2bk \sin 2k \ell} \right) G_{2M-1} \pm \left( \frac{1}{2ai k} + \frac{\cos 2k \ell \mp 1}{2bk \sin 2k \ell} \right) G_{2M}, \]

(2.18c)

where the kernels are

\[K^\pm(y) \equiv K_1(y) \pm K_2(y) = \sum_{n=1}^{\infty} \left( \frac{1}{a q_n} \cos a_n y + \frac{\cosh 2r_n \ell \mp 1}{b r_n \sinh 2r_n \ell} \cos b_n y \right). \]

(2.19)

Now, if $h^\pm(t)$ are two (real and even) functions that solve the ‘auxiliary’ integral equations (3)

\[
\int_{-b}^{b} K^\pm(t - y)h^\pm(t) dt = 1 \quad \left( H^\pm \equiv \int_{-b}^{b} h^\pm(t) dt \right), \]

(2.20)

we get by linearity

\[
g_1(y) \pm g_2(y) = \left[ -2 + \left( \frac{1}{2ai k} + \frac{\cos 2k \ell \mp 1}{2bk \sin 2k \ell} \right) G_1 \pm \left( \frac{\cot k \delta}{2ak} + \frac{\cos 2k \ell \mp 1}{2bk \sin 2k \ell} \right) G_2 \pm \frac{1}{2ak \sin k \delta} G_3 \right] h^\pm(y); \]

(2.21a)

\[
g_{2m-1}(y) \pm g_{2m}(y) = \left[ -2 - \frac{1}{2ak \sin k \delta} G_{2m-2} + \left( \frac{\cot k \delta}{2ak} + \frac{\cos 2k \ell \mp 1}{2bk \sin 2k \ell} \right) (G_{2m-1} \pm G_{2m}) \right] h^\pm(y). \]
\[ h^\pm(y), \quad m = 2, \ldots, M - 1; \]  
\[ g_{2M-1}(y) \pm g_2(y) = \left( -\frac{1}{2ak \sin k\delta} G_{2M-2} + \left( \frac{\cot k\delta}{2ak} + \frac{\cos 2k\ell \mp 1}{2bk \sin 2k\ell} \right) G_{2M-1} \right) \] 
\[ \pm \left( \frac{1}{2ai k} + \frac{\cos 2k\ell \mp 1}{2bk \sin 2k\ell} \right) G_{2M} \right] h^\pm(y). \]  
(2.21c)

An integration of (2.21) over \(|y| < b\) finally leads to the following square \((2M \times 2M)\) system of linear algebraic equations for the unknowns \(G_1, \ldots, G_{2M}\) (\(M\) equations with upper sign and \(M\) equations with lower sign):

\[
\begin{aligned}
&\left[ 1 - \left( \frac{1}{2ai k} + \frac{\cos 2k\ell \mp 1}{2bk \sin 2k\ell} \right) H^\pm \right] G_1 \pm \left[ 1 - \left( \frac{\cot k\delta}{2ak} + \frac{\cos 2k\ell \mp 1}{2bk \sin 2k\ell} \right) H^\pm \right] G_2 \\
&\pm \frac{H^\pm}{2ak \sin k\delta} G_3 = -2H^\pm; \\
&\frac{H^\pm}{2ak \sin k\delta} G_{2m-2} + \left[ 1 - \left( \frac{\cot k\delta}{2ak} + \frac{\cos 2k\ell \mp 1}{2bk \sin 2k\ell} \right) H^\pm \right] (G_{2m-1} \pm G_{2m}) \\
&\pm \frac{H^\pm}{2ak \sin k\delta} G_{2m+1} = 0, \quad m = 2, \ldots, M - 1; \\
&\frac{H^\pm}{2ak \sin k\delta} G_{2m-2} + \left[ 1 - \left( \frac{\cot k\delta}{2ak} + \frac{\cos 2k\ell \mp 1}{2bk \sin 2k\ell} \right) H^\pm \right] G_{2m-1} \\
&\pm \left[ 1 - \left( \frac{1}{2ai k} + \frac{\cos 2k\ell \mp 1}{2bk \sin 2k\ell} \right) H^\pm \right] G_{2M} = 0.
\end{aligned}
\]  
(2.22)

Once solved (2.20) and system (2.22), functions \(g(y)\) can be determined by means of (2.21); hence, (2.10) and (2.11)—along with (2.12)—can be used in (2.4) to get the scattered field throughout the structure, including all the evanescent wave modes (of order \(n \geq 1\) in (2.4)) arising near the screens. In particular, we are interested in the reflection and transmission parameters that are given by formulas (2.10a) (so that only the integrals \(H^\pm\) of functions \(h^\pm(t)\) are concerned). Of course, since the kernels \(K^\pm\) of (2.20) contain the wave number through \(q_n\) and \(r_n\), the dependence of such parameters on frequency remains still implicit.

Note finally that, in the limiting case \(\ell \to 0\) (namely, when the screens reduce to \(M\) arrays of collinear slit-type cracks), (2.20) with plus sign becomes equal to (2.10) in (2), and, putting \(G_{2m-1} = G_{2m}, m = 1, \ldots, M\), the (half) part of system (2.22) with upper sign becomes the \(M \times M\) system (2.13) of that paper.

3. Solution of the auxiliary integral equations and algebraic system

As explained in (3), the (even) solutions \(h^\pm(t)\) of (2.20), along with their integrals \(H^\pm\), can be deduced by solving—numerically—six integral equations each one independent of frequency. These are obtained from (2.20) by means of a mild approximation that is closely related with the (one-mode) regime of propagation here considered and leads from kernels \(K^\pm\), as given in (2.19), to
kernels deprived of wave number. The approximation is based on the (quasi-)equations

$$q_n \approx \pi n/a, \quad r_n \approx \pi n/b, \quad \text{for } n = 2, 3, \ldots, \quad (3.1)$$

which can be claimed to hold good since $k < \pi/a$; the values of $q_1$ and $r_1$ are taken exact (see (3, section 4) and forthcoming Appendix B). Thus, we got an analytic representation for $H^{\pm}$ (named $S^{\pm}$ in that paper) in which certain coefficients, depending on geometric parameters, must be calculated numerically, but the dependence on frequency appears in explicit form, that is of great worth for our aims. A sketch of the procedure is reported in Appendix A. The advantage of this (‘semi-analytical’) approach with respect to a direct numerical treatment applied to (2.20) is that such equations, whose kernels contain $k$, should be solved each time anew for each new value of $k$ in the range (2.6), while the six integral equations alluded to above can be solved once for all independently on $k$.

On its side, writing the unknowns as

$$G_{2m-1} \pm G_{2m} \equiv G_m^{\pm}, \quad m = 1, \ldots, M, \quad (3.2a)$$

system (2.22) can be put in the following (‘block tri-diagonal’) form:

$$\begin{pmatrix}
    P_1 & C_1 & 0 & \cdots & 0 & 0 \\
    C_2 & F & C_1 & \cdots & 0 & 0 \\
    0 & C_2 & F & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & F & C_1 \\
    0 & 0 & 0 & \cdots & C_2 & P_2
\end{pmatrix}
\begin{pmatrix}
    G_1 \\
    G_2 \\
    G_3 \\
    \vdots \\
    G_{M-1} \\
    G_M
\end{pmatrix} = \begin{pmatrix}
    \Lambda \\
    0 \\
    0 \\
    \vdots \\
    0 \\
    0
\end{pmatrix}, \quad (3.2b)$$

where we have defined

$$P_1 = \begin{pmatrix} a^+ + b^+ & a^+ - b^+ \\ a^- - b^- & a^- + b^- \end{pmatrix}, \quad P_2 = \begin{pmatrix} a^+ + b^+ & b^+ - a^+ \\ b^- - a^- & a^- + b^- \end{pmatrix}, \quad F = \begin{pmatrix} 2b^+ & 0 \\ 0 & 2b^- \end{pmatrix},$$

$$C_1 = \begin{pmatrix} c^+ & c^+ \\ -c^- & -c^- \end{pmatrix}, \quad C_2 = \begin{pmatrix} c^+ & -c^+ \\ c^- & -c^- \end{pmatrix}, \quad G_m = \begin{pmatrix} G_m^+ \\ G_m^- \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \lambda^+ \\ \lambda^- \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.2c)$$

along with

$$a^\pm = 1 - \left( \frac{1}{2ak} + \frac{\cos 2k\ell \mp 1}{2bk \sin 2k\ell} \right) H^\pm, \quad b^\pm = 1 - \left( \frac{\cot k\delta}{2ak} + \frac{\cos 2k\ell \mp 1}{2bk \sin 2k\ell} \right) H^\pm,$$

$$c^\pm = \frac{H^\pm}{2ak \sin k\delta}, \quad \lambda^\pm = -4H^\pm. \quad (3.2d)$$
Let us now denote by $J_M$ the full determinant of the system and introduce, for $N \geq 2$, the following five determinants of order $2N$:

$$I_N^{(k)} = \begin{vmatrix} A_k & C_1 & 0 & \cdots & 0 & 0 \\ B_k & F & C_1 & \cdots & 0 & 0 \\ 0 & C_2 & F & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & F & C_1 \\ 0 & 0 & 0 & \cdots & C_2 & P_2 \end{vmatrix}, \quad k = 0, \ldots, 4, \quad (3.3a)$$

where $A_0 = F$, $B_0 = C_2$ and

$$A_1 = \begin{pmatrix} -c^+ & 0 \\ -c^- & 2b^- \end{pmatrix}, \quad A_2 = \begin{pmatrix} -c^+ & 2b^+ \\ -c^- & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} c^+ & 0 \\ c^- & 2b^- \end{pmatrix}, \quad A_4 = \begin{pmatrix} c^+ & 2b^+ \\ c^- & 0 \end{pmatrix}, \quad B_1 = B_3 = \begin{pmatrix} 0 & -c^+ \\ 0 & -c^- \end{pmatrix}, \quad B_2 = B_4 = \begin{pmatrix} 0 & c^+ \\ 0 & c^- \end{pmatrix}. \quad (3.3b)$$

It is not difficult to realise that all the above determinants can be expressed by recurrence relations as follows:

$$I_N^{(0)} = +4b^+b^-I_{N-1}^{(0)} + 2b^+c^-(I_{N-1}^{(1)} - I_{N-1}^{(2)}) - 2b^-c^+(I_{N-1}^{(3)} - I_{N-1}^{(4)}),$$

$$I_N^{(1)} = -2b^-c^+I_{N-1}^{(0)} - 2c^+c^-(I_{N-1}^{(1)} - I_{N-1}^{(2)}),$$

$$I_N^{(2)} = +2b^+c^-I_{N-1}^{(0)} - 2c^+c^-(I_{N-1}^{(3)} - I_{N-1}^{(4)}), \quad (N \geq 2), \quad (3.4a)$$

$$I_N^{(3)} = -I_N^{(1)}, \quad I_N^{(4)} = -I_N^{(2)}$$

and

$$J_M = 2(a-b^+ + a^+b^-)I_{M-1}^{(0)} + [(a^- - b^-)c^+ + (a^+ + b^+)c^-](I_{M-1}^{(1)} - I_{M-1}^{(2)}) - [(a^- + b^-)c^+ + (a^+ - b^+)c^-](I_{M-1}^{(3)} - I_{M-1}^{(4)})$$

$$= 2(a-b^+ + a^+b^-)I_{M-1}^{(0)} + 2(a^-c^+ + a^+c^-)(I_{M-1}^{(1)} - I_{M-1}^{(2)}), \quad (3.4b)$$

where we conventionally put

$$I_1^{(0)} = +2(a-b^+ + a^+b^-), \quad I_1^{(1)} = -2a^-c^+, \quad I_1^{(2)} = +2a^+c^-,$$

$$I_1^{(3)} = -I_1^{(1)}, \quad I_1^{(4)} = -I_1^{(2)}. \quad (3.5)$$

What is worth noting now is that recurrence relations (3.4a), with initial conditions (3.5), can be resolved in a closed form: if we set

$$\alpha^\pm = \sqrt{(b^\pm)^2 - (c^\pm)^2} \quad (3.6a)$$

and

$$\beta_\pm = 2(b^+b^- - c^+c^- \pm a^+a^-) \quad (3.6b)$$
(which are the only nonvanishing eigenvalues of the matrix of coefficients of (3.4a)), the solution is given by

\[
I_N^{(0)} = \frac{1}{\alpha + \alpha^-} \left[ (\alpha^+ b^- + \alpha^- b^+) (\alpha^+ a^- + \alpha^- a^+) \beta_+^{N-1} 
  - (\alpha^+ b^- - \alpha^- b^+) (\alpha^+ a^- - \alpha^- a^+) \beta_-^{N-1} \right],
\]

\[
I_N^{(1)} = -\frac{c^+}{\alpha^+} \left[ (\alpha^+ a^- + \alpha^- a^+) \beta_+^{N-1} + (\alpha^+ a^- - \alpha^- a^+) \beta_-^{N-1} \right], \quad N \geq 1
\]

\[
I_N^{(2)} = -\frac{c^-}{\alpha^-} \left[ (\alpha^+ a^- + \alpha^- a^+) \beta_+^{N-1} - (\alpha^+ a^- - \alpha^- a^+) \beta_-^{N-1} \right],
\]

\[
I_N^{(3)} = -I_N^{(1)}, \quad I_N^{(4)} = -I_N^{(2)}
\]

(that is also extended to \(N = 1\) in view of (3.5)).

Going back to the main system (3.2), the solution for the first \(2(M-1)\) unknowns is directly obtained by the Cramer’s rule as follows:

\[
G_1^+ = \{+[(a^- + b^-)\lambda^+ - (a^+ - b^+)\lambda^-] I_{M-1}^{(0)} + (c^- \lambda^+ + c^+ \lambda^-) (I_{M-1}^{(1)} - I_{M-1}^{(2)})\}/J_M,
\]

\[
G_1^- = \{-(a^- - b^-)\lambda^+ - (a^+ + b^+)\lambda^-] I_{M-1}^{(0)} + (c^- \lambda^- + c^+ \lambda^-) (I_{M-1}^{(1)} - I_{M-1}^{(2)})\}/J_M,
\]

\[
G_m^+ = +2^{m-1}(a^- \lambda^+ - a^+ \lambda^-)(b^+ c^- - b^- c^+)^{m-1} I_{M-m+1}^{(1)}/J_M, \quad m = 2, \ldots, M-1,
\]

\[
G_m^- = -2^{m-1}(a^- \lambda^- + a^+ \lambda^-)(b^+ c^- - b^- c^+)^{m-1} I_{M-m+1}^{(2)}/J_M
\]

while the solution for \(G_M^+, G_M^-\) is achieved by solving the last two equations of the system (and using (3.8a) or (3.8b)):

\[
G_M^+ = -\frac{[(a^- + b^-)c^+ + (a^+ - b^+)c^-](G_{M-1}^+ - G_{M-1}^-)}{2(a^- b^- + a^+ b^+)}
\]

\[
G_M^- = -\frac{[(a^- - b^-)c^+ + (a^+ + b^+)c^-](G_{M-1}^+ - G_{M-1}^-)}{2(a^- b^- + a^+ b^+)}.
\]

All the above formulas (including those for \(H^\pm\) in Appendix A) give to (2.10a) an explicit expression with respect to frequency, as we claimed. We note that the dependences on number \(M\) and distance \(\delta\) also are explicit.

### 4. Wave properties of the structure

We chose a fixed geometry of the grating by taking \(b/a = \ell/a = 0.5\) and \(\delta/a = 2\). Thus, we have studied the behaviour of coefficients \(R, T\) with respect to frequency parameter \(ak/\pi \in (0, 1)\) (see (2.6)) for two values of number \(M\) and with respect to \(M \in [2, 20]\) for four values of \(ak/\pi\).

First of all, it is worth noting that the balance of powers

\[
|R|^2 + |T|^2 = 1,
\]

valid for the one-mode far-field propagation here concerned (11), appears to be ‘identically’ verified, since all calculations of the left-hand term in the considered cases have always given the value 1.
(within the computational errors) (cf. (2), where the greater simplicity of the formulas given for \( R \) and \( T \) allows a direct proof of (4.1)). This fact suggests that the present procedure implies the analytical fulfillment of that balance, differently from other (numerical) approaches in which (4.1) is assumed \textit{a priori} and typically used to control the results obtained. As a consequence, we can study only one of the above coefficients, namely that of transmission \( T \).

Looking at the curves in the enclosed figures, one can recognise the general qualitative characteristics of wave penetration through cascading structures of the type in concern: an oscillating (quasi-)periodic behaviour of the scattering parameters with respect to frequency providing great amounts of transmission, spaced by intervals with a rapid decay of the transmission. This physically means that, in the one-mode regime, the structure can exhibit ‘passing bands’ of frequency, where the wave is transmitted through the screens without much loss of amplitude, or ‘stopping bands’, where the wave undergoes somewhat sudden breakdown. See Figs 2 and 3. Note that the position of the bands does not depend on number \( M \) (though the loss or gain of transmission are more rapid

![Fig. 2](image1)

**Fig. 2** Transmission parameter \(|T|\) versus frequency \((b/a = l/a = 0.5, \delta/a = 2; M = 10)\)

![Fig. 3](image2)

**Fig. 3** Transmission parameter \(|T|\) versus frequency \((b/a = l/a = 0.5, \delta/a = 2; M = 20)\)
for greater $M$), while the period of oscillations actually decreases with $M$. Further comments and physical insights on these matters can be found in (8 to 10, 13).

In our context, the transitions from passing to stopping bands or vice-versa are analytically determined by the (‘complex’ or ‘real’) eigenvalues defined in (3.6). More precisely, when $\beta_{\pm}$ is—or becomes—complex (namely, the frequency is such that \((b^+)^2 - (c^+)^2[(b^-)^2 - (c^-)^2] < 0\)), the relevant formulas imply an oscillating behaviour with respect to both frequency and number $M$; on the contrary, when $\beta_{\pm}$ becomes real (namely, \((b^+)^2 - (c^+)^2[(b^-)^2 - (c^-)^2] > 0\)), we have rapidly decaying behaviour. By solving the inequality in brackets, we found the critical values of frequency governing such transitions; these are (for any $M$) $ak/\pi = 0.247, 0.383, 0.595, 0.657, 0.837$ and are clearly reflected in Figs 2 and 3.

These properties can be also emphasised by showing the dependence of the transmission parameter versus the number $M$ of screens, which, in some sense (for fixed $\delta$), represents the ‘distance’ through the structure; see Fig. 4. For values of frequency in the passing bands, the behaviour is strictly periodic without great change of amplitude (see upper lines); on the contrary, for values in the stopping bands, transmission rapidly decreases with distance (see lower lines). This fact sharply distinguishes the periodic structure here concerned from the so-called ‘nonorganised’ structures, where obstacles are distributed in a random way: in the latter ones, the transmission decreases with distance ‘for any’ (even very low) frequency (14).

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References

APPENDIX A

By using (3.1), the kernel of (2.20) becomes

\[ K^\pm(y) \approx K^\pm_0(y) - A(k) \cos \frac{\pi y}{a} - B^\pm(k) \cos \frac{\pi y}{b}, \]  

(A.1a)

where

\[ A(k) = \frac{1}{\pi} - \frac{1}{aq_1}, \quad B^\pm(k) = \frac{\cosh(2\pi \ell/b) \mp 1}{\pi \sinh(2\pi \ell/b)} - \frac{\cosh(2r \ell) \mp 1}{br_1 \sinh(2r \ell)} \]  

(A.1b)

and

\[ K^\pm_0(y) = \frac{1}{\pi} \sum_{n=1}^\infty \left( \frac{1}{n} \cos a_n y + \frac{\cosh(2\pi n \ell/b) \mp 1}{n \sinh(2\pi n \ell/b)} \cos b_n y \right) \]  

(A.1c)

(which are kernels not containing the wave number). By substituting (A.1a) into (2.20) and considering the evenness of \( h^\pm \), we get

\[ \int_{-b}^b K^\pm_0(t - y)h^\pm(t) \, dt = 1 + A(k)H^\pm_a \cos \frac{\pi y}{a} + B^\pm(k)H^\pm_b \cos \frac{\pi y}{b}, \]  

(A.2a)

where

\[ \begin{pmatrix} H^\pm_a \\ H^\pm_b \end{pmatrix} = \int_{-b}^b h^\pm(t) \begin{pmatrix} \cos(\pi t/a) \\ \cos(\pi t/b) \end{pmatrix} \, dt \]  

(A.2b)
denotes unknown quantities. As a consequence, by linearity, functions $h^\pm$ can be represented as

$$h^\pm(y) = h^\pm_0(y) + A(k)H^\pm_a h^\pm_a(y) + B^\pm(k)H^\pm_b h^\pm_b(y), \quad |y| < b,$$

where functions $h^\pm_0$, $h^\pm_a$ and $h^\pm_b$ solve the following six integral equations:

$$\int_{-b}^{b} K^\pm_{00}(t - y) \begin{pmatrix} h^\pm_0(t) \\ h^\pm_a(t) \\ h^\pm_b(t) \end{pmatrix} dt = \begin{pmatrix} 1 \\ \cos(\pi y/a) \\ \cos(\pi y/b) \end{pmatrix}, \quad |y| < b,$$  \hspace{1cm} (A.4)

each one independent on wave number $k$.

By integrating (A.3) over $|y| < b$, we obtain an expression for $H^\pm$ as follows:

$$H^\pm = H^\pm_{00} + A(k)H^\pm_a H^\pm_{a0} + B^\pm(k)H^\pm_b H^\pm_{b0},$$  \hspace{1cm} (A.5a)

where

$$\begin{pmatrix} H^\pm_{00} \\ H^\pm_{a0} \\ H^\pm_{b0} \end{pmatrix} = \int_{-b}^{b} \begin{pmatrix} h^\pm_0(t) \\ h^\pm_a(t) \\ h^\pm_b(t) \end{pmatrix} dt.$$  \hspace{1cm} (A.5b)

In (A.5a), the unknowns (containing $k$) are $H^\pm_a$ and $H^\pm_b$, while $H^\pm_{00}$, $H^\pm_{a0}$ and $H^\pm_{b0}$ are computable constants (free of $k$) once (A.4) is solved. To calculate these unknowns, we multiply (A.3) first by $\cos(\pi y/a)$ and second by $\cos(\pi y/b)$, and then integrate over $|y| < b$, to get

$$\begin{align*}
H^\pm_a &= H^\pm_{0a} + A(k)H^\pm_a H^\pm_{aa} + B^\pm(k)H^\pm_b H^\pm_{ab}, \\
H^\pm_b &= H^\pm_{0b} + A(k)H^\pm_a H^\pm_{ab} + B^\pm(k)H^\pm_b H^\pm_{bb},
\end{align*}$$  \hspace{1cm} (A.6a)

where

$$\begin{align*}
H^\pm_{0a} &= \int_{-b}^{b} h^\pm_0(t) \begin{pmatrix} \cos(\pi t/a) \\ \cos(\pi t/b) \end{pmatrix} dt, \\
H^\pm_{0b} &= \int_{-b}^{b} h^\pm_0(t) \begin{pmatrix} \cos(\pi t/a) \\ \cos(\pi t/b) \end{pmatrix} dt, \\
H^\pm_{ba} &= \int_{-b}^{b} h^\pm_a(t) \begin{pmatrix} \cos(\pi t/a) \\ \cos(\pi t/b) \end{pmatrix} dt, \\
H^\pm_{bb} &= \int_{-b}^{b} h^\pm_b(t) \begin{pmatrix} \cos(\pi t/a) \\ \cos(\pi t/b) \end{pmatrix} dt.
\end{align*}$$  \hspace{1cm} (A.6b)

Equation (A.6) gives a linear system for $H^\pm_a$ and $H^\pm_b$ since all other quantities (defined in (A.6b)) also are computable constants (free of $k$) once (A.4) is solved. Thus, substituting the solution of this system into (A.5a) leads to the sought representation of $H^\pm$, quoted in section 3, in which the frequency is only contained in coefficients $A(k)$, $B^\pm(k)$.

**APPENDIX B**

Apart from what was already observed about the approximations employed (recall (2.15) and (3.1)), we can get a complete estimate of our results by solving numerically the system of integral equations arising from (2.14) without any approximation, neither of type (2.15) nor of type (3.1). To this end, let us substitute the (exact) values of the Fourier constants from (2.10) and (2.11) into (2.14). In the case $M = 2$, we obtain a $4 \times 4$ system in the unknowns $g_1, \ldots, g_4$ which can be uncoupled into two $2 \times 2$ systems, in the unknowns
$g_{1,4}^+ \equiv g_1 + g_4$, $g_{2,3}^+ \equiv g_2 + g_3$ the first one and $g_{1,4}^- \equiv g_1 - g_4$, $g_{2,3}^- \equiv g_2 - g_3$ the second one. These two systems are, in $|y| < b$ (one with upper sign and one with lower sign),

$$\int_{-b}^{b} K_1(t - y)g_{1,4}^+(t)dt + \int_{-b}^{b} K_2(t - y)g_{2,3}^+(t)dt = -2 + \left(\frac{1}{2ak} + \frac{\cot 2k\ell}{2bk}\right)(G_1 \pm G_4) - \frac{1}{2bk \sin 2k\ell}(G_2 \pm G_3),$$  \hspace{1cm} \text{(B.1a)}

$$\int_{-b}^{b} K_2(t - y)g_{1,4}^+(t)dt + \int_{-b}^{b} \sum_{n=1}^{\infty} \left[\frac{\cosh q_n \delta + \frac{1}{aq_n \sinh q_n \delta}}{aq_n \sinh q_n \delta} \cos a_n(t - y) + \frac{\coth(2r_n \ell)}{br_n} \cos b_n(t - y)\right]g_{2,3}^+(t)dt = -\frac{1}{2bk \sin 2k\ell}(G_1 \pm G_4) + \left(\frac{\cos k\delta + \frac{1}{2ak \sin k\delta} + \frac{\cot 2k\ell}{2bk}}{2ak \sin k\delta + \frac{\cot 2k\ell}{2bk}}\right)(G_2 \pm G_3).$$  \hspace{1cm} \text{(B.1b)}

Such systems of integral equations, with kernels now depending on distance $\delta$ (besides $k$), have been solved by means of a well-known numerical algorithm in the theory of boundary element methods, more precisely, by a suitable form of the ‘collocation’ technique (15). This easily gives rise to values for $G_1 \pm G_4$, and thus for $G_1$ and $G_4$, which can be used to calculate parameters $R$ and $T$ by (2.10a). Of course, as already said, to study the behaviour of any result with respect to frequency, the integral equations must be solved numerically anew for each new value of the frequency parameter.

First, we note that, in all the computations performed, the balance of powers (4.1) has been always verified to hold to a very good extent: actually, when calculating the left-hand term, we found that the difference from 1 is greater than $2 \times 10^{-3}$ only in the final part of the frequency range (2.6), where it equals about $4 \div 5 \times 10^{-3}$. As an example, by choosing the same values for parameters $b/a$, $\ell/a$ and $\delta/a$ (as above) and taking more than 100 values for $ak/\pi$ in $(0, 1)$, we have built up a (solid) line for coefficient $|T|$ reflecting the (‘exact’) numerical solution: see Fig. 5, where (by dashed line) the graph of $|T|$ as given by our analytical approach for the same example is also reported (recall that $M = 2$). We can see that the agreement between solid and dashed lines is quite excellent: these are practically coincident almost throughout, apart from small final discrepancies which probably are due to the error implied by approximation (3.1) for increasing frequency.

Fig. 5 Comparison among numerical (solid line), analytical (dashed line) and wide-spacing (dotted line) solutions ($b/a = l/a = 0.5$, $\delta/a = 2$; $M = 2$)
It is also interesting to compare the results just obtained with those arising from an approximation of ‘wide-spacing’ type directly applied to our approach (valid in the one-mode regime). By disregarding \textit{a priori} all evanescent wave modes (of order greater than zero in (2.4)), (2.14) gives rise to a $2M \times 2M$ linear system of algebraic equations in the unknowns $G$ (instead of integral equations in the $g$'s); the solution of such a system, which is clearly explicit with respect to $k$, allows to plot the scattering parameters versus frequency via (2.10a). The line obtained for $|T|$ and $M = 2$ is reported (as a dotted line) in Fig. 5: we can see that the quoted approximation does not perform well when used in the present procedure, the greatest defect being in our opinion the large difference from zero near the (cut-off) value $ak/\pi = 1$. Indeed, it was already noticed in the literature—see (7, 10) for instance—that ignoring evanescent modes cannot lead to good results when only a single propagating mode is assumed.

Finally, in order to control how the approximation inferred from (2.7) behaves when the ratio $\delta/a$ becomes smaller than 2 (which is the chosen value), we have fixed one frequency in the middle of (2.6) and plotted a line for coefficient $|T|$ versus parameter $\delta/a$ from 0.5 to 3.5 by using our explicit formulas. Parallely, we have built up a line reflecting the corresponding numerical solution of the exact system (B.1): one could see that the two lines are practically coincident from $\delta/a \approx 1.5$ onwards, which confirms the choice we made. Compare with (2, 6, 7, 12) where it was already shown that positions like (2.7) in the ambit of cascading structures give good results even if that ratio holds 2, which amounts to an interscreen distance just equal to the vertical period of the grating or, equivalently, to the (minimum) wavelength of the incident wave.