EVOLUTION, STABILITY AND EQUILIBRIUM SHAPES
OF ROTATING DROPS WHICH ARE CHARGED OR
SUBJECT TO ELECTRIC FIELDS

by V. J. GARCÍA-GARRIDO, M. A. FONTELOS†
(Instituto de Ciencias Matemáticas (CSIC - UAM - UC3M - UCM), C/Nicolás Cabrera,
13-15, Campus de Cantoblanco, 28049 Madrid, Spain)
and
U. KINDELÁN
(Departamento de Matemática Aplicada y Met. Inf., Universidad Politécnica de Madrid,
Alenza 4, 28003 Madrid, Spain)

[Received 2 August 2012. Revise 27 May 2013. Accepted 20 July 2013]

Summary

In this article, we study by means of the boundary element method the effect that rotation at constant angular momentum $L$ has on the evolution of a conducting and viscous drop when it holds an amount of charge $Q$ on its surface or is immersed in an external electric field of magnitude $E_\infty$ acting in the direction of the rotation axis. This droplet is considered to be contained in another viscous and insulating fluid. Our numerical simulations and stability analysis show that the Rayleigh fissibility ratio $\chi$ at which charged drops become unstable decreases with angular momentum. For neutral drops subject to an electric field, the critical value of the field which destabilizes the drop increases with rotation. Concerning equilibrium shapes, approximate spheroids and ellipsoids are obtained and the transition values between these two families of solutions is described. When the drop becomes unstable, a two-lobed structure forms where a pinch-off occurs in finite time or dynamic Taylor cones (in the sense of [Betelú et al., Phys. Fluids. 18 (2006)]) develop, whose semiangle, for small $L$, remains the same as if there was no rotation in the system.

1. Introduction

Electrohydrodynamics is an area of fluid mechanics which has become fundamental to many industrial and technological applications in recent years. The development of techniques such as electrospinning, electrospraying and the design of Field Emission Electric Propulsion (FEEP) colloid thrusters for space vehicles and satellites (1) is clear evidence of its importance. We should also mention its contributions to electrowetting and electronic paper (2) and the promising microfluidic chips, where electric fields are used to control chemicals inside a small device with very thin channels (3, 4). Concerning physical applications, the understanding of coalescence and fission processes for charged droplets is crucial to study how thunderstorm clouds are formed (5).

Since the pioneering works in 1843 by J. Plateau (6), where an oil drop immersed in a neutral buoyancy tank was rotated by turning a shaft, many scientists have become interested in studying
the behaviour of droplets subject to different forces such as rotation, gravity, surface tension and electromagnetic fields. The rotating drop problem for example was used by Chandrasekhar in the field of astrophysics to determine the shape of self-gravitating masses (6). A detailed numerical study of equilibrium configurations and their stability for rotating drops was first undertaken by Brown and Scriven in (8) and improved by Heine (9). From the point of view of the evolution problem, a detailed analysis with boundary element method (BEM) that considers also rotation at constant angular momentum has recently been presented in (10).

The effects of charge on droplets was first investigated by Lord Rayleigh in 1882 (11). With an energy stability analysis he showed that, for a spherical, isolated and conducting drop of radius \( R \) and surface tension \( \gamma \) surrounded by an insulating medium of permittivity \( \varepsilon_0 \), a loss of stability occurs at critical values of the charge \( Q^c_n = 4\pi\sqrt{\varepsilon_0\gamma}R^3(n + 2) \) to shape perturbations given by the \( n \)-th order Legendre polynomial (number of lobes on the perturbed shape). At each critical value \( Q^c_n \), the sphere is neutrally stable and a family of \( n \)-lobed shapes branch. These families of \( n \)-lobed shapes were described numerically in 1989 by Basaran and Scriven using finite elements (12).

Regarding conducting drops under the influence of electric fields, Taylor obtained in 1964 a family of approximate prolate solutions and identified a critical value of the electric field for which these configurations become unstable and develop Taylor cones (13, 14). Numerical evolution with BEM of charged or neutral viscous and conducting drops subject to uniform electric fields is addressed in (15).

The aim of this article is to describe the effects that rotation at constant angular momentum has on the evolution of charged or neutral viscous and conducting drops subject to an electric field. This is done by means of BEM, which is a suitable choice for tracking interfaces in Stokes regimes. An interesting aspect is that our algorithm is implemented to be adaptive, since a good resolution is needed in regions of the mesh where singularities develop. The rotating and charged drop model became relevant when Bohr and Wheeler proposed it in 1939 as a simplified version of the mechanism of nuclear fission (16). Our main motivation for this work comes from the existent discrepancy in the values between theory (14) and laboratory experiments (5) for the measured semiangle of Taylor cones and Rayleigh’s critical charge. We will establish how the critical charge or electric field a drop can sustain before becoming unstable varies with angular momentum. We also present several methods to obtain approximate equilibrium solutions and compare them with those introduced by Rosenkilde and Randall (17, 18), who used an appropriate extension of Chandrasekhar’s virial method when rotation takes place at constant angular velocity.

We start by introducing in Section 2 the mathematical equations that govern the physical system and dedicate Section 3 to briefly describe the numerical method implemented to solve the evolution problem. We follow with Section 4, where a theoretical and numerical approach to compute axisymmetric solutions of charged rotating droplets is presented. Section 5 discusses axisymmetric rotating drops in electric fields and Section 6 studies the effects of rotation on dynamic Taylor cones. To finish, in Sections 7 and 8, a 3D stability analysis is conducted for charged rotating drops or rotating drops immersed in an electric field.

2. Mathematical formulation

We are interested in the evolution, stability and equilibrium configurations of a viscous and conducting drop surrounded by another viscous and insulating fluid. Both fluids rotate about a common axis (e.g. the \( z \) axis) with constant angular momentum \( \mathcal{L} \) and are assumed to be incompressible. We consider the following two situations: in the first, the droplet has an amount
of charge $Q$ distributed over its surface, and in the second, the drop is neutral and subject to a uniform external electric field of magnitude $E\infty$ along its axis of rotation. Working in a rotating frame of reference, Navier-Stokes equations for a fluid with viscosity $\mu_i$, density $\rho_i$, pressure $p^{(i)}$ and velocity field $\mathbf{u}^{(i)}$ have the form\[\text{(1)}\]:

$$\begin{cases}
\frac{\partial \mathbf{u}^{(i)}}{\partial t} + \mathbf{u}^{(i)} \cdot \nabla \mathbf{u}^{(i)} = -\nabla p^{(i)} + \mu_i \Delta \mathbf{u}^{(i)} - 2\Omega_i \mathbf{w} \times \mathbf{u}^{(i)} - \mathbf{w} \times (\mathbf{w} \times \mathbf{r}) , & \text{in } \mathcal{D}_1(t), \\
\nabla \cdot \mathbf{u}^{(i)} = 0, & \text{in } \mathcal{D}_1(t),
\end{cases}$$

where $\mathbf{w} \equiv (0, 0, w)$ is the angular velocity and $\mathcal{D}_1(t)$ is the region enclosed by the droplet and $\mathcal{D}_2(t)$ that of the surrounding fluid. Observe that, since we will work in regimes where the Reynolds number is small, there will be time for diffusion of vorticity to occur provided that deformation is sufficiently slow, thus leading to an almost solid-body rotation. These allow to neglect the fictitious force arising from the variable rate of rotation of the frame of reference. The terms $\mathbf{w} \times \mathbf{u}^{(i)}$ and $\mathbf{w} \times (\mathbf{w} \times \mathbf{r})$ represent the Coriolis and centrifugal forces respectively. We can write the centrifugal force as:

$$-\mathbf{w} \times (\mathbf{w} \times \mathbf{r}) = \omega^2 r_{\text{axis}} \mathbf{e}_r = \nabla \left( \frac{1}{2} \omega^2 r_{\text{axis}}^2 \right).$$

(2.2)

where $r_{\text{axis}}$ is the orthogonal distance from a point at the surface of the drop to the axis of rotation. Suppose now that the density of the surrounding fluid, $\rho_2$, is very small with respect to the density of the droplet, $\rho_1$, and set $\rho_2 = 0$. If we define reduced pressures as:

$$\Pi^{(1)} = \rho^{(1)} - p^{(1)} \frac{L^2}{2 r_{\text{axis}}^2}, \quad \Pi^{(2)} = p^{(2)},$$

(2.3)

and the drop’s moment of inertia:

$$I = \rho_1 \int_{\mathcal{D}_1(t)} r_{\text{axis}}^2 \, dV,$$

(2.4)

then (2.1) becomes:

$$\begin{cases}
\frac{\partial \mathbf{u}^{(i)}}{\partial t} + \mathbf{u}^{(i)} \cdot \nabla \mathbf{u}^{(i)} = -\nabla \Pi^{(i)} + \mu_i \Delta \mathbf{u}^{(i)} - 2\Omega_i \mathbf{w} \times \mathbf{u}^{(i)} , & \text{in } \mathcal{D}_1(t), \\
\nabla \cdot \mathbf{u}^{(i)} = 0, & \text{in } \mathcal{D}_1(t),
\end{cases}$$

(2.5)

In order to non-dimensionalize (2.5) we introduce a characteristic length $l$, a typical velocity $U$ and a characteristic time scale $\tau = l/U$ so that:

$$\mathbf{u}^{(i)} = \frac{\mathbf{u}^{(i)}}{U}, \quad \mathbf{r} = \frac{\mathbf{r}}{l}, \quad \Pi^{(i)} = \frac{1}{\mu_1 U^2} \Pi^{(i)}, \quad \omega = \omega z, \quad \lambda = \frac{\mu_2}{\mu_1}, \quad \zeta = \frac{\rho_2}{\rho_1}.$$

Omitting overbars to simplify notation, the non-dimensional problem is:

$$\begin{cases}
\zeta^{-1} Re \frac{\partial \mathbf{u}^{(i)}}{\partial \tau} + \mathbf{u}^{(i)} \cdot \nabla \mathbf{u}^{(i)} = -\nabla \Pi^{(i)} + \lambda^{-1} \Delta \mathbf{u}^{(i)} - 2\zeta^{-1} \mathbf{z} \times \mathbf{u}^{(i)}, & \text{in } \mathcal{D}_1(t), \\
\nabla \cdot \mathbf{u}^{(i)} = 0, & \text{in } \mathcal{D}_1(t),
\end{cases}$$

(2.6)
Two dimensionless parameters arise: $Re$ is the Reynolds number (measures the relative importance between inertial and viscous forces) and $Ek$ is the Ekman number (characterizing the relation between Coriolis and viscous forces). They are defined as:

$$Re = \frac{\rho_1 U_l}{\mu_1}, \quad Ek = \frac{\mu_1}{\rho_1 \omega l^2}.$$  \hfill (2.7)

In this article we will study the case where viscous forces dominate inertial and Coriolis forces. This regime is known as Stokes flow and is characterized by $Re \ll 1$ and $Ek \gg 1$. Thus we have Stokes system:

$$\begin{cases}
-\nabla \Pi^{(i)} + \lambda^{i-1} \Delta \mathbf{u}^{(i)} = 0, & \text{in } D_1 (t) \\
\nabla \cdot \mathbf{u}^{(i)} = 0, & \text{in } D_1 (t),
\end{cases}$$

(2.8)

to which we impose that the normal component of the velocity is continuous across the boundary:

$$\mathbf{u}^{(1)} \cdot \mathbf{n} = \mathbf{u}^{(2)} \cdot \mathbf{n} \equiv \mathbf{u} \cdot \mathbf{n},$$

(2.9)

and the kinematic condition:

$$v_n = \mathbf{u} \cdot \mathbf{n}, \quad \text{on } \partial D (t),$$

(2.10)

with $v_n$ being the normal component of the free boundary.

Now, since the drop is considered an ideal conductor, the potential $\mathcal{V}$ must be constant inside and at the drop’s surface, and all the charge is located at the boundary. The electric potential satisfies the Laplace equation:

$$\begin{cases}
\Delta \mathcal{V} = 0, & \text{in } D_2 (t) \\
\mathcal{V} = \mathcal{V}_0, & \text{in } \partial D (t), \\
\mathcal{V} \to -\mathcal{E}_\infty z + O(|r|^{-1}), & \text{as } |r| \to \infty
\end{cases}$$

(2.11)

and $\mathcal{V}_0$ has to be chosen so that the total charge is $Q$. At the boundary of the drop the surface charge density $\sigma$ is given by the normal derivative of the potential, $\sigma = -\varepsilon_0 \frac{\partial \mathcal{V}}{\partial n}$, with $\varepsilon_0$ the permittivity of the surrounding fluid and $\mathbf{n}$ is the outward unit normal to the surface of the droplet. At the surface of a conductor, the repulsive electrostatic force per unit area is:

$$\mathbf{F}_e = \frac{\varepsilon_0}{2} \left( \frac{\partial \mathcal{V}}{\partial n} \right)^2 \mathbf{n} = \frac{\sigma^2}{2\varepsilon_0} \mathbf{n}.$$

To solve (2.8) we set a balance between viscous stresses and capillary, electrostatic and centrifugal forces at the interface of both fluids:

$$\left( T^{(2)} - T^{(1)} \right) \mathbf{n} = 2\gamma \mathcal{H} - \sigma \frac{E^2}{2\varepsilon^2} r^2 \frac{\partial^2}{\partial z^2} + \sigma^2 \frac{E^2}{2\varepsilon^2} \mathbf{n}, \quad \text{on } \partial D_1 (t),$$

(2.12)

where $\gamma$ is the surface tension and $\mathcal{H}$ the mean curvature and, for Newtonian fluids:

$$T^{(k)}_{ij} = -\Pi^{(k)} \delta_{ij} + \mu_k \left( \frac{\partial \mathbf{u}^{(k)}}{\partial x_j} + \frac{\partial \mathbf{u}^{(k)}}{\partial x_i} \right), \quad k = 1, 2.$$
To make the boundary condition (2.12) dimensionless we introduce the characteristic scales:

$$l = \sqrt{V}, \quad U = \frac{\gamma}{\mu_1}, \quad \tau = \frac{l}{U},$$

and the quantities:

$$\mathcal{T}^{(i)} = \frac{l}{\gamma} T^{(i)}, \quad \mathcal{H} = l H, \quad \mathcal{T} = \frac{T}{\mathcal{P}_1}, \quad \sigma = \sqrt{l \gamma \sigma_0},$$

where \(V\) is the volume occupied by the drop. By defining:

$$L = \frac{L}{\sqrt{\mathcal{P}_1 \gamma}}, \quad \chi = \frac{Q^2}{48 \pi} = \frac{Q^2}{48 \pi \gamma_0 \mathcal{E}_0 \mathcal{L}}, \quad E_\infty = \sqrt{\frac{\epsilon_0 l}{\gamma} E_\infty},$$

we get for the boundary condition, omitting overbars:

$$\left( T^{(2)} - T^{(1)} \right) n = \left( 2H - \frac{L^2}{2\mathcal{L}} l_{axis}^2 - \frac{\sigma^2}{2} \right) n, \quad \text{on } \partial \mathcal{D}(t).$$

In what follows we will describe the results in terms of the parameters in (2.16) where \(\chi\) is known as Rayleigh’s fissibility ratio. Notice that one can easily recover the physical values by solving for \(L, Q\) and \(E_\infty\).

Concerning the stationary problem (where \(u^{(i)} \equiv 0\)), equilibrium solutions can be calculated by solving the modified Young–Laplace equation:

$$\delta \Pi = 2\gamma H - \mathcal{P}_1 \frac{L^2}{2\mathcal{L}} l_{axis}^2 - \frac{\sigma^2}{2}, \quad \text{on } \partial \mathcal{D},$$

where \(\mathcal{D}\) is the region enclosed by the drop and \(\delta \Pi = \Pi^{(1)} - \Pi^{(2)}\) is the reduced pressure difference across the drop’s surface. It is important to note that this equilibrium equation can be seen as the Euler–Lagrange equation of a particular energy functional. In this variational framework, our problem reduces to the minimization of the total energy for the closed system, which we can write as:

$$E_{\text{total}} = E_{\text{area}} + E_{\text{kinetic}} + E_{\text{electrostatic}} - \delta \Pi (V - V_0),$$

where \(\delta \Pi (V - V_0)\) is the constraint (\(\delta \Pi\) plays the role of a Lagrange multiplier) so the volume of the drop is \(V = V_0\). \(E_{\text{area}}\) is the energy due to surface area, \(E_{\text{kinetic}}\) is the rotational kinetic energy for constant angular momentum rotation and \(E_{\text{electrostatic}}\) is the electrical energy. These energies can be obtained from:

$$E_{\text{area}} = \gamma \text{Area} \left( \partial \mathcal{D} \right), \quad E_{\text{kinetic}} = \frac{L^2}{2\mathcal{L}}, \quad E_{\text{electrostatic}} = \frac{\gamma_0}{2} \int_{\mathcal{D} \setminus \partial \mathcal{D}} |\mathbf{E}|^2 \, dV.$$  

Since the electric field verifies \(\mathbf{E} = -V\), after integrating by parts one can write the electrostatic energy as:

$$E_{\text{electrostatic}} = \frac{\gamma_0}{2} \int_{\mathcal{D} \setminus \partial \mathcal{D}} |\mathbf{E}|^2 \, dV = \frac{\gamma_0}{2} \int_{\partial \mathcal{D}} V \frac{\partial V}{\partial n} \, dS = \frac{1}{2} \int_{\partial \mathcal{D}} V_0 \sigma \, dV = \frac{1}{2} Q V_0 = \frac{Q^2}{2C}.$$
As the drop evolves, the capability of the mesh to approximate the free boundary might worsen. To find the velocity, we will use (2.19) and the minimization argument in Sections 4 and 7 to find approximate spheroidal equilibrium shapes and the transition curve from these solutions to ellipsoidal configurations.

3. Numerical method

The BEM has proved to be a very efficient and powerful tool to simulate the evolution of drops under Stokes regimes. Many authors in the literature apply this numerical technique to deal with free-surface problems involving drop dynamics. With BEM, one can find a boundary integral equation that describes the velocity of the drop’s interface at each time step. The way to proceed is to use Green’s functions for Stokes equation in combination with several properties of Dirac’s delta function (for a comprehensive explanation on how to derive this condition see [21] and [23]). One arrives at:

\[ 4\pi (1 + \lambda) u_j (x_0) = - \int_{\partial D(i)} G_{ij}(x, x_0) f_j(x) \, dS - (1 - \lambda) \int_{PV} u_i(x) T_{ijk}(x, x_0) n_k(x) \, dS, \]

where:

\[ G_{ij}(x, x_0) = \frac{\delta_{ij}}{|x - x_0|} + \frac{(x_i - x_{0,i})(x_j - x_{0,j})}{|x - x_0|^3}, \quad i, j \in \{1, 2, 3\}, \quad (3.2) \]

\[ T_{ijk}(x, x_0) = -6 \frac{(x_i - x_{0,i})(x_j - x_{0,j})(x_k - x_{0,k})}{|x - x_0|^5}, \quad i, j, k \in \{1, 2, 3\}, \quad (3.3) \]

\[ f_j(x) = \left[ 2 \mathcal{H}(x) - \frac{L^2}{2T^2} \mathbf{e}_{axis} - \frac{\sigma^2(x)}{2} \right] n_j(x), \quad i \in \{1, 2, 3\}. \quad (3.4) \]

and \( x_0 \) is a point on the boundary. Notice that both terms on the right-hand side of (3.1) present a singularity at \( x = x_0 \). To remove these singularities and increase the stability of our method we use a technique proposed in [21] based on the properties:

\[ \int_{\partial D(i)} G_{ij}(x, x_0)n_j(x) \, dS = 0, \quad \int_{\partial D(i)} T_{ijk}(x, x_0)n_k(x) \, dS = -4\pi \delta_{ij}. \quad (3.5) \]

Given a time \( t > 0 \), we approximate \( \partial D(t) \) by a triangle mesh with \( N \) vertices and \( M \) triangles. To find the velocity \( u \) we need to calculate the mean curvature and the surface charge density at each node of the mesh as well as the moment of inertia of the drop. Once the velocity field is known, we can move the boundary mesh in the normal direction (the tangential component of velocity is not considered because it only redistributes the nodes over the boundary) with an Euler explicit scheme:

\[ x_i(t_{j+1}) = x_i(t_j) + \nu_0 (x_i, t_j) n_i(x_i) \Delta t, \quad i \in \{1, \ldots, N\}. \quad (3.6) \]

As the drop evolves, the capability of the mesh to approximate the free boundary might worsen with time. This could happen for instance when the geometry of the interface starts to develop...
singularities, such as dynamic Taylor cones. To address this issue we have implemented several regularization algorithms that use Delaunay remeshing and a locally uniform distribution of nodes (relaxation technique). Even with this improved node distribution, one has to take into account that boundary regions presenting high curvature values need a larger density of nodes in order to resolve those areas with enough precision. To achieve this requirement, we apply local refinement to those triangles whose area is larger than a certain multiple of the area of an equilateral triangle with edges equal to the local radius of curvature. For a detailed description on the adaptive meshing we have used see [24] and [22].

The approach taken to compute the mean curvature is introduced in [20], where the mean curvature at each vertex of the mesh is approximated by the mean curvature of the paraboloid that best fits its neighbours in an iterative process. This algorithm also provides us with the unit normal at each node. Regarding the moment of inertia, for each triangle of the mesh we create a tetrahedron whose vertices are determined by the three triangle vertices and the origin and add its contribution to the inertia as shown in [25]. This is a very simple and efficient method with quadratic convergence.

Finally, the equation for the surface charge density at point $x_0 = (x_0, y_0, z_0)$ on the drop’s surface is:

$$V_0 + E_\infty z_0 = \frac{1}{4\pi \varepsilon_0} \int_{\partial D(t)} \frac{\sigma(x)}{|x - x_0|} \, dS.$$  (3.7)

If we write $\sigma = V_0 \sigma_0 + \sigma_{ind}$, where $\sigma_{ind}$ is the surface charge density induced by the external electric field, then:

$$E_\infty z_0 = \frac{1}{4\pi \varepsilon_0} \int_{\partial D(t)} \frac{\sigma_{ind}(x)}{|x - x_0|} \, dS, \quad 1 = \frac{1}{4\pi \varepsilon_0} \int_{\partial D(t)} \frac{\sigma_0(x)}{|x - x_0|} \, dS,$$  (3.8)

and $V_0$ is determined by the condition:

$$Q = V_0 \int_{\partial D(t)} \sigma_0(x) \, dS + \int_{\partial D(t)} \sigma_{ind}(x) \, dS.$$  (3.9)

To numerically invert the equations appearing in (3.8) we have also used the BEM.

For our research we have implemented two different numerical codes. One deals with the axisymmetric problem, for which (3.1) reduces to a 1D integral equation, since all the integrals involving the azimuthal angle can be explicitly expressed in terms of elliptic functions [21]. All equilibrium solutions discussed in the next three sections were generated with this version of the code. Then, in Sections 7 and 8, a stability analysis on these axisymmetric equilibrium configurations is performed with a 3D version of the code.

4. Equilibrium solutions for charged rotating drops

In this section, we present two techniques to approximate equilibrium configurations for a conducting drop with charge $Q$ rotating at constant angular momentum $L$ and compare them with numerical experiments. Numerical evidence shows that rotation leads to an imperfect bifurcation [24]. This imperfection, in the language of bifurcation theory, ruptures the bifurcation point attained at $\chi = 1$ for non-rotating charged drops [27], resulting in two distinct families of equilibrium shapes as shown in Fig. 1. We start by deriving an analytical second-order approximation in the small parameter $L^2$ similarly to the work in [28], where rotation is considered to take place at constant angular velocity...
Fig. 1  Comparison of equilibrium shapes for different methods. Deformation is defined as $D = \frac{r_{\text{equat}} - r_{\text{polar}}}{r_{\text{equat}} + r_{\text{polar}}}$, where $r_{\text{polar}}$ represents the semiaxis in the direction of the rotation axis (z-axis) and $r_{\text{equat}}$ is the semiaxis on the OXY plane

ω and the ratio of the deformation amplitude to the radius of the initial spherical shape is used as the small parameter. Then, we benefit from the variational formulation (2.19) to introduce oblate and prolate spheroidal solutions that minimize the energy.

4.1 Expansion for small $L$

To compute the equilibrium shapes one has to solve the pressure balance equation on the free surface:

$$\delta \Pi = 2yH - \vartheta_1 \frac{L^2}{2\pi} r_{\text{axis}}^2 - \frac{\epsilon_0}{2} \left( \frac{\partial V}{\partial n} \right)^2, \text{ on } \partial \mathcal{D},$$  

(4.1)

It is well known that for $L = 0$, the sphere is a solution of (4.1) for all values of $Q$. This suggests that one could search for solutions as expansions in powers of $L^2 \ll 1$ about the sphere of radius $R$.

By imposing axial symmetry, one can use spherical coordinates to write the desired solution as:

$$S : r (\theta) = R + f (\theta) L^2 + g (\theta) L^4 + O \left( L^6 \right), \quad \theta \in [0, \pi],$$  

(4.2)

where $f$ and $g$ can be expressed in terms of spherical harmonics as:

$$f (\theta) = \sum_{l=0}^{\infty} f_{2l} Y_{2l,0} (\theta), \quad g (\theta) = \sum_{l=0}^{\infty} g_{2l} Y_{2l,0} (\theta),$$  

(4.3)

under the additional assumption of equatorial symmetry. To calculate approximate equilibrium solutions we will use the formulas derived in the Appendix by setting $L^2 \equiv \varepsilon$ and $f + g L^2 \equiv h$. 


We have that:

\[ H(S) = \frac{1}{R} + \mathcal{L}_H(f) \mathcal{L}^2 + \left( \mathcal{L}_H(g) + \frac{1}{2} Q_H(f, f) \right) \mathcal{L}^4 + O \left( \mathcal{L}^6 \right), \]

\[ \left( \frac{\partial \mathcal{V}}{\partial n}(S) \right)^2 = \left( \frac{\mathcal{Q}}{4\pi \varepsilon_0 R^2} \right)^2 + \mathcal{L}_V(f) \mathcal{L}^2 + \left( \mathcal{L}_V(g) + \frac{1}{2} Q_V(f, f) \right) \mathcal{L}^4 + O \left( \mathcal{L}^6 \right). \]

(4.4)

where \( \mathcal{L}_H, \mathcal{L}_V, Q_H \) and \( Q_V \) are operators verifying (A.12). Define a new pressure:

\[ \delta P = \frac{R}{2\gamma} \delta \Pi, \]

(4.5)

to introduce Rayleigh’s fissibility ratio and expand the pressure in powers of \( \mathcal{L}^2 \) to get:

\[ \delta P = \delta P_0 + \delta P_1 \mathcal{L}^2 + \delta P_2 \mathcal{L}^4 + O \left( \mathcal{L}^6 \right). \]

(4.6)

At this point we can start to compare terms in (4.1). To zeroth-order we have:

\[ \delta P_0 = 1 - \chi. \]

(4.7)

For the first-order contribution collect all terms with \( \mathcal{L}^2 \) and use the property (A.15) to yield:

\[ \delta P_1 + \left( \frac{15}{16\pi \sqrt{\gamma \varrho} R^2} \right)^2 \sin^2 \theta = R \mathcal{L}_H(f) - \frac{\epsilon_0 R}{4\gamma} \mathcal{L}_V(f) = - \frac{1}{R} \left( f + \frac{1}{2} \Delta_0 f \right) + \]

\[ + \frac{2\chi}{R} \left( 2 f - \sum_{l=0}^{\infty} (2l + 1) f_{2l} Y_{2l,0} \right) = \frac{1}{R} \sum_{l=0}^{\infty} a(\chi, l) f_{2l} Y_{2l,0}. \]

(4.8)

where:

\[ a(\chi, l) = (2\chi - 1) + (1 - 4\chi) l + 2l^2, \]

(4.9)

are the eigenvalues associated to spherical harmonics \( Y_{2l,0} \) for the operator:

\[ K = R^2 \mathcal{L}_H - \frac{\epsilon_0 R^2}{4\gamma} \mathcal{L}_V, \]

since:

\[ K(Y_{2l,0}) = a(\chi, l) Y_{2l,0}. \]

Applying (A.4) and volume conservation (A.5), we equate the \( Y_{0,0} \) component with the pressure:

\[ \delta P_1 = - \frac{75}{128 \varrho_1 \gamma \pi^2 R^7}, \quad 0 = \frac{1}{R} \sum_{l=1}^{\infty} a(\chi, l) f_{2l} Y_{2l,0} + \frac{3\sqrt{125}}{64 \varrho_1 \gamma \pi^2 R^7} Y_{2,0}. \]

(4.10)
These linear combinations of $Y$ lead to:  
$$f (\theta) = f_2 Y_{2,0} (\theta), \quad f_2 = - \frac{15 \sqrt{3}}{128 \rho_1 \gamma \pi^2 R^6 (1 - \chi)}. \quad (4.11)$$

and by virtue of incompressibility one obtains:

$$g_0 = - \frac{1}{2 \sqrt{\pi} R^2} = - \frac{1125}{215 \rho_1^2 \gamma^2 \pi^2 R^{13} (1 - \chi)^2}.$$  

This proves that our equilibrium shape is an oblate or prolate spheroid to first-order approximation depending on whether $\chi < 1$ or $\chi > 1$ respectively. To balance second-order terms:

$$\delta P_2 = R \mathcal{L}_H (g) - \frac{\varepsilon R}{4 \gamma} \mathcal{L}_V (g) + \frac{R}{2} \mathcal{Q}_H (f, f) - \frac{\varepsilon R}{8 \gamma} \mathcal{Q}_V (f, f) -$$  

$$- \left( \frac{15}{16 \pi \sqrt{\gamma \rho_1 R^4}} \right)^2 \left( 2 f + \frac{5}{\sqrt{\pi}} \right) \sin^2 \theta, \quad (4.12)$$

use that the quadratic operators $\mathcal{Q}_H$ and $\mathcal{Q}_V$ verify in combination with:

$$\mathcal{Q}_H (f, f) = f_2^2 \mathcal{Q}_H (Y_{2,0}, Y_{2,0}), \quad \mathcal{Q}_V (f, f) = f_2^2 \mathcal{Q}_V (Y_{2,0}, Y_{2,0}),$$

and that:

$$Y_{2,0}^2 = \frac{1}{\sqrt{\pi}} \left( \frac{3}{7} Y_{4,0} + \frac{1}{7} \sqrt{3} Y_{2,0} + \frac{1}{2} Y_{0,0} \right).$$

These linear combinations of $Y_{0,0}, Y_{2,0}$ and $Y_{4,0}$ indicate that to satisfy we need:

$$g (\theta) = g_0 Y_{0,0} (\theta) + g_2 Y_{2,0} (\theta) + g_4 Y_{4,0} (\theta). \quad (4.13)$$

Under this assumption we have:

$$R \mathcal{L}_H (g) - \frac{\varepsilon R}{4 \gamma} \mathcal{L}_V (g) = \frac{1}{R} \sum_{l=0}^{2} a (\chi, l) g_2 Y_{2l,0}$$

$$= \frac{(1 - 2 \chi) f_2^2}{2 \sqrt{\pi} R^2} Y_{0,0} + \frac{2(1 - \chi) g_2}{R} Y_{2,0} + \frac{3(3 - 2 \chi) g_4}{R} Y_{4,0}.$$  

Equating the pressure with the terms involving $Y_{0,0}$ as before we get:

$$\delta P_2 = - \left( \frac{(\chi + 4) f_2^2}{2 \sqrt{\pi} R^2} + \frac{15 \sqrt{3} f_2}{16 \rho_1 \gamma \pi^2 R^8} \right) Y_{0,0}. \quad (4.14)$$

Then, in order to satisfy the coefficients multiplying $Y_{2,0}$ and $Y_{4,0}$ must be zero, which leads to:

$$g_2 = \frac{5 \sqrt{3} f_2^2 (1 + \chi)}{14 \sqrt{\pi} R (1 - \chi)} = \frac{15 f_2}{448 \rho_1 \gamma \pi^2 R^4 (1 - \chi)}, \quad g_4 = \frac{5 f_2^2 (1 - \chi)}{7 \sqrt{\pi} R (3 - 2 \chi)} - \frac{15 \sqrt{3} f_2}{224 \rho_1 \gamma \pi^2 R^4 (3 - 2 \chi)}.$$  

$$\quad (4.15)$$
and our approximate solution to second-order is:

\[ r(\theta) = R + f_2 Y_{2,0}(\theta) L^2 + \left( g_{0} Y_{0,0}(\theta) + g_{2} Y_{2,0}(\theta) + g_{4} Y_{4,0}(\theta) \right) L^4. \]  

(4.16)

Observe that this solution presents singularities at the eigenvalues (4.19) of \( K \), which are the bifurcation points for non-rotating charged drops (27). Comparing approximate solutions (4.16) with numerical experiments we have seen that, for small values of angular momentum \( L \), a good agreement is achieved, provided that Rayleigh’s fissibility ratio \( \chi \) is far from the bifurcation points. To be quantitative, the deviation on the spheroid deformation between solutions obtained by simulation and those described by (4.16) is less than 1% for \( 0 \leq \chi \leq 0.485 \), and less than 5% when \( 0.485 \leq \chi \leq 0.6765 \). This is shown in Fig. 1.

4.2 Spheroidal model

A different approach to compute equilibrium configurations is to find the states that minimize the total energy (2.19) of the system. Until now, we have already seen that spheroids are good approximate solutions for small values of rotation and charge. This encourages us to look for oblate and prolate spheroids that minimize the energy.

Given the spheroid with semiaxes \( a \) and \( c \):

\[ \frac{(x^2 + y^2)}{a^2} + \frac{z^2}{c^2} = 1, \]  

(4.17)

we can write \( c \) in terms of \( a \) using that the volume of the spheroid is \( V = \frac{4\pi}{3} a^2 c \), reducing the energy to a function of one variable, say \( a \). For oblate (\( a > c \)) and prolate (\( a < c \)) spheroids the moment of inertia is:

\[ I_{\text{oblate}} = I_{\text{prolate}} = \frac{2}{5} \rho_1 a^2, \]

and we also have exact formulas to calculate the surface area and the capacitance [24–31]:

\[ A_{\text{prolate}} = 2\pi a^2 \left( 1 + \frac{\arcsin e}{e \sqrt{1 - e^2}} \right), \quad C_{\text{prolate}} = 8\pi \varepsilon_0 e \frac{e}{\log \left( \frac{1 + e}{1 - e} \right)} \frac{a^2}{c^2}, \]

\[ A_{\text{oblate}} = 2\pi a^2 \left( 1 + \frac{1}{e} \arctanh e \right), \quad C_{\text{oblate}} = 4\pi \varepsilon_0 a \frac{e}{\arcsin e} \frac{a^2}{c^2}, \]

Therefore, the energy function for oblate spheroids is:

\[ E_{\text{total}}(a) = 2\pi a^2 \gamma \left( 1 + \frac{1 - e^2}{e} \arctanh e \right) + \frac{5L^2}{4\rho_1 a^2} + \frac{Q^2}{8\pi \varepsilon_0 e} \arcsin e, \]  

(4.18)

and for prolate spheroids:

\[ E_{\text{total}}(a) = 2\pi a^2 \gamma \left( 1 + \frac{\arcsin e}{e \sqrt{1 - e^2}} \right) + \frac{5L^2}{4\rho_1 a^2} + \frac{a^2 Q^2}{12\varepsilon_0 e} \log \left( \frac{1 + e}{1 - e} \right). \]  

(4.19)

Notice that the total energy as well as the eccentricity depend on the semiaxis \( a \), which makes it very lengthy to deal with the minimization problem analytically. Consequently, given a pair of values
(L, Q), to compute the critical points of these energy functionals we proceeded in the usual way: first we compute the first and second derivatives of the energy with respect to a and then, critical points are determined straightforwardly using Maple©. The solutions obtained are compared in Fig.[I] with numerical experiments and the approximate configurations derived in the previous subsection.

4.3 Numerical simulations and comparison with approximate models

In order to validate the models introduced in this section, we have compared them with numerical results obtained by simulating, with the axisymmetric version of the code, the evolution of a small oblate perturbation of a spherical droplet with volume V = 1. All simulations were run with λ = 0.1 until a stationary oblate-like profile for the drop’s interface is reached or, if the charge is large enough, dynamic Taylor cones develop. We show in Figs. [I] and [II] that oblate spheroids provide, for large values of χ, a better approximation to the numerical equilibrium shapes than the solutions corresponding to the asymptotic expansion method. Moreover, Fig. [I] also demonstrates that the effect of rotation on the system is to remove the bifurcation point that charged non-rotating drops present at χ = 1. Observe that the prolate-like family of solutions derived with the asymptotic expansion method is discontinuous at χ = 1.5, which is an eigenvalue of K. To make the curve continuous, one could expand the solution about that point and match it with the one obtained in Section 4.1. It is important to remark that prolate configurations never show up in our numerical experiments, indicating that they are naturally unstable.

5. Equilibrium shapes for rotating drops in uniform electric fields

When an uncharged conducting drop is subject to a uniform electric field it elongates in the direction of the field and prolate spheroids become good approximations for the equilibrium shapes (14). If the electric field is strong enough, self-similar conical tips develop at the poles (15) and, at some
point, thin fluid jets of microdroplets emerge from these tips. If we also include rotation about an axis in the direction of the field, the resulting centrifugal forces tend to flatten the droplet and counterbalance electrostatic forces. The combination of these two forces creates a new family of equilibrium configurations, oblate spheroids. Surprisingly, uncharged conducting drops can remain spherical for a particular ratio of the angular momentum to the applied electric field.

Before presenting our results concerning the evolution towards oblate and prolate equilibrium shapes we will show that spheres are not equilibrium solutions for a rotating drop holding a non-zero charge and subject to an electric field of magnitude . The electric potential:

\[ V(r) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{\varepsilon_\infty R^3}{r^2} \cos \theta - \varepsilon_\infty r \cos \theta, \quad r \geq R, \]

solves the Laplace equation:

\[
\begin{cases}
\Delta V = 0, & \text{in } \mathbb{R}^3 - B_R(0), \\
V = V_0, & \text{in } \partial B_R(0), \\
V \to -\varepsilon_\infty z + O(|r|^{-1}), & \text{as } |r| \to \infty
\end{cases}
\]

where \( B_R(0) \) is the solid sphere of radius \( R \) centered at \( x_0 = 0 \). The moment of inertia of a solid sphere is:

\[ I = \frac{8\pi}{15}\rho_1 R^5, \]

and the normal derivative of the potential and mean curvature at the surface of the sphere take the values:

\[ \frac{\partial V}{\partial n} \bigg|_{r=R} = -\frac{Q}{4\pi\varepsilon_0 R^2} - 3\varepsilon_\infty \cos \theta, \quad \mathcal{H} = \frac{1}{R}. \]

Then, from \( 4.1 \) we get:

\[ \delta \Pi = \frac{2\gamma}{R} - \frac{225L^2}{128\rho_1 \pi^2 R^8} \sin^2 \theta - \frac{\varepsilon_0}{2} \left( -\frac{Q}{4\pi\varepsilon_0 R^2} - 3\varepsilon_\infty \cos \theta \right)^2. \]

Rearranging:

\[ \delta \Pi + \frac{2\gamma}{R} (\chi - 1) + \frac{9\varepsilon_0 \varepsilon_\infty^2}{2} = \left( \frac{9\varepsilon_0 \varepsilon_\infty^2}{2} - \frac{225L^2}{128\rho_1 \pi^2 R^8} \right) \sin^2 \theta - \frac{3Q\varepsilon_\infty}{4\pi R^2} \cos \theta, \]

which holds for all \( \theta \), provided that \( \varepsilon_\infty \neq 0 \), when \( Q = 0 \) and:

\[ \frac{9\varepsilon_0 \varepsilon_\infty^2}{2} = \frac{225L^2}{128\rho_1 \pi^2 R^8} \Leftrightarrow E_\infty = \frac{5}{6} \sqrt{\frac{4\pi}{3} L}, \]

giving the linear relationship between angular momentum and electric field for spherical shapes.

Theoretical works to study spheroidal equilibrium configurations and their stability were initiated in 1974 by Rosenkilde and Randall. Using Chandrasekhar’s tensor virial method they obtained a linear relationship between \( L^2 \) and \( E_\infty \) for approximate spheroidal solutions with the same aspect ratio.
ratio, \( \alpha = \frac{r_{\text{polar}}}{r_{\text{equat}}} \). Although their results apply to an isolated, conducting, incompressible, inviscid and rotating droplet with constant angular velocity, we have compared our numerical experiments (rotation taking place at constant angular momentum) to those derived by Rosenkilde et al. to see that they are in close agreement for small values of \( L \) and \( E_\infty \) as displayed in Fig. 3. For this task we have simulated the evolution of an oblately perturbed spherical drop of volume \( V = 1 \), viscosity ratio \( \lambda = 0.1 \), and values for \( L \) taken between 0 and 1.05 at intervals of 0.05 and for \( E_\infty \) between 0 and 1 at intervals of 0.025 respectively. All solutions obtained in this way are depicted in Fig. 4.

Consider the spheroid (4.17) and its non-dimensional semiaxes:

\[
a_1 = a_2 = \frac{a}{R}, \quad a_3 = \frac{c}{R},
\]

where \( R \) is the radius of the sphere with the same volume as the spheroid. The formula deduced in (18) that gives the relationship between the angular momentum and the electric field is:

\[
\frac{5}{6} \sqrt{\frac{4\pi}{3}} L^2 = a_1^2 (A_3 - A_1) - \frac{2\sqrt{3}}{3A_3} \sqrt{\frac{3}{4\pi}} \left( a_1^2 A_{13} - 3a_3^2 A_{33} \right) E_\infty^2,
\]

where the coefficients \( A_i, A_{ij} \) and \( A_i \) are defined as follows:

\[
A_i = \int_0^\infty \frac{dt}{\sqrt{(a_1^2 + t) (a_2^2 + t) (a_3^2 + t)}}, \quad i \in \{1, 2, 3\},
\]

\[
A_{ij} = \int_0^\infty \frac{dt}{\sqrt{(a_1^2 + t) (a_2^2 + t) (a_3^2 + t)}}, \quad i, j \in \{1, 2, 3\},
\]
Fig. 4  Equilibrium configurations for different values of $L$

$$A_i = \int_0^\infty \frac{dt}{\sqrt{(a_i^1 + t^2)(a_i^2 + t^2)(a_i^3 + t^2)}}, \quad i \in \{1, 2, 3\},$$

and satisfy the properties:

$$\sum_{i=1}^3 A_i = \frac{2}{a_1a_2a_3}, \quad 3A_i + \sum_{j \neq i} A_{ij} = \frac{2}{a_1a_2a_3a_i^2}, \quad A_{ij} = -\frac{A_i - A_j}{a_i^2 - a_j^2} \quad \text{when } j \neq i.$$

For spheroids with semiaxes $a_1 = a_2 \neq a_3$ we have explicit formulas \[33, 34\] in terms of their eccentricity:

$$A_i^{\text{oblate}} = \frac{1}{a_i^3} \left( \frac{\arcsin e - e\sqrt{1-e^2}}{e} \right), \quad A_3^{\text{oblate}} = \frac{2}{a_1^3} \left( \frac{e}{\sqrt{1-e^2}} - \arcsin e \right),$$

$$A_1^{\text{oblate}} = \frac{1}{2a_1^3e} \left( (1 + e^2) \frac{\arctanh e}{e} - 1 \right), \quad A_3^{\text{oblate}} = \frac{1}{a_1^3e^2} \left( \frac{1}{1 - e^2} - \arctanh e \right),$$

$$A_1^{\text{prolate}} = \frac{1}{a_1^3e^3} \left( \frac{e}{1 - e^2} - \frac{1}{2} \log \left( \frac{1 + e}{1 - e} \right) \right), \quad A_3^{\text{prolate}} = \frac{1}{a_3^3e^3} \left( \log \left( \frac{1 + e}{1 - e} \right) - 2e \right),$$

$$A_1^{\text{prolate}} = \frac{\sqrt{1 - e^2}}{2a_1^3e^2} \left( \sqrt{1 - e^2} - \left( 1 - 2e^2 \right) \frac{\arcsin e}{e} \right), \quad A_3^{\text{prolate}} = \frac{1}{a_3^3e^2\sqrt{1 - e^2}} \left( \arcsin e \frac{e}{e} - \sqrt{1 - e^2} \right).$$
If we define the aspect ratio, \( \alpha = \frac{a_3}{a_1} \), then (5.4) can be rewritten in the form:

\[
E_\infty^2 = h(\alpha) + g(\alpha) L^2,
\]

(5.5)

yielding for each \( \alpha \) the equation of a line. An approximate polynomial representation of functions \( h \) and \( g \) is:

\[
h(\alpha) = 4.6774\alpha^3 - 17.213\alpha^2 + 21.823\alpha - 9.2863
\]

(5.6)

\[
g(\alpha) = -0.62694\alpha^3 + 2.7119\alpha^2 - 4.7114\alpha + 4.4308
\]

Even though our numerical results do not match the Rosenkilde and Randall approximate theoretical solutions for large values of the parameters, they are still linearly related in the \((L^2, E_\infty^2)\)-space (see Fig. 3). Therefore, we can obtain from numerical results representations of the functions \( h \) and \( g \) appearing in (5.5) that are more accurate than those given by Rosenkilde and Randall. Using standard interpolation one finds that:

\[
h(\alpha) = 4.9586\alpha^3 - 18.135\alpha^2 + 22.835\alpha - 9.6572
\]

(5.7)

\[
g(\alpha) = -2.2864\alpha^3 + 8.3977\alpha^2 - 11.369\alpha + 7.0616
\]

(5.8)

describe the isolines verified by the numerical solutions for any value of the aspect ratio. Expressions (5.7) and (5.8) provide us with an easy and accurate way to compute approximate spheroidal solutions for droplets that rotate at constant angular momentum and are subject to a uniform external electric field.

Suppose that we start from a situation where no electric field is present and we set a value for the angular momentum \( L > 0 \) for which the corresponding equilibrium solution is oblate. As we increase the magnitude of the electric field, electrostatic pressure counterbalances the flattening effect of rotation (which tends to thicken the equatorial region) by stretching the droplet in the direction of the applied field and, consequently, a smooth transition from oblate to prolate solutions takes place. In this process, and as the variation of the parameters is continuous, a spherical solution is attained, marking the point where oblate configurations meet prolate ones. If we keep increasing the electric field, prolate spheroids eventually become unstable and develop dynamic Taylor cones. From experiments, the values where this change in stability takes place is shown in Fig. 5.

6. Analysis of dynamic Taylor cones

The destabilizing effect that a conducting fluid droplet undergoes when it bears an amount of charge above Rayleigh’s limit or is immersed in a strong electric field gives rise to cone-like singularities known in the literature as dynamic Taylor cones. This phenomenon was first discovered by Taylor (14) in 1964 and takes place in very short time scales. In his works, Taylor computed theoretically a value for the cone semiangle of 49.3°, which has turned out to be very different from the average value of about 30° obtained numerically (15) and from experiments. Moreover, this angle seems to depend slightly on the ratio of viscosities of the two fluids, \( \lambda \), and interestingly it develops in a self-similar way (15).

To understand the role that rotation at constant angular momentum has in the cone formation process, we have studied its contribution to charged conducting drops and separately to uncharged...
Conducting droplets under the influence of electric fields. Comparing the centrifugal and surface tension stresses near the region where the tips develop we have found that the effects of rotation are negligible and, therefore, the cone formation mechanism behaves as if there was no rotation. This implies in particular that we still get self-similar evolution in the form:

\[
\kappa_{\text{tip}} = O \left( (t_0 - t)^{-\frac{1}{2}} \right), \quad \sigma_{\text{tip}} = O \left( (t_0 - t)^{-\frac{1}{2}} \right), \quad u_{z,\text{tip}} = O \left( (t_0 - t)^{-\frac{1}{2}} \right),
\]

(6.1)

where \(\kappa_{\text{tip}}, \sigma_{\text{tip}}\) and \(u_{z,\text{tip}}\) are respectively the mean curvature, the surface charge density and the \(z\) component of velocity at the tip, and \(t_0\) is the time for which the singularity (conical tips) occurs. This asymptotic behaviour was proved in \((13)\) numerically and from dimensional considerations.

In Fig. 6 drop profiles are superimposed for several values of the angular momentum to show that the angle of the conical tips does not vary significantly with angular momentum. The reason is that centrifugal forces become subdominant with respect to surface tension at the tip. If we follow the evolution of the tangent to the interface at a point located a distance \(r\) from the rotation axis, as \(t \to t_0\), we find a clear tendency to form a conical tip whose semiangle \(\beta\) is roughly independent of \(L\) as demonstrated in Fig. 7 (\(\beta\) is measured by considering the angle at which the line that best fits the node located at the rotation axis and its neighbours intersects the vertical \(z\) axis).

7. Evolution and stability in 3D of charged rotating drops

After studying the equilibrium behaviour of axisymmetric drops, we focus now on their stability. To perform this analysis, following \((10)\), we start from an axisymmetric equilibrium configuration of a charged rotating droplet and perturb it with a combination of spherical harmonics, \(\varepsilon_1 Y_{2,0} + \varepsilon_2 Y_{2,1}\),
Singular profiles for increasing values of the angular momentum. Observe that the cone semiangle is the same for all profiles with $\epsilon_1 = \epsilon_2 = 0.05$, and find that the droplet returns to its axisymmetric configuration if Rayleigh fissibility ratio is within a certain range of values $0 \leq \chi \leq \chi_{\text{axi}}(L)$. The upper limit of this interval, $\chi_{\text{axi}}(L)$, will be established approximately using an energy argument and confirmed numerically by simulation (see Fig. 8). The theoretical derivation of this limit goes as follows: suppose that there exists an ellipsoidal family of equilibrium configurations to our problem that is energetically more favourable than the oblate spheroidal family (later we show numerically that this ellipsoidal family indeed exists). The energy of the system for oblate spheroidal solutions is given by:

$$E_{\text{oblate}}(a, Q, L) = \gamma A_{\text{oblate}}(a) + \frac{L^2}{2I_{\text{oblate}}(a)} + \frac{Q^2}{2C_{\text{oblate}}(a)},$$

where $a$ is the semimajor axis and the semiminor axis, $c$, can be obtained from the volume condition in terms of $a$. We find then $a^* \equiv a^*(Q, L)$ such that attains a minimum and consider a general
**Fig. 7** Angle formed by the tangent with respect to the $z$ axis as a function of $r$ at various times. Notice the convergence towards a semiangle at the tip of $\beta^* \approx 0.535$ rad ($= 30.6^\circ$). Discontinuous curves correspond to times $t = 0.8, 0.9, 1, 1.1, 1.2, 1.25, 1.3, 1.325, 1.33$.

**Fig. 8** Comparison of numerical results with the theoretical spheroid/ellipsoid transition curve
behaviour and in Fig. 10 we plot the four geometric configurations for a drop that is rotating with neck between two smaller droplets. In Fig. 9 we depict the bifurcation diagram that summarizes this configurations: a drop developing dynamic Taylor cones in the equatorial plane or a drop forming a.

Increasing \( \chi \), the amount of charge that a drop can hold before it becomes unstable in the presence of rotation, the amount of charge that a drop can hold before it becomes unstable is lower than Rayleigh’s limit for a non-rotating drop, getting smaller as the angular momentum increases.
Finally, it is important to remark that, when dynamic Taylor cones develop, the local form of the tip remains circularly symmetric as in the case for axisymmetric droplets. Consequently, rotation plays no main role in the geometric shaping of these kind of singularities.

8. Evolution and stability in 3D of rotating drops in electric fields

In this last section, we perform a 3D stability analysis for rotating drops at constant angular momentum immersed in an uniform electric field parallel to the rotation axis. This is done by evolving an initially spherical drop with the full 3D code and studying all different configurations attained during this process. The values for angular momentum $L$ and electric field $E_\infty$ are taken from 0 to 0.95 at 0.05 intervals and the ratio of viscosities is set to $\lambda = 0.1$. Our simulations show oblate- and prolate-like solutions (the same ones obtained with the axisymmetric model) together with spherical equilibrium shapes according to (32). A transition curve between spheroidal- and ellipsoidal-equilibrium shapes is determined together with a stability limit curve separating equilibrium configurations from Taylor cone formation (see Fig. 11). The stability curve increases at first, showing that rotating drops are more stable to higher electric field strengths, but eventually turns back on itself, implying that too much rotation penalizes the stability instead of contributing to it. Above the spheroid–ellipsoid transition curve, ellipsoidal-like equilibrium configurations appear (ellipsoids and peanut shapes, consisting of two lobes connected by a stable neck (8)) and for large values of the electric field, a two-lobed structure forms where a pinch-off occurs in finite time. All these behaviours are depicted in Fig. 12.
Fig. 10  Evolution of a charged rotating drop at $L = 0.3$ for different values of the charge. Picture 10(a) corresponds to $\chi = 0.7981$: the initial axisymmetric configuration remains stable. In 10(b), $\chi = 0.8795$: the drop has evolved to a stable ellipsoid configuration. In 10(c), $\chi = 0.9528$: the drop becomes unstable, it develops a neck between two smaller droplets and eventually breaks. Finally, for 10(d) $\chi = 1.0179$: the drop is also unstable, it develops dynamic Taylor cones in the equatorial plane. Colour gradation represents mean curvature (grey level in printed version) and the axis of rotation lies along the vertical $z$ axis.

9. Conclusions

In this article, we have studied the evolution and equilibrium shapes of conducting and viscous droplets immersed in a viscous and insulating fluid by means of the BEM. For charged and rotating drops with constant angular momentum, we present two theoretical methods to calculate stationary solutions, one based on asymptotic expansions and the other on spheroidal approximations that minimize the energy. Numerical data combined with these models shows that when rotation is present, a rupture of the bifurcation point attained by the family of charged non-rotating drops occurs. Concerning the stability of charged rotating drops, for each value of the angular momentum there exists a range of values for the charge for which solutions are stable to non-symmetric perturbations. Over this critical value of the charge, a family of ellipsoidal solutions branches and extends until a second critical value is reached, where singularity behaviours such as dynamic Taylor cones and two-lobed drop breakup appear. An important consequence that
can be drawn from simulations is that rotating drops can hold less charge (below Rayleigh’s limit) than non-rotating ones before they become unstable. When we consider conducting neutral drops rotating with constant angular momentum and subject to a uniform electric field parallel to the axis of rotation, oblate- and prolate-like spheroidal solutions are obtained. In this case, numerical experiments indicate that equilibrium solutions with the same aspect ratio are linearly related in the \((L^2, E_{\infty})\) diagram. These simulations determine a curve in the \((L, E_{\infty})\) plane defining the stability limit for equilibrium configurations which eventually turns back on itself. Our analysis concludes from this curve that rotating drops show more stability to higher values of the electric field strength, unless the angular momentum becomes very high and penalizes stability instead of contributing to it. Additionally, a transition curve between spheroidal- and ellipsoidal-like solutions is computed numerically. Finally, and for both situations described in this article, the opening semiangle obtained for dynamic Taylor cones remains the same as the ones corresponding to the system without rotation, provided that the angular momentum is small.

Many questions arise for further research concerning charged and rotating droplets under the influence of an electric field. Among them, our interest focuses on the description of the \((\chi, E_{\infty})\) diagram under the presence of the drop’s angular momentum and the role that rotation plays in the overall stability of this complex system. Another important point concerns the contribution of inertial terms when the Reynolds number is large \((Re \gg 1)\) and the way they influence evolution and stability. These issues will be discussed in future publications.
Fig. 12  Evolution of a neutral rotating drop subject to an electric field in the direction of the rotation axis.  
Fig. 12(a) corresponds to $L = 0.55$ and $E_{\infty} = 0.4$. In 12(b) the parameters are $L = 0.45$ and $E_{\infty} = 0.75$. 
Fig. 12(c) is obtained when $L = 0.75$ and $E_{\infty} = 0.3$. For $L = 0.95$ and $E_{\infty} = 0.55$ a two-lobed structure 
forms leading to a pinch-off in finite time. Finally, for 12(e) we have $L = 0.05$ and $E_{\infty} = 0.6$. Colour gradation 
represents mean curvature (grey level in printed version). The axis of rotation and the applied electric field lie 
along the vertical $z$ axis.
Acknowledgements

This work has been financially supported by the project MTM2011-26016 from the Ministerio de Ciencia e Innovación of Spain. The authors thankfully acknowledge the computer resources provided by the Centro de Supercomputación de Galicia (CESGA). We would also like to thank the anonymous referee for his/her valuable suggestions.

References


A. APPENDIX: USEFUL MATHEMATICAL FORMULAS FOR SECTION 4

Given a sphere of radius $R$, take an axially symmetric second-order perturbation of amplitude $\varepsilon$:

$$S_\varepsilon : r(\theta) = R + f(\theta) \varepsilon + g(\theta) \varepsilon^2 + O(\varepsilon^3), \quad \theta \in [0, \pi],$$

(A.1)

where:

$$f(\theta) = \sum_{l=0}^{\infty} f_{2l} Y_{2l,0}(\theta), \quad g(\theta) = \sum_{l=0}^{\infty} g_{2l} Y_{2l,0}(\theta).$$

(A.2)
are expanded in terms of spherical harmonics:

$$Y_{l,m}(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} P_l^m(\cos \theta) e^{im\phi},$$  \hspace{1cm} (A.3)

and $P_l^m$ are the associated Legendre polynomials. The volume of the perturbed shape is:

$$V(S_\epsilon) = \int_{S_\epsilon} dV = \int_0^{2\pi} \int_0^\pi \int_0^{r(\theta)} \rho^2 \sin \theta \, d\rho \, d\theta \, d\phi = \frac{1}{3} \int_0^{2\pi} \int_0^\pi r^3(\theta) \sin \theta \, d\theta \, d\phi =$$

$$\frac{4\pi}{3} R^3 + R^2 \int_0^{2\pi} \int_0^\pi f(\theta) \sin \theta \, d\theta \, d\phi + R^2 \int_0^{2\pi} \int_0^\pi \left( f^2(\theta) + Rg(\theta) \right) \sin \theta \, d\theta \, d\phi + O(\epsilon^3),$$

Substituting (A.2) and considering:

$$\sin^2 \theta = \frac{4\sqrt{\pi}}{3} \left( Y_{0,0} - \frac{\sqrt{\pi}}{5} Y_{2,0} \right),$$  \hspace{1cm} (A.4)

we get:

$$V(S_\epsilon) = \frac{4\pi}{3} R^3 + 2\sqrt{\pi} R^2 f_0 \epsilon + \left( R \sum_{l=0}^\infty f_{2l}^2 + 2\sqrt{\pi} g_0 R^2 \right) \epsilon^2 + O(\epsilon^3),$$

and thus, if we want the volume to remain constant (since the fluid forming the drop is incompressible) at first and second order we need:

$$f_0 = 0, \quad g_0 = -\frac{1}{2\sqrt{\pi} R} \sum_{l=0}^\infty f_{2l}^2.$$  \hspace{1cm} (A.5)

For the moment of inertia:

$$I(S_\epsilon) = \frac{\rho_1}{5} \int_{S_\epsilon} r^2_{\text{axis}} \, dV = \frac{\rho_1}{5} \int_0^{2\pi} \int_0^\pi \int_0^{r(\theta)} \rho^2 \sin^3 \theta \, d\rho \, d\theta \, d\phi =$$

$$= \frac{\rho_1}{5} \int_0^{2\pi} \int_0^\pi \int_0^{r(\theta)} r^3(\theta) \sin^3 \theta \, d\theta \, d\phi + \frac{8\rho_1 R^5}{15} + \rho_1 R^3 \epsilon \int_0^{2\pi} \int_0^\pi f(\theta) \sin^3 \theta \, d\theta \, d\phi +$$

$$+ \rho_1 \epsilon^2 \int_0^{2\pi} \int_0^\pi \left( 2f^2(\theta) + Rg(\theta) \right) \sin^3 \theta \, d\theta \, d\phi + O(\epsilon^3),$$

and using (A.3) this yields:

$$I(S_\epsilon) = \frac{8\rho_1 R^5}{15} - \frac{4\rho_1 R^4}{3} \sqrt{\frac{\pi}{5}} f_2 \epsilon + \frac{4\rho_1 R^3}{3} \left( \sum_{l,k=1}^\infty f_{2l} f_{2k} A_{2l,2k} - R \sqrt{\frac{\pi}{5}} g_2 \right) \epsilon^2,$$  \hspace{1cm} (A.6)

with:

$$A_{l,k} = \sqrt{(2l + 1)(2k + 1)} \left( c_{l,k,0,0}^0 \right)^2 - \frac{1}{5} \left( c_{l,k,2,0}^0 \right)^2 - \frac{\delta_{lk}}{2},$$  \hspace{1cm} (A.7)
where $e_{m_1,m_2,m}^{j_1,j_2,j}$ are the Clebsch–Gordan coefficients and $\delta_{jk}$ is the Kronecker delta symbol. The equilibrium condition requires the term $\frac{1}{2}\mathcal{L}^2$, so we expand it only to first order. Using asymptotics:

$$
\frac{1}{(a + bx)^2} \approx \frac{1}{a^2} \left(1 - \frac{2b}{a}\right), \quad x \ll 1,
$$

giving:

$$
\frac{1}{\mathcal{L}^2(S_p)} = \left(\frac{15}{8\pi \theta_1 R^5} \right)^2 \left(1 + \frac{1}{R}\sqrt{\frac{\pi}{2} f_2 e}\right) + O\left(e^2\right).
$$

The distance to the axis of rotation has the form:

$$
r_{\text{axis}}^2 = r^2(\theta) \sin^2\theta = R^2 \sin^2\theta + 2Rf \sin^2\theta + O\left(e^2\right),
$$

and thus:

$$
\frac{r_{\text{axis}}^2}{\mathcal{L}^2} = \left(\frac{15}{8\pi \sqrt{2\theta_1 R^5}} \right)^2 \left(R^2 + \sqrt{2Rf + R\sqrt{\frac{\pi}{2} f_2 e}^2}\right) \sin^2\theta + O\left(e^3\right),
$$

(A.8)

Now, to find the mean curvature and the normal derivative of the potential for a perturbation of the sphere:

$$
\mathcal{P}_\varepsilon : r(\theta, \varphi) = R + h(\theta, \varphi) \varepsilon
$$

(A.10)

we can expand them in powers of $\varepsilon$ to yield:

$$
\mathcal{H}(\mathcal{P}_\varepsilon) = \frac{1}{R} + \mathcal{L}_\mathcal{H}(h) \varepsilon + \frac{1}{2} \mathcal{Q}_{\mathcal{H}}(h, h) \varepsilon^2 + O\left(e^3\right),
$$

(A.11)

$$
\left(\frac{\partial \mathcal{V}}{\partial n}(\mathcal{P}_\varepsilon)\right)^2 = \left(\frac{Q}{4\pi \varepsilon_0 R^3}\right)^2 + \mathcal{L}_\mathcal{V}(h) \varepsilon + \frac{1}{2} \mathcal{Q}_{\mathcal{V}}(h, h) \varepsilon^2 + O\left(e^3\right),
$$

(A.12)

which $\mathcal{L}_\mathcal{H}$, $\mathcal{L}_\mathcal{V}$, and $\mathcal{Q}_{\mathcal{H}}$, $\mathcal{Q}_{\mathcal{V}}$ are linear and quadratic operators. The linear operators have the form:

$$
\mathcal{L}_\mathcal{H}(h) = -\frac{1}{R^2} (h + \frac{1}{2} \Delta_\omega h), \quad \mathcal{L}_\mathcal{V}(h) = -\frac{2}{R} \left(\frac{Q}{4\pi \varepsilon_0 R^3}\right)^2 \left(2h - \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l + 1) h_{l,m} Y_{l,m}\right),
$$

with:

$$
h(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} h_{l,m} Y_{l,m}(\theta, \varphi), \quad \Delta_\omega h = \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \left(\frac{\partial h}{\partial \theta}\right) + \frac{1}{\sin \theta} \frac{\partial^2 h}{\partial \varphi^2}.
$$

(A.13)

In addition, the quadratic operators verify, as shown, that:

$$
\mathcal{Q}_\mathcal{H}(Y_{2,0}, Y_{2,0}) = -\frac{5}{\sqrt{\pi} R^3}\left(0, 0, 2\sqrt{\frac{3}{2}} Y_{2,0} + 6\frac{3}{4} Y_{4,0}\right),
$$

(A.14)

$$
\mathcal{Q}_\mathcal{V}(Y_{2,0}, Y_{2,0}) = \frac{8 \gamma \chi}{\sqrt{\pi} \varepsilon_0 R^3}\left(-\frac{5}{2} Y_{0,0} + \frac{5}{2} Y_{2,0} - \frac{15}{7} Y_{4,0}\right).
$$

To finish, notice that the Laplace operator on the unit sphere, $\Delta_\omega$, satisfies the property:

$$
\Delta_\omega Y_{l,m} = -l(l + 1) Y_{l,m},
$$

implying that the spherical harmonics $Y_{l,m}$ are eigenfunctions of $\Delta_\omega$ with eigenvalues $-l(l + 1)$.

(A.15)