Cognitive levels and approaches taken by students failing written examinations in mathematics

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A study was conducted at the Technical University Berlin involving students who twice failed the written examination in the first semester course Linear Algebra for Engineers in order to better understand the reasons behind their failure. The study considered student understanding in terms of Bloom's taxonomy and the ways in which students approached problem solving. The results indicate that students rely on lower-order thinking processes and these processes are linked to solution approaches. Thus, by investigating solution strategies in homework sets and in classwork, an instructor can easily identify students at risk of not understanding at the appropriate level. In this contribution, the study is related to the framework set forth by the European Society for Engineering Education (SEFI) Mathematics Working Group.

1. Introduction

One of the goals of the European Commission (Europe 2020) is for at least 40% of the population between the age of 30 and 34 to complete a college education. To achieve this goal, recruiting must also be combined with developing measures to improve retention rates. The crucial phase in a student’s university experience is generally the transition from secondary to tertiary education. Especially in educational facilities offering courses with large lectures, the anonymity and lack of accountability can lead to undesirable results (see e.g. O’Shea, 2005).

The course Linear Algebra for Engineers at the Technical University (TU) Berlin serves over 3500 incoming freshmen each year. Eight lecturers (five in winter and three in summer semesters) cover the same material to provide students with the theoretical foundations. The lecturers are in general professors or post-docs, the latter of whom often have little teaching experience. The weekly 2-h lecture is supplemented by a weekly 2-h tutorial. Many of the tutors are also inexperienced pedagogically and mathematically. Students can apply to become a tutor in their third semester at the university, and the first-semester engineering mathematics courses are typically assigned to new tutors. Thus, the challenge is not only to support students but also to support instructors at different levels. Several measures, including a reorganization of the tutorials to promote more active involvement of the students as well as tutor training, took place in 2006 (see e.g. Roegner, 2008), coinciding with the introduction of...
bachelor degree programmes at the TU Berlin. Although the situation has improved since then, with success rates of 42–47% instead of 35–40%, it is far from ideal.

One of the key issues in investigating how to further improve the course lies in the area of assessment. In interviews with students who twice failed the written examination, many claimed that they invested a lot of time in studying for the tests. In fact, most of these students faithfully attended weekly office hours in preparation for the oral examination. The students are thus industrious. The question is where do things go wrong? One contributing factor to the low success rates lies in the fact that there are no lower-level courses offered (for credit) in which students can be placed according to their abilities. Not being able to change this structural condition, other factors need to be investigated to improve the overall situation further.

The results of a study (Roegner, 2012) with students in an oral examination suggest that the depth at which these students learn is insufficient at the tertiary level. Their fixation on memorization and step-by-step procedures hinders them from succeeding in university-level mathematics. In terms of the SEFI’s Framework for Mathematics Curricula in Engineering Education (SEFI MWG, 2011), which will subsequently be referred to as the Framework, the students are trapped in the Reproduction mode. The results also suggest that students’ approaches to solving problems can serve as an indicator for inappropriate depths of learning, thereby supplying tutors with methods for identifying students at risk of failing.

2. Method of Investigation

Over the course of 2 years, examination questions were formulated and tested during oral examinations to ensure that various approaches were not only possible but also taken by students. The actual study, which took place during 2009, considers the answers of 43 participants to the following two problems.

**Problem 1.**

Given is the matrix $A := \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 3 \\ 0 & -2 & 3 \end{bmatrix} \in \mathbb{R}^{3,3}$.

(a) Determine the eigenvalues and associated eigenspaces of the matrix $A$.
(b) Is the matrix $A$ diagonalizable? If so find a diagonal matrix $D$ and an invertible matrix $S$ so that $A = SDS^{-1}$. (In the course, the columns of $S$ are appropriate eigenvectors of $A$.)
(c) Is the matrix mapping $A : \mathbb{R}^3 \to \mathbb{R}^3$, $\overline{x} \mapsto A\overline{x}$ injective?

**Problem 2.**

The (ordered) basis $B := \{B_1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_2 := \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B_3 := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B_4 := \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\}$ of $\mathbb{R}^{2,2}$ and the linear mapping

$L : \mathbb{R}^{2,2} \to \mathbb{R}^{2,2}; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a + b & 0 \\ c + d & 0 \end{bmatrix}$

are given.

(a) Determine the representing matrix $L_B$ for the linear mapping $L$ with respect to the basis $B$.
(b) Is the linear mapping $L$ injective?
(c) Can you find an eigenvalue and associated eigenvector of $L$?

Each part of Problem 1 has been practised using algorithms in the course but can also be at least partially solved using other methods or chains of reasoning. In Problem 2, the first part is more or less
procedural, but is somewhat tricky for students due to the underlying vector space. The last two parts of the second problem are not practised in the course, although clearly the concepts are related to those in Problem 1.

The students were asked the problems and their sub-problems in the same order in most cases. Variations occurred when students anticipated follow-up questions or offered information relating to properties that they happened to notice along the way. In computational portions of the examination, the correct answers were agreed upon before the student went on to the next step.

The scoring chosen reflects the categories in Bloom’s hierarchical taxonomy adapted to the case of mathematics (see e.g. Riegeluth and Moore, 1999): Knowledge (K), Comprehension (C), Application of Knowledge (AK), Analysis (A), Synthesis and Evaluation. A point was awarded in the positive (respectively negative) column of a given category or subcategory for each correct (respectively incorrect) written or oral statement. Furthermore, the ways in which the students approached the problems or steps therein were taken into account. Two contrasting pairs were used for this purpose: global versus local and conceptual versus procedural. A global approach was noted when a student viewed the problem holistically, at least to some degree. A local approach was recorded when the student was so focused on the current step that they failed to draw upon other information, such as previous steps. A conceptual approach was noted when a student used a theorem instead of applying the expected algorithm. Procedural was noted when the student applied the algorithm instead of a theoretical argument that a first-semester engineering student could reasonably be expected to make and which would have made the problem easier to solve. To some extent, the recording of the strategies thus provides some measure as to the efficiency of the solution, which, as Schoenfeld (2000) has suggested, students reflect upon too little in examination situations.

In terms of the Framework, the competencies Thinking Mathematically (TM), Reasoning Mathematically (RM), Problem Solving (PS), Communication (Com), Representing Mathematical Entities (RME) and Handling Mathematical Symbols and Formalism (HMSF) are addressed. It was reasonable for students to demonstrate the levels Reproduction and Connections within each of these categories and Reflection in the categories RM, PS and RME. For the convenience of the reader, the competencies and levels are depicted in Table 1.

The students in the study were not selected rather they specifically came to the author to be examined. Although the sample is therefore not random, much can be learned by the performance of these students.

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1 As the problems were constructed for the purpose of exploring the first four categories, Synthesis and Evaluation will be henceforth ignored.
3. Student performance and findings

Before beginning with a revised summary of the results presented in Roegner (2012), it should be mentioned that AK is used a little differently than in Bloom’s original sense. The students may be able to recite or explain an algorithm, but that does not mean that they can carry out the details, even though they may have seen similar problems. Carrying out the details is for most students easier than understanding how or why the algorithm works, which is part of the Category C. For this reason, the alteration in the meaning of AK switches the ordering in the hierarchy, so that AK comes before instead of after C.

For the evaluation, students were awarded positive and negative points in the following manner. For each correct statement, the student was awarded a positive point in one or more of Bloom’s categories. For each incorrect statement, they received likewise negative points. In Table 2, the average positive and negative points earned by students are given along with the percentage of points earned for each category. Furthermore, reference values are given that were obtained by combining ‘good’ answers given by students throughout the study. For the sake of clarity, the results have been rounded to integral values. Although this sometimes skew the data a little, the trends are still clear.

The students were overall successful in demonstrating their competency in the Categories K and AK. On the average, 87% of their total positive points in Problem 1 (77% in Problem 2) were earned in these two categories (see the rows with the lighter grey background). Compared to the corresponding reference values (66% and 54%, respectively), their reliance on skills in these categories is quite high. The understanding demonstrated by the students corresponds well with the Reproduction levels of TM, PS and Com in the Framework. Only 11% and 2%, of the positive points earned in Problem 1 were in the Categories C and A, respectively (11% and 11% in Problem 2). The positive scoring demonstrated by the students in these last two categories corresponds fairly well to the Connections level of TM, RM, PS, Com and RME.

The main portion of students’ errors occurred in K (25%) and AK (38%) for Problem 1 and in C (33%) for Problem 2. Typical errors within each of these categories are given in Table 3 with the percentage of occurrence for each problem.

### Table 2. Student scoring and reference values

<table>
<thead>
<tr>
<th>Category</th>
<th>K</th>
<th>AK</th>
<th>C</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1—Average student points (+)</td>
<td>11</td>
<td>12</td>
<td>3</td>
<td>0.4</td>
</tr>
<tr>
<td>Reference values</td>
<td>17</td>
<td>11</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>% in a category (Students +)</td>
<td>42</td>
<td>45</td>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>% in a category (Reference)</td>
<td>40</td>
<td>26</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>Average student points (–)</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>% in a category (Students –)</td>
<td>25</td>
<td>38</td>
<td>25</td>
<td>13</td>
</tr>
<tr>
<td>Problem 2—Average student points (+)</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Reference values</td>
<td>4</td>
<td>8</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>% in a category (Students +)</td>
<td>22</td>
<td>55</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>% in a category (Reference)</td>
<td>18</td>
<td>36</td>
<td>9</td>
<td>23</td>
</tr>
<tr>
<td>Average student points (–)</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>% in a category (Students –)</td>
<td>17</td>
<td>17</td>
<td>33</td>
<td>33</td>
</tr>
</tbody>
</table>

2 The data are skewed a bit due to rounding errors. The students had a slightly higher percentage of errors in the Category C than in A. Answers to questions dealing with Category A were for the most part avoided by students, so that the relevance is perhaps not so great.
Students considered problems locally 1.6 times as often as globally and procedurally 10 times as often as conceptually in Problem 1, with the corresponding numbers in Problem 2 being 2 and 1.6. Students were categorized according to their favoured approaches. A student is said to favour a global approach if his ratio of global to local points was at least as high as the average student. Otherwise the student is considered as favouring a local approach. Analogously, a student is said to favour a conceptual approach if his conceptual to procedural point ratio was at least as high as the average student in the study. Otherwise the student is said to favour a procedural approach. The ratio of average positive scoring to average negative scoring within a category was calculated. Thus, a value of 1.0 indicates that the student made as many correct responses as incorrect responses. A value greater than 1.0 means that the student made more correct responses than incorrect ones and a value less than 1.0 vice versa. These ratios were calculated with respect to the solution approaches of global (g) versus local (l) and conceptual (c) versus procedural (p) and are gathered in Table 4, which requires some explanation for interpreting the results.

Consider the value 1.5 in the Category K. The plus to minus ratio of students who favoured a global approach was thus 1.5 times higher in this category than that of students who favoured a local approach. That means that the average student favouring a global approach made 1.5 times more correct responses per incorrect response than a student favouring a local approach. A value of 1.0 thus indicates that the students had similar performance with respect to the number of correct responses per error. Thus, the students favouring global (respectively conceptual) approaches outscores the students favouring local (respectively procedural) approaches in all but four instances (numbers lower than 1.0).

The best performance in Category C is by students favouring global approaches, whereas in Category A, students favouring a conceptual approach had a much higher ratio of right to wrong answers. Students favouring both global and conceptual approaches substantially outperformed the students favouring local and procedural approaches in nearly all categories in each problem.

The data were conditioned to avoid division by zero and to differentiate between students with a few or no correct responses as opposed to many incorrect responses. The correlation coefficients comparing the ratios of positive to negative scoring in each category with the ratio of global to local approaches as well as the ratio of conceptual to procedural approaches were then calculated. For \(g/l\) in the Category C, the correlation coefficient is 0.55 (p-value 0.0001) for Problem 1, 0.60 (p-value 0.0001) for Problem 2, and 0.34 (p-value .04) combined. The corresponding results for \(c/p\) in the
Category A are 0.54 (p-value 0.0002) for Problem 1, 0.64 (p-value < 0.0001) for Problem 2, and 0.53 (p-value 0.0007) when combined. The correlation between g/l and C is thus somewhat weak although significant, whereas that between c/p and A are rather strong.

### 4. Discussion and conclusions for education

The study indicates that students were unsuccessful twice in the written examination because they had not learned to learn mathematics at a deeper level. They tended to be rule learners and shied away from attempting new solution strategies unless they were heavily prompted to do so. Their main focus during the oral examination was on Reproduction skills in nearly all competencies set forth in the Framework. Most students who attempted to demonstrate competencies at the level of Connections, again generally because they were prompted, failed to do so in a satisfactory manner. Helping shallow learners transform to deep learners is, thus, a key issue for students in making the transition from school to the university. The challenges are first of all how to identify and then how to help these students.

These findings provide at least a partial answer to the first challenge. The approaches students took in solving problems were classified into two different contrasting pairs. Students taking a global or holistic approach to the problem demonstrated many more instances of positive scoring as compared to negative scoring in Bloom’s category Comprehension. The students taking a conceptual approach instead of the usual procedural approach were able to demonstrate more connections in Bloom’s category Analysis. The correlation values provided in the previous section support these findings. Thus, even for inexperienced tutors, who incidentally have the most contact with the first-semester students in the course, analysing solution approaches during class discussions or in homework can help them to identify which students may be learning at too low a level. Of course, the pilot study presented here should be extended to a more heterogeneous group of students, and different phases in the learning process need to be taken into account. Nevertheless, the results are promising. The second challenge (how to help these students once they have been identified) is an entirely different problem altogether that still requires novel ideas and much attention.

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REFERENCES


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