AXISYMMETRIC CONTACT OF A RIGID
INCLUSION EMBEDDED AT THE INTERFACE OF A
PIEZOELECTRIC BIMATERIAL

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Summary

The axisymmetric contact problem of a rigid inclusion embedded in the piezoelectric bimaterial
frictionless interface subjected to simultaneous far-field compression and electric displacement
is addressed. With the aid of a robust technique, the coupled governing integral equations of
this mixed boundary-value problem are reduced to decoupled Fredholm integral equations with
a constraint equation. A useful limiting case for the contact problem of transversely isotropic
bimaterials is addressed. The present solution is analytically in agreement with the existing
solution for an isotropic bimaterial. Selected numerical results of interest to engineering
applications including the radius of separation zone, contact pressure and contact electric
displacement are plotted to portray the effects of precompression, piezoelectric coupling and
material properties.

1. Introduction

Following Hertz (1), who examined the contact of elastic solids, many researchers have presented
extensive fundamental results for the contact problems of isotropic media. Gladwell (2) and
Johnson (3) performed a comprehensive review of history and literature and summarized the pre-
sented contributions corresponding to the contact problems of isotropic materials.

With the development of modern technologies, the application of anisotropic materials such as
composites, piezocomposites and magnetics has increased rapidly because of their high perfor-
man ce and advantages. One may refer to Ding (4) to see a number of classical contact problems,
such as two elastic bodies in contact, rigid indenters, frictional Hertzian contact, for transversely
isotropic materials. Among anisotropic materials, transversely isotropic piezoelectric ceramics are
widely used in areas such as electronics, lasers, supersonics, microwave, infrared, navigation and
biology. There are many papers considering the classical contact problems of piezoelectric materi-
als. Fan et al. (5) studied the two-dimensional contact on a piezoelectric half-plane by employing
Stroh’s formalism. Maceri and Bisegna (6) considered numerical evaluation of the unilateral fric-
tionless contact of a piezoelectric body with a rigid support using variational principles. Ding et al.
(7, 8) studied a series of common three-dimensional contact including spherical contact, a conical
indenter, elliptical Hertzian contact and an upright circular flat punch on a transversely isotropic
piezoelectric half-space for both smooth and frictional cases. Later, many researchers studied the
related subjects pertinent to contact problems of piezoelectric solids, for instance, see (9 to 20).

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One of the interesting contact problems is the unilateral contact of an inclusion embedded between two elastic bodies. This model can be used to examine the extent of separation induced by a proppant (a granular porous medium) injected at a prefractured resource bearing geological formation (21). Moreover, imperfect manufacturing of layered media usually accompanies inclusions between two adjacent layers. Therefore, it is necessary to make theoretical analysis and accurate quantitative description of the effects of an inclusion on interfacial elastic fields and the extent of separation zone induced by it. Alblas solved the contact problem for a small rigid obstacle pressed between two identical elastic half-planes (22) and layers (23) for the plane-strain state. Gladwell (24) presented the solution for two dissimilar half-planes. This work was extended to three-dimensional problems by Gladwell and Hara (25) for the separation of a smooth precompressed bimaterial elastic interface due to insertion of an axisymmetric rigid inclusion with an oblate spheroidal shape, which is non-symmetric about the interface plane. They used conventional Hertzian assumption and presented numerical results for the case of a symmetric obstacle compressed between two dissimilar half-spaces. Selvadurai (21, 26) examined the separation at a precompressed smooth bimaterial elastic interface due to the existence of a rigid disc-shaped inclusion at the interface. Following Selvadurai's works, Gladwell (27) showed that for a rigid flat inclusion of arbitrary shape compressed between two media, three conditions hold:

1. the disc indentations in corresponding half-spaces are related by

\[ \vartheta_1 \Delta_1 = \vartheta_2 \Delta_2, \]  

where \( \Delta_1 + \Delta_2 \) is the total thickness of the disc and \( \vartheta_1 \) and \( \vartheta_2 \) are constants pertaining to material properties of the half-spaces;

2. the stress distributions along the interfaces between the half-spaces and the rigid inclusion are identical; and

3. the displacement normal to the interface between the half-spaces are equal to zero.

He argued that these results are also valid for transversely isotropic bimaterials. The focus of the work of Gladwell (27) is the results (1), (2) and (3) above and it should be emphasized that he was not concerned with the complete solution of the mixed boundary-value problem, which is necessary for determination of the extent of the separation zone. To the best of the authors' knowledge, however, this problem has not been investigated so far for piezoelectric media, which is the motivation of this study.

In this work, the axisymmetric contact problem of a rigid inclusion embedded in the piezoelectric bimaterial frictionless interface is presented. The system is subjected to simultaneous far-field compression and electric displacement. Employing potential function introduced by Ding et al. (28) and Hankel transform, the mixed boundary-value problem is written in an equivalent form of coupled integral equations. By virtue of a robust and efficient technique, these equations are reduced to two decoupled Fredholm integral equations for the interfacial stress and the electric displacement. Two limiting cases for transversely isotropic bimaterial and isotropic bimaterial are also presented. It is noted that there is a missing term in the formulation of Selvadurai (21) who considered isotropic bimaterial. The radius of separation zone, contact pressure and contact electric displacement are calculated for different material combinations. The effects of precompression, piezoelectric coupling and mismatch properties of materials on system response are discussed.
2. Governing equations and potential function

The axisymmetric governing equilibrium equations for a linear piezoelectric material in the absence of body forces and free charges can be expressed as

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r} = 0,
\]

\[
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0,
\]

\[
\frac{\partial D_r}{\partial r} + \frac{\partial D_z}{\partial z} + \frac{D_r}{r} = 0,
\]

where \(\sigma_{rr}, \sigma_{\theta \theta}, \sigma_{zz}\) and \(\sigma_{rz}\) denote the components of the stress tensor and \(D_r\) and \(D_z\) are the components of electric displacement. The corresponding constitutive relation for transversely isotropic piezoelectric materials with the \(z\)-axis as the axis of symmetry may be written as follows:

\[
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\theta \theta} \\
\sigma_{zz} \\
\sigma_{rz}
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 \\
c_{12} & c_{11} & c_{13} & 0 \\
c_{13} & c_{13} & c_{33} & 0 \\
0 & 0 & 0 & c_{44}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{rr} \\
\varepsilon_{\theta \theta} \\
\varepsilon_{zz} \\
2\varepsilon_{rz}
\end{bmatrix} -
\begin{bmatrix}
0 & 0 & e_{31} & 0 \\
0 & 0 & e_{31} & 0 \\
e_{15} & 0 & e_{33} & 0 \\
2e_{rz}
\end{bmatrix}
\begin{bmatrix}
E_r \\
E_\theta \\
E_z
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
D_r \\
D_z
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & e_{15} \\
e_{31} & e_{31} & e_{33} & 0 \\
0 & 0 & e_{33} & 0 \\
2e_{rz}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{rr} \\
\varepsilon_{\theta \theta} \\
\varepsilon_{zz} \\
2\varepsilon_{rz}
\end{bmatrix} +
\begin{bmatrix}
e_{11} & 0 \\
0 & e_{33}
\end{bmatrix}
\begin{bmatrix}
E_r \\
E_z
\end{bmatrix},
\]

in which \(\varepsilon_{rr}, \varepsilon_{\theta \theta}, \varepsilon_{zz}\) and \(\varepsilon_{rz}\) are the components of the strain tensor; \(E_r\) and \(E_z\) the components of electric field; \(c_{11}, c_{12}, c_{13}, c_{33}\) and \(c_{44}\) the elastic moduli constants; \(e_{11}\) and \(e_{33}\) the dielectric constants and \(e_{15}, e_{31}\) and \(e_{33}\) the piezoelectric constants. The relations between the components of strain tensor and displacement vector, \(u_r\) and \(u_z\), are given by

\[
e_{rr} = \frac{\partial u_r}{\partial r}, \quad e_{\theta \theta} = \frac{u_r}{r}, \quad e_{zz} = \frac{\partial u_z}{\partial z}, \quad e_{rz} = \frac{u_r}{r} + \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right).
\]

The electric field components are related to the electric potential \(\phi\) by

\[
E_r = -\frac{\partial \phi}{\partial r}, \quad E_z = -\frac{\partial \phi}{\partial z}.
\]

In order to solve the coupled system of partial differential equations (2.1), the potential function introduced by Ding et al. (28) is used

\[
u_r = -\left( c_{13} + c_{44} \right) \left( e_{11} \Lambda + e_{33} \frac{\partial^2}{\partial z^2} \right) + \left( e_{15} + e_{31} \right) \left( e_{15} \Lambda + e_{33} \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2 F}{\partial r \partial z},
\]

\[
u_z = \left[ \left( c_{11} \Lambda + c_{44} \frac{\partial^2}{\partial z^2} \right) \left( e_{11} \Lambda + e_{33} \frac{\partial^2}{\partial z^2} \right) + \left( e_{15} + e_{31} \right)^2 \Lambda \frac{\partial^2}{\partial z^2} \right] F,
\]

\[
\phi = \left[ \left( c_{11} \Lambda + c_{44} \frac{\partial^2}{\partial z^2} \right) \left( e_{15} \Lambda + e_{33} \frac{\partial^2}{\partial z^2} \right) - \left( c_{13} + c_{44} \right) \left( e_{15} + e_{31} \right) \Lambda \frac{\partial^2}{\partial z^2} \right] F,
\]
The coefficients $a_i$ in the above equation, where

$$
\Lambda = \frac{\partial^2}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r}.
$$

(2.7)

Substituting (2.6) into (2.1) and using (2.2)–(2.5) result in a governing partial differential equation for the potential function $F$

$$
(L + \frac{\partial^2}{\partial z^2})\left(\Lambda + \frac{\partial^2}{\partial z^2}\right) F = 0,
$$

(2.8)

where $z_i^2 = s_i^2 z^2$ (i = 1, 2, 3) and $s_i^2$ are the three roots of the equation

$$
as s^6 - bs^4 + cs^2 - d = 0.
$$

(2.9)

The coefficients $a$, $b$, $c$ and $d$ appearing in (2.9) are expressed in terms of the material properties as

$$
a = c_{44}(e_{33}^2 + c_{33}e_{33}), \quad d = c_{11}(e_{15}^2 + c_{44}e_{11}),
$$

$$
b = c_{33}[c_{44}e_{11} + (e_{15} + e_{31})^2] + e_{33}[c_{11}c_{33} + c_{44}^2 - (c_{13} + c_{44})^2]
$$

$$+ e_{33}[2c_{44}e_{15} + c_{11}e_{33} - 2(c_{13} + c_{44})(e_{15} + e_{31})],
$$

$$c = c_{44}[c_{11}e_{33} + (e_{15} + e_{31})^2] + e_{11}[c_{11}c_{33} + c_{44}^2 - (c_{13} + c_{44})^2]
$$

$$+ e_{15}[2c_{11}e_{33} + c_{44}e_{15} - 2(c_{13} + c_{44})(e_{15} + e_{31})].
$$

The three roots $s_i$ are chosen such that their real parts be always non-negative, $\text{Re}(s_i) > 0$.

Employing the zeroth-order Hankel transform pair with respect to the radial coordinate,

$$
\tilde{F}^0(\xi, z) = \int_0^\infty F(r, z)r J_0(r \xi) \, dr, \quad F(r, z) = \int_0^\infty \tilde{F}^0(\xi, z)\xi J_0(r \xi) \, d\xi,
$$

in which $\xi$ is the Hankel transform parameter and $J_0$ is the Bessel function of first kind of order zero, the solution of (2.8) for distinct values of $s_i^2$ can be obtained as

$$
F(r, z) = \sum_{i=1}^3 \int_0^\infty (A_i(\xi)e^{-s_i \xi |z|} + B_i(\xi)e^{s_i \xi |z|}) J_0(r \xi) \, d\xi.
$$

(2.10)

In the above equation, $A_i(\xi)$ and $B_i(\xi)$ are unknown functions to be determined from the boundary conditions. Utilizing (2.10), one can find the following useful expressions:

$$
u_r(r, z) = \frac{z}{|z|} \sum_{i=1}^3 \eta_i \int_0^\infty \xi^4 (A_i(\xi)e^{-s_i \xi |z|} - B_i(\xi)e^{s_i \xi |z|}) J_1(r \xi) \, d\xi,
$$

$$
u_z(r, z) = \sum_{i=1}^3 \nu_i \int_0^\infty \xi^4 (A_i(\xi)e^{-s_i \xi |z|} + B_i(\xi)e^{s_i \xi |z|}) J_0(r \xi) \, d\xi,
$$

$$
\phi(r, z) = \sum_{i=1}^3 \lambda_i \int_0^\infty \xi^4 (A_i(\xi)e^{-s_i \xi |z|} + B_i(\xi)e^{s_i \xi |z|}) J_0(r \xi) \, d\xi,
$$

where $\eta_i$, $\nu_i$, and $\lambda_i$ are unknown coefficients to be determined from the boundary conditions.
\[ \sigma_{rz}(r, z) = -\sum_{i=1}^{3} \kappa_i \int_{0}^{\infty} \xi^5 (A_i(\xi)e^{-s_i \xi |z|} + B_i(\xi)e^{s_i \xi |z|}) J_1(r \xi) \, d\xi, \]

\[ \sigma_{zz}(r, z) = \frac{z}{|z|} \sum_{i=1}^{3} \int_{0}^{\infty} \xi^5 (A_i(\xi)e^{-s_i \xi |z|} - B_i(\xi)e^{s_i \xi |z|}) J_0(r \xi) \, d\xi, \]

\[ D_z(r, z) = \frac{z}{|z|} \sum_{i=1}^{3} \gamma_i \int_{0}^{\infty} \xi^5 (A_i(\xi)e^{-s_i \xi |z|} - B_i(\xi)e^{s_i \xi |z|}) J_0(r \xi) \, d\xi, \]

where

\[ \eta_i = [(c_{13} + c_{44})(\epsilon_{11} - \epsilon_{33}s_i^2) + (e_{15} + e_{31})(e_{15} - e_{33}s_i^2)]s_i, \]

\[ \lambda_i = (c_{11} - c_{44}s_i^2)(e_{15} - e_{33}s_i^2) + (c_{13} + c_{44})(e_{15} + e_{31})s_i^2, \]

\[ v_i = (c_{11} - c_{44}s_i^2)(\epsilon_{11} - \epsilon_{33}s_i^2) - (e_{15} + e_{31})s_i^2, \]

\[ \kappa_i = e_{15}\lambda_i + c_{44}s_i\eta_i + c_{44}v_i, \]

\[ r_i = c_{13}\eta_i - s_i(e_{33}v_i + e_{33}\lambda_i), \]

\[ \gamma_i = e_{31}\eta_i - s_i(e_{33}v_i - e_{33}\lambda_i), \quad i = 1, 2, 3. \]

### 3. Statement of the problem

Consider a rigid circular disc shape ideal conductor with radius \( a \) and thickness \( \Delta \) embedded in the piezoelectric bimaterial frictionless interface as shown in Fig. 1. This system is subjected to uniform far-field compression \( \sigma_0 \) and electric displacement \( D_0 \) normal to the interface. A cylindrical coordinate system is set at the interface in such a way that \( z \)-axis is normal to the interface and is the axis of symmetry for both lower and upper half-spaces. For convenience, the superscript I or II over a quantity indicates that the quantity is pertinent to the upper half-space (Region I, \( z > 0 \)) or lower half-space (Region II, \( z < 0 \)), respectively.

The conductivity assumption of disc inclusion and the considerable gap width between two half-spaces imply using impermeable model for the separation zone (29). Employing superposition technique and denoting the radius of separation zone by \( b \), one may write the mixed boundary conditions at the interface as

\[ u^I_z(r, 0) = \Delta^I, \quad u^II_z(r, 0) = -\Delta^II, \quad \phi^I(r, 0) = \phi^II(r, 0) = \phi^d, \]

(3.1)

for the disc region \( (0 \leq r \leq a) \);

\[ \sigma^I_{rz}(r, 0) = \sigma^II_{rz}(r, 0) = \sigma_0, \quad D^I_z(r, 0) = D^II_z(r, 0) = D_0, \]

(3.2)

for the separation zone \( (a < r < b) \);

\[ u^I_z(r, 0) = u^II_z(r, 0), \quad \phi^I(r, 0) = \phi^II(r, 0), \]

(3.3)

\[ \sigma^I_{zz}(r, 0) = \sigma^II_{zz}(r, 0), \quad D^I_z(r, 0) = D^II_z(r, 0), \]

(3.4)

from the continuity conditions at the bimaterial contact region \( (b < r < \infty) \); and the frictionless contact assumption implies

\[ \sigma^I_{rz}(r, 0) = \sigma^II_{rz}(r, 0) = 0, \quad 0 < r < \infty. \]

(3.5)
Here, $\phi^d$ is the electric potential of the disc inclusion and $\Delta^I$ and $\Delta^{II}$ are the values of the disc penetration into Regions I and II, respectively. From Fig. 1, it is obvious that
\[ \Delta^I + \Delta^{II} = \Delta. \] (3.6)

In addition to the above conditions, one must satisfy the following regularity conditions
\[ \lim_{|z| \to \infty} u^j = \lim_{|z| \to \infty} u^j = \lim_{|z| \to \infty} \phi^j = 0, \quad j = I, II. \] (3.7)

The regularity conditions (3.7) imply that $B^I_i = B^{II}_i = 0$ ($i = 1, 2, 3$). Define two new functions $\tilde{\sigma}(\xi)$ and $\tilde{D}(\xi)$ as
\[ \tilde{\sigma}^j(\xi) = \xi^4 \sum_{i=1}^{3} t^j_i A^j_i(\xi), \quad \tilde{D}^j(\xi) = \xi^4 \sum_{i=1}^{3} \gamma^j_i A^j_i(\xi), \quad j = I, II. \] (3.8)

Now, by applying the frictionless contact condition (3.5), the boundary conditions (3.1)–(3.4) can
be expressed in the following integral forms
\[
\int_0^\infty [a_i^j \tilde{\sigma}^\xi (\zeta) + \beta_i^j \tilde{D}^\xi (\zeta)] J_0 (r \zeta) \, d\zeta = (1 - i)(-1)^j \phi^i + (2 - i) \Delta^j, \quad i = 1, 2, \quad j = I, II, \quad (3.9)
\]
for \( 0 \leq r < a \);
\[
\int_0^\infty \tilde{\sigma}^I (\zeta) J_0 (r \zeta) \, d\zeta = \sigma_0, \quad \int_0^\infty \tilde{D}^I (\zeta) J_0 (r \zeta) \, d\zeta = D_0, \quad j = I, II, \quad (3.10)
\]
for \( a < r < b \) and
\[
\int_0^\infty [a_i^j \tilde{\sigma}^\xi (\zeta) + \beta_i^j \tilde{D}^\xi (\zeta) + a_i^{II} \tilde{\sigma}^\xi (\zeta) + \beta_i^{II} \tilde{D}^\xi (\zeta)] J_0 (r \zeta) \, d\zeta = 0, \quad i = 1, 2, \quad (3.11)
\]
\[
\int_0^\infty \tilde{\sigma}^I (\zeta) J_0 (r \zeta) \, d\zeta = \int_0^\infty \tilde{D}^I (\zeta) - \tilde{D}^{II} (\zeta) J_0 (r \zeta) \, d\zeta = 0, \quad (3.12)
\]
for \( b < r < \infty \), where
\[
\begin{align*}
\vartheta a_1 &= v_1 (\kappa_2 \gamma_3 - \kappa_3 \gamma_2) + v_2 (\kappa_3 \gamma_1 - \kappa_1 \gamma_3) + v_3 (\kappa_1 \gamma_2 - \kappa_2 \gamma_1), \\
\vartheta a_2 &= \lambda_1 (\kappa_2 \gamma_3 - \kappa_3 \gamma_2) + \lambda_2 (\kappa_3 \gamma_1 - \kappa_1 \gamma_3) + \lambda_3 (\kappa_1 \gamma_2 - \kappa_2 \gamma_1), \\
\vartheta \beta_1 &= v_1 (t_2 \kappa_3 - t_3 \kappa_2) + v_2 (t_3 \kappa_1 - t_1 \kappa_3) + v_3 (t_1 \kappa_2 - t_2 \kappa_1), \\
\vartheta \beta_2 &= \lambda_1 (t_2 \kappa_3 - t_3 \kappa_2) + \lambda_2 (t_3 \kappa_1 - t_1 \kappa_3) + \lambda_3 (t_1 \kappa_2 - t_2 \kappa_1), \\
\vartheta &= \gamma_1 (t_2 \kappa_3 - t_3 \kappa_2) + \gamma_2 (t_3 \kappa_1 - t_1 \kappa_3) + \gamma_3 (t_1 \kappa_2 - t_2 \kappa_1).
\end{align*}
\]
For simplicity, the superscript \( j \) has been omitted in the above definitions of \( \vartheta, \alpha_i \) and \( \beta_i \) \( (i = 1, 2) \).

In order to reduce the triple integral equations (3.9)–(3.12) to Fredholm-type integral equations, an extension of the method introduced by Noble (30) and Cooke (31) is used. To this end assume that
\[
\int_0^\infty \tilde{\sigma}^I (\zeta) J_0 (r \zeta) \, d\zeta = \begin{cases} f_1^j (r), & 0 < r < a, \\ f_2 (r), & b < r < \infty \end{cases}, \quad j = I, II, \quad (3.13)
\]
\[
\int_0^\infty \tilde{D}^I (\zeta) J_0 (r \zeta) \, d\zeta = \begin{cases} g_1^j (r), & 0 < r < a, \\ g_2 (r), & b < r < \infty \end{cases}, \quad j = I, II. \quad (3.14)
\]
With the aid of (3.13) and (3.14), and the following Abel integral equations,
\[
F_1^j (r) = \int_r^a \frac{sf_1^j (s)}{\sqrt{s^2 - r^2}} \, ds, \quad G_1^j (r) = \int_r^a \frac{sg_1^j (s)}{\sqrt{s^2 - r^2}} \, ds, \quad 0 < r < a, \quad j = I, II, \quad (3.15)
\]
\[
F_2^j (r) = \int_b^r \frac{sf_2 (s)}{\sqrt{r^2 - s^2}} \, ds, \quad G_2^j (r) = \int_b^r \frac{sg_2 (s)}{\sqrt{r^2 - s^2}} \, ds, \quad b < r < \infty, \quad (3.16)
\]
one can rewrite (3.9) as

\[
\alpha_i^j \left\{ F_1^j(r) + \sigma_0 \sqrt{b^2 - r^2} \right\} + \frac{2}{\pi} \int_0^\infty \frac{s F_2(s)}{s^2 - r^2} ds \right] + \frac{2}{\pi} \int_0^\infty \frac{s G_2(s)}{s^2 - r^2} ds \right\} \\
+ \beta_i^j \left\{ G_1^j(r) + D_0 \sqrt{b^2 - r^2} \right\} + \frac{2}{\pi} \int_0^\infty \frac{s G_2(s)}{s^2 - r^2} ds \right\} \\
= (1 - i)(-1)^j \phi^d + (2 - i) \Delta^j, \quad 0 < r < a, \quad i = 1, 2, \quad j = I, II. \tag{3.17}
\]

After some manipulations, (3.17) yields the following decoupled integral equations for the stress and electric displacement distributions on the faces of the disc (0 < r < a)

\[
F_1^j(r) + \sigma_0 \sqrt{b^2 - r^2} \right\} + \frac{2}{\pi} \int_0^\infty \frac{s F_2(s)}{s^2 - r^2} ds = \left( \frac{-1)^j \phi^d \beta_1^j + \Delta^j} {\beta_1^j \alpha_1^j - \beta_1^j \alpha_2^j} \right), \tag{3.18}
\]

\[
G_1^j(r) + D_0 \sqrt{b^2 - r^2} \right\} + \frac{2}{\pi} \int_0^\infty \frac{s G_2(s)}{s^2 - r^2} ds = \left( \frac{-1)^j \phi^d \alpha_1^j + \Delta^j} {\beta_1^j \alpha_1^j - \beta_1^j \alpha_2^j} \right), \tag{3.19}
\]

\[
\int_0^a \left[ \sigma_{zz}^I(r, 0) - \sigma_{zz}^{II}(r, 0) \right] dr = 0, \quad \int_0^a \left[ D^I_z(r, 0) - D^{II}_z(r, 0) \right] dr = 0, \tag{3.20}
\]

\[
\Delta^I = [(a_2^I + a_2^{II}) \beta_1^I - (\beta_1^I + \beta_2^I) \alpha_1^I] \Delta / C, \quad \Delta^{II} = [(a_2^I + a_2^{II}) \beta_1^{II} - (\beta_1^I + \beta_2^I) \alpha_1^{II}] \Delta / C, \tag{3.21}
\]

\[
\phi^d = (a_2^{II} \beta_1^I - a_2^I \beta_2^I) \Delta / C, \quad C = (a_2^I + a_2^{II})(\beta_1^I + \beta_1^{II}) - (a_1^I + a_1^{II})(\beta_1^I + \beta_2^I), \tag{3.22}
\]

\[
F_2(r) = \sigma_0 \sqrt{b^2 - r^2} \right\} - \frac{2r}{\pi} \int_0^a \frac{F_1(s)}{r^2 - s^2} ds, \quad b < r < \infty. \tag{3.23}
\]

\[
G_2(r) = D_0 \sqrt{b^2 - r^2} \right\} - \frac{2r}{\pi} \int_0^a \frac{G_1(s)}{r^2 - s^2} ds, \quad b < r < \infty. \tag{3.24}
\]

Eventually, substituting \( F_2(r) \) and \( G_2(r) \) into (3.18) and (3.19) leads to a pair of decoupled Fredholm integral equations of the second kind, as follows

\[
F_1(r) + \int_0^a K(r, \zeta) F_1(\zeta) d\zeta = - (\beta_2^I + \beta_2^{II}) \Delta / C + \sigma_0 \mathcal{F}(r), \quad 0 < r < a, \tag{3.25}
\]

\[
G_1(r) + \int_0^a K(r, \zeta) G_1(\zeta) d\zeta = (a_2^I + a_2^{II}) \Delta / C + D_0 \mathcal{F}(r), \quad 0 < r < a, \tag{3.26}
\]

where the differentiable kernel of the integral equations is

\[
K(r, \zeta) = \frac{2}{\pi^2 (r^2 - \zeta^2)} \left\{ r \ln \left[ \frac{b - r}{b + r} \right] - \zeta \ln \left[ \frac{b - \zeta}{b + \zeta} \right] \right\} \nonumber \\
= \frac{4}{\pi^2 (\zeta^2 - r^2)} \left\{ r \tanh^{-1} \left[ \frac{r}{b} \right] - \zeta \tanh^{-1} \left[ \frac{\zeta}{b} \right] \right\}
\]
and
\[
\mathcal{F}(r) = \frac{2}{\pi} \left\{ \sqrt{a^2 - r^2} \tan^{-1} \sqrt{\frac{b^2 - a^2}{a^2 - r^2}} - \sqrt{b^2 - a^2} \right\}.
\] (3.27)

Finally, in order to determine the length of separation zone \(b\), one may consider the fact that the stress intensity factor must vanish at \(r = b\),
\[
K_I = \lim_{r \to b} \sqrt{2\pi(r - b)\sigma_{zz}(r, 0)} = 0,
\] (3.28)
which is equivalent to
\[
\frac{2b}{\pi \sigma_0} \int_0^a \frac{F_1(s)}{b^2 - s^2} ds + \sqrt{b^2 - a^2} = 0.
\] (3.29)
It is deduced from (3.29) that the extent of separation zone is independent of the applied far-field electric displacement, \(D_0\).

Now, it is interesting to determine the normal displacement and the electric potential along the bimaterial smooth interface region \(b < r < \infty\). It is easily obtained that
\[
\begin{align*}
\alpha_l \left\{ F_2(r) + \sigma_0[\sqrt{r^2 - a^2} - \sqrt{r^2 - b^2}] + \frac{2r}{\pi} \int_0^a \frac{F_1(s)}{r^2 - s^2} ds \right\} \\
+ \beta_l \left\{ G_2(r) + D_0[\sqrt{r^2 - a^2} - \sqrt{r^2 - b^2}] + \frac{2r}{\pi} \int_0^a \frac{G_1(s)}{r^2 - s^2} ds \right\} \\
- \frac{d}{dr} \left\{ (i - 2) \int_r^\infty \frac{s u_z(s, 0) ds}{\sqrt{s^2 - r^2}} - (i - 1) \int_r^\infty \frac{s \phi(s, 0) ds}{\sqrt{s^2 - r^2}} \right\} = 0, \quad b < r < \infty, \quad i = 1, 2.
\end{align*}
\]
Therefore,
\[
\int_r^\infty \frac{s u_z(s, 0) ds}{\sqrt{s^2 - r^2}} = c_1, \quad \int_r^\infty \frac{s \phi(s, 0) ds}{\sqrt{s^2 - r^2}} = c_2, \quad b < r < \infty,
\] (3.30)
where \(c_1\) and \(c_2\) are constants. Employing the property of Abel’s integral equation, it is deduced that the electric potential and normal displacement are identically zero along the bimaterial contact region beyond the separation zone,
\[
u_z(r, 0) = 0 \quad \text{and} \quad \phi(r, 0) = 0, \quad b < r < \infty.
\] (3.31)

4. Limiting cases
The Fredholm integral equation for two degenerate problems are derived as the limiting cases of the present theory. These two cases are inferred: (i) when the piezoelectric coupling is absent, which corresponds to the transversely isotropic problem, and (ii) when both media are considered to be isotropic, a problem considered by Selvadurai (21).
4.1 Transversely isotropic bimaterial

When $e_{15} = e_{31} = e_{33} = 0$, the piezoelectric coupling is absent and the problem reduces to the transversely isotropic bimaterial. This assumption leads to the following relations

$$
\Delta^I = \frac{a_1^I \Delta}{a_1^I + a_1^{II}}, \quad \Delta^{II} = \frac{a_1^{II} \Delta}{a_1^I + a_1^{II}},
$$

(4.1)

$$
F_1(r) + \int_0^a K(r, \xi) F_1(\xi) d\xi = \frac{\Delta}{a_1^I + a_1^{II}} + \sigma_0 F(r), \quad 0 < r < a,
$$

(4.2)

where

$$
\alpha_1 = -\frac{c_{33}(c_{13} + c_{44})s_1s_2 (s_1 + s_2)}{c_{44} \left(c_{33}s_1^2 + c_{13}\right) \left(c_{33}s_2^2 + c_{13}\right)}.
$$

(4.3)

Following Lekhnitskii (32), $s_i^2$, ($i = 1, 2$) are the roots of

$$
c_{33}c_{44}s^4 - [c_{11}c_{33} + c_{44}^2 - (c_{13} + c_{44})^2]s^2 + c_{11}c_{44} = 0.
$$

(4.4)

As stated before, this important result can be used for modelling of proppant since the geological materials have transversely isotropic behaviour rather than isotropic behaviour. This problem has not been considered in the literature.

4.2 Isotropic bimaterial

The material constants for an isotropic medium become

$$
c_{11} = c_{33} = \frac{2G(1 - \nu)}{1 - 2\nu}, \quad c_{12} = c_{13} = \frac{2G\nu}{1 - 2\nu}, \quad c_{44} = G,
$$

(4.5)

where $G$ and $\nu$ are the elastic shear modulus and Poisson’s ratio, respectively. Substituting (4.5) into (4.1)–(4.4) leads to

$$
\Delta^I = \frac{\Delta}{1 + \alpha}, \quad \Delta^{II} = \frac{\alpha \Delta}{1 + \alpha},
$$

(4.6)

$$
F_1(r) + \int_0^a K(r, \xi) F_1(\xi) d\xi = -\frac{G^I \Delta}{(1 - \nu^I)(1 + \alpha)} + \sigma_0 F(r), \quad 0 < r < a,
$$

(4.7)

where

$$
\alpha = \frac{G^I(1 - \nu^{II})}{G^{II}(1 - \nu^I)}.
$$

(4.8)

In the formulation of Selvadurai (21), the last term in (3.27), $\sqrt{b^2 - a^2}$, is missing.

5. Numerical results and discussion

By virtue of the numerical method given by Atkinson (33), the governing Fredholm integral equations (3.25) and (3.26) can be converted to a set of linear algebraic equations. In what follows, the numerical solution procedure is only discussed for the integral equation (3.25), but similar approach is also valid for the solution of $G_1(r)$ in (3.26).
For numerical purposes, it is useful to introduce the dimensionless parameters
\[ \psi(\zeta) = \frac{F_1(r)}{a\sigma_0}, \quad \zeta = \frac{r}{a}, \quad c = \frac{b}{a}. \]  
(5.1)

In view of these definitions, (3.25) can be written in the discretized form
\[ \sum_{n=1}^{N} A_{mn} \psi_n = \varrho_m, \quad m = 1, 2, \ldots, N, \]  
(5.2)

where \( \psi_n = \psi(\zeta_n), \zeta_n = (2n-1)/(2N), A_{mn} = \delta_{mn} + (a/N)K(a\zeta_m, a\zeta_n) \) and \( \varrho_n = -(\beta^I + \beta^I_2)\Delta/(a\sigma_0 c) + F(a\zeta_n)/a. \)

In these, \( \{\zeta_n\} \) is a set of nodal points and \( \delta_{mn} \) denotes the Kronecker delta. Furthermore, one can recast condition (3.29), which is utilized for the determination of \( c \), in the following discretized form
\[ \varepsilon = \frac{2c}{\pi N} \sum_{n=1}^{N} \frac{\psi_n}{c^2 - \zeta_n^2} + \sqrt{c^2 - 1}, \]  
(5.3)

where \( \varepsilon \) denotes the error. An iterative process is employed to solve the discretized system. The number of iterations is increased until the error becomes admissible, \( |\varepsilon| < 10^{-6} \).

With the aid of the introduced numerical approach, the numerical results can be obtained for different material combinations. Consider a full space made of two specific piezoelectric materials, poled lead zirconate titanate (PZT-4) ceramic for Medium I and for Medium II, with their properties given in Table 1. In order to illustrate the effects of piezoelectric coupling, two different cases are taken into account: (i) when both media are piezoelectric and (ii) when the piezoelectric coupling is absent and the problem reduces to transversely isotropic bimaterial.

For Case (i), the calculated values of indentation of the disc inclusion into the half-space Regions I and II are \( 0 \cdot 531/a \) and \( 0 \cdot 469/a \), respectively. Also, the electric potential of the disc inclusion has been obtained, \( \phi^d = 1 \cdot 027 \times 10^9 \Delta(\text{Vm}^{-1}). \) Moreover, \( \Delta^I = 0 \cdot 617\Delta \) and \( \Delta^{II} = 0 \cdot 383\Delta \) are calculated for Case (ii). Therefore, the effect of piezoelectric coupling on the indentation values is significant. In the remaining results, the applied stress, \( \sigma_0 \), and electric displacement, \( D_0 \), are normalized with respect to \( c_{44}^0 \) and \( e_{33}^0 \), respectively; where the superscript ‘0’ indicates that the constants are pertinent to PZT-4.

Fig. 2 depicts the extent of the zone of separation versus the magnitude of far-field compression for two different values of disc inclusion relative thickness, \( \Delta/a = 10^{-1}, 10^{-2} \). As stated before, the extent of the zone of separation is independent of the applied far-field electric displacement, \( D_0 \). It is observed that the piezoelectric coupling effect results in increase of the radius of separation zone.

Table 1  Material properties of PZT-4 and BaTiO₃

<table>
<thead>
<tr>
<th></th>
<th>Elastic constants (GPa)</th>
<th>Piezoelectric constants (Cm⁻²)</th>
<th>Dielectric constants ( \times 10^{-10}\text{Fm}^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( c_{11} )</td>
<td>( c_{12} )</td>
<td>( c_{13} )</td>
</tr>
<tr>
<td>PZT-4</td>
<td>139</td>
<td>77.8</td>
<td>74.3</td>
</tr>
<tr>
<td>BaTiO₃</td>
<td>166</td>
<td>77.0</td>
<td>78.0</td>
</tr>
</tbody>
</table>
The normal stress distribution on the face of the disc inclusion is shown in Fig. 3. Near the disc edge, the value of the stress tends to infinity and becomes singular at this point. It is realized that the piezoelectric effect causes the entire disc surface to experience higher contact pressure. This is a manifestation of the size of the separation zone.

**Fig. 2** The extent of separation zone versus applied far-field compression for bimaterial full spaces

**Fig. 3** The contact pressure distribution along the radius of the disc inclusion
Figure 4 shows that the normal electric displacement on the disc surface, like normal stress distribution, exhibits singularity at the edge of the disc. It is observed that larger values of \( \Delta / a \) that corresponds to larger separation zone result in higher electric displacement on the surface of the disc.

Fig. 4 The normal electric displacement distribution along the radius of the disc inclusion

Fig. 5 The extent of separation zone versus applied far-field compression for homogeneous full spaces \((\Delta / a = 10^{-2})\)
In order to illustrate the effects of the material mismatch on the response of the system, the radius of separation zone versus the far-field compression is plotted for different homogeneous full spaces in Fig. 5. The results shown in Fig. 5 are obtained for $\Delta/a = 10^{-2}$. Since BaTiO$_3$ is stiffer than PZT-4, the radius of separation zone for BaTiO$_3$ full space is larger than that of PZT-4 for both Cases (i) and (ii). The symmetry of the homogeneous full-space problem implies zero electric potential in the disc inclusion and equal indentations for both lower and upper half-spaces. It is seen that the results for the bimaterial full spaces (shown in Fig. 2) are bounded by the corresponding results of single-phase full spaces displayed in Fig. 5.

References