

# Spreading and Contact Resistance Formulae Capturing Boundary Curvature and Contact Distribution Effects

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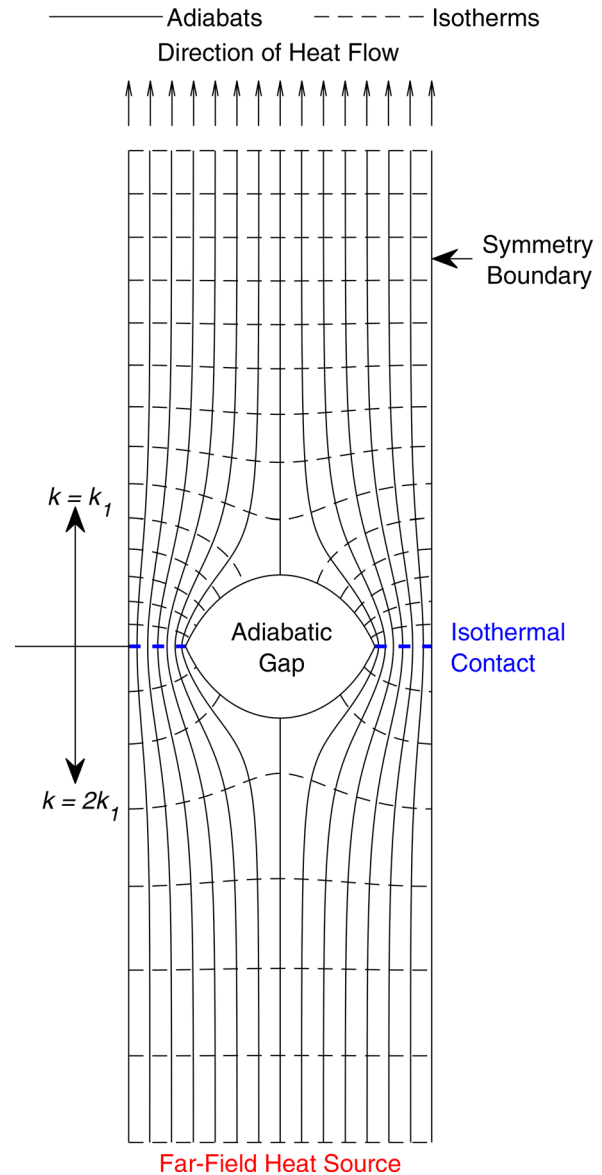
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There is a substantial and growing body of literature which solves Laplace's equation governing the velocity field for a linear-shear flow of liquid in the unwetted (Cassie) state over a superhydrophobic surface. Usually, no-slip and shear-free boundary conditions are applied at liquid–solid interfaces and liquid–gas ones (menisci), respectively. When the menisci are curved, the liquid is said to flow over a “bubble mattress.” We show that the dimensionless apparent hydrodynamic slip length available from studies of such surfaces is equivalent to (i) the dimensionless spreading resistance for a flat, isothermal heat source flanked by arc-shaped adiabatic boundaries and (ii) the dimensionless thermal contact resistance between symmetric mating surfaces with flat contacts flanked by arc-shaped adiabatic boundaries. This is important because real surfaces are rough rather than smooth. Furthermore, we demonstrate that this observation provides a significant source of new and explicit results on spreading and contact resistances. Significantly, the results presented accommodate arbitrary solid-to-solid contact fraction and arc geometry in the contact resistance problem for the first time. We also provide formulae for the case when each period window includes a finite number of no-slip (or isothermal) and shear free (or adiabatic) regions and extend them to the case when the latter are weakly curved. Finally, we discuss other areas of mathematical physics to which our results are directly relevant. [DOI: 10.1115/1.4039993]

## 1 Introduction

The phenomena of spreading (constriction) and contact resistance are illustrated in Fig. 1, a contour plot of adiabats and isotherms within contacting materials of thermal conductivities  $k_1$  and  $2k_1$  prepared using the solution by Crowdy [1] discussed below for a linear-shear flow over a superhydrophobic surface. The same rate of heat conducts between adjacent pairs of adiabats, and due to the spreading resistance, the temperature gradients are sharpest near the heat source. Often, the majority of the heat flow is through the contacts as the gaps are filled with a low conductivity gas and radiation heat transfer across them is relatively weak [2]. Then, the assumption of adiabatic gaps made here is valid. The net effect of surface roughness is a larger temperature difference required to transfer a



**Fig. 1** Adiabats and isotherms for a two-dimensional temperature field in materials of thermal conductivities  $k_1$  and  $2k_1$  assuming flat contacts and adiabatic circular arcs in nominal plane of contact as per adaptation of a  $O((1 - \phi)^4)$  solution by Crowdy [1]

specified heat rate through the materials relative to the idealized limit of flat interfaces on account of the coupled effects of a reduction of the cross-sectional area of the domain and spreading resistance. This is accounted for in the heat transfer literature by contact resistance. A comprehensive review of spreading and contact resistance was provided by Yovanovich and Marotta [2] in 2003 and subsequent work was reviewed by Razavi et al. [3] in 2016.

Cooper et al. [4] published a seminal paper on contact resistance in 1969. Therein, an estimate of the contact resistance was obtained based upon the solution for the (axisymmetric) temperature field in a cylinder, where the boundary conditions were an “almost isothermal” contact surrounded by an adiabatic annulus along its base, an adiabatic outer radius, and a uniform far-field heat flux. Contrary to the topography of rough surfaces, the analysis by Cooper et al. [4] and the vast majority of subsequent ones assume a flat plane of contact with periodic isotherms and adiabats. Notably, Das and Sadhal [5] considered the Cartesian-geometry problem for flat contacts between semi-infinite materials of different thermal conductivities adjacent to sparsely distributed,

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circular-arc geometry gaps, not assumed to be of equal protrusion angle, filled with a third material of finite thermal conductivity. Bipolar coordinates were used and further details are provided by Das [6]. Das and Sadhal [7] then used their solution to the single-gap problem as a basis for the (symmetric) multigap one to capture the interactions between gaps. To preserve analytical tractability, it was further assumed that the gaps were thin, which simplified the temperature distribution in them and their interaction. The temperature field was computed to  $O((1-\phi)^6)$ , where  $\phi$  is the contact fraction, from which the contact resistance follows, although the closed-form expression is not provided due to its length.

The principal contribution of this paper is to adapt (and in one case extend) to the problem at hand existing studies from the fluid dynamics literature for the flow of liquid in the Cassie state over ridge-type structures oriented parallel to the flow direction. The mathematical similarity of the problems means that recent developments accounting for meniscus curvature can be immediately applied here to provide the change in spreading and contact resistances relative to a planar interface. Indeed, when appropriately nondimensionalized, the spreading resistance, contact resistance, and apparent hydrodynamic slip length are identical. They depend upon two dimensionless geometric parameters for a periodic domain with one flat, isothermal (no-slip) boundary condition and one adiabatic (shear free) arc one along its base. These are contact (solid) fraction and protrusion angle ( $\alpha$ ), i.e., that tangent to the arc emanating from where it contacts the solid and defined to be positive when material is removed from the baseline Cartesian domain. Dimensionless contact resistance is further dependent on the thermal conductivity ratio for the mating materials.

This paper proceeds as follows: In Sec. 2, we provide definitions for spreading and contact resistances in domains with arc-geometry, adiabatic boundaries. In Sec. 3, we nondimensionalize the (apparent hydrodynamic) slip length for linear-shear flows over a superhydrophobic surface, spreading resistance, and contact resistance such that they are all equal to the identical function of solid (contact) fraction and protrusion angle. In Sec. 4, we provide a suite of expressions for this parameter which are collectively valid for nearly the complete range of solid fraction and protrusion angle and provide illustrative results. Section 5 considers the case when each period window contains a finite number of no-slip and shear-free slots and extends the existing literature such that the latter may be weakly curved. Here, we also provide an expression for spreading resistance when the boundary condition along the flat portion of the domain is one of constant heat flux rather than temperature. The implications of our results are discussed in Sec. 6. Conclusions are provided in Sec. 7.

## 2 Definitions of Spreading and Contact Resistances

Figure 2 shows an isothermal boundary condition imposed along all or part of the base of a semi-infinite domain. In the one-dimensional (1D) conduction case shown in Fig. 2(a), the (flux

based) thermal resistance ( $R''$ ) is given by  $L/k$ , where  $L$  is the height of the domain. It is instructive to next consider the two-dimensional problem where an adiabatic boundary condition, of width  $2a$ , is adjacent to an isothermal heat source, of width  $2(d-a)$ , along a flat domain base as per Fig. 2(b). (The heat source or adiabat may be placed centrally, as Fig. 2 may be interpreted as one period of an alternating periodic pattern.) Decomposing the temperature field into 1D and perturbative (p) components such that  $T = T_{1d} + T_p$ , the total resistance is

$$\frac{\overbrace{T_s - T_\infty}^{R''_t}}{\bar{q}''} = \frac{\overbrace{T_{s,1d} - T_{\infty,1d}}^{R''_{1d}}}{\bar{q}''} + \frac{\overbrace{T_{s,p} - T_{\infty,p}}^{R''_{sp}}}{\bar{q}''} \quad (1)$$

where the subscripts sp and s denote spreading and heat source, respectively, and  $\bar{q}'' = \phi \bar{q}''_s$ , where  $\phi = (d-a)/d$  and  $\bar{q}''_s$  is the mean heat flux leaving the source. The temperature of the source is zero; therefore, the spreading resistance is

$$R''_{sp} = -\frac{\bar{T}_{\infty,p}}{\bar{q}''} \quad (2)$$

As per the linear superposition of the one-dimensional and perturbation problems, the mean heat flux through the perturbation domain is zero. This implies that the temperature averaged over the width of the domain, denoted by  $\bar{T}$ , in the perturbation problem is constant; hence, the spreading resistance is often expressed as

$$R''_{sp} = -\frac{\bar{T}_{y=0,p}}{\bar{q}''} \quad (3)$$

Proceeding to the case of an adiabatic arc along the base of the domain as per Fig. 2(c) invalidates the foregoing linear superposition-based approach to determining the spreading resistance. However, continuing to define the one-dimensional resistance as that for a one-dimensional domain, i.e.,  $L/k$ , the spreading resistance is

$$R''_{sp} = \lim_{y \rightarrow \infty} \left( -\frac{T}{\bar{q}''} - \frac{y}{k} \right) \quad (4)$$

We note that negative protrusion angles (see Fig. 2(c)) and positive protrusion angles decrease and increase total thermal resistance relative to a flat, adiabatic boundary (see Fig. 2(b)), respectively. For example, in the case of a negative protrusion angle, the additional cross-sectional area added to the domain enables a fraction of the heat from the source to conduct along a downward path absent in the case of a flat, adiabatic interface. Insofar as contact resistance, only the solution for a positive protrusion angle is relevant as area is removed from both sides of a nominally flat interface.

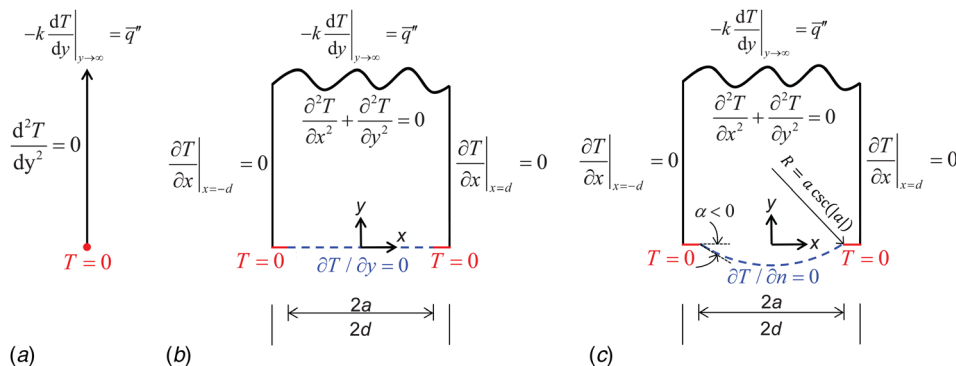
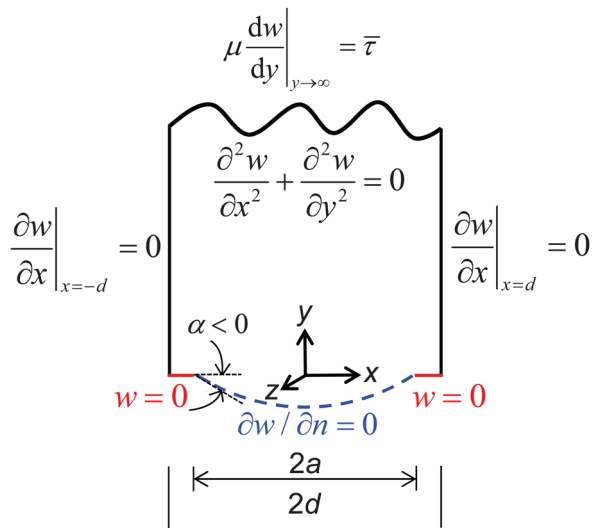


Fig. 2 Heat conduction from an isothermal, flat heat source into a semi-infinite domain: (a) one-dimensional case, (b) two-dimensional case with flat, adiabatic boundary, and (c) two-dimensional case with adiabatic-arc boundary



**Fig. 3** Linear shear flow of liquid in the Cassie state over ridges oriented parallel to the flow

Considering heat conduction from material 1 to material 2 in the representative domain shown in Fig. 1, it follows from the symmetry arguments discussed in Cooper et al. [4] and generalized by Das and Sadhal [5] that the contact regions are isothermal. Moreover, for an arbitrary number of non-uniformly spaced contacts in a period window, all the contacts may be shown to remain isothermal. Hence

$$R''_{t,1} = R''_{sp,1} + \frac{\lim_{y \rightarrow \infty}(y)}{k_1} \quad \text{material 1} \quad (5)$$

$$R''_{t,2} = \left[ R''_{sp,1} + \frac{\lim_{y \rightarrow \infty}(y)}{k_1} \right] \frac{k_1}{k_2} \quad \text{material 2} \quad (6)$$

where  $R''_{t,1} = \lim_{y \rightarrow \infty}(-T_1/\bar{q}'')$ ,  $R''_{t,2} = \lim_{y \rightarrow \infty}(T_2/\bar{q}'')$  and  $\lim_{y \rightarrow \infty}(y/k_1)$  is understood to be the (infinite) 1D resistance. The contact resistance between materials 1 and 2 becomes

$$R''_{tc} = R''_{sp,1} \left( 1 + \frac{k_1}{k_2} \right) \quad (7)$$

### 3 Conversion Between Hydrodynamic and Thermal Problems

The mathematical problem depicted in Fig. 2(c) may be considered in the context of a linear-shear flow of liquid in the Cassie-state over ridges oriented parallel to the flow as per Fig. 3. Here,  $\mu$

is the viscosity of the liquid,  $w$  is its streamwise ( $z$  direction) velocity, and  $\bar{\tau}$  is the far-field shear stress or, equivalently, that across any horizontal plane in the domain. A no-slip boundary condition is applied along the solid-liquid interfaces (ridge tips) and the (curved) meniscus is considered shear free. The apparent hydrodynamic slip length, denoted by  $b$  and having dimensions of length, is defined such that when it is multiplied by  $\bar{\tau}/\mu$ , it provides the perturbation to the far-field velocity relative to a one-dimensional flow over a flat, no slip boundary. Thus

$$b = \lim_{y \rightarrow \infty} \left( \frac{w\mu}{\bar{\tau}} - y \right) \quad (8)$$

We nondimensionalize lengths by  $d$  and velocity by  $\bar{\tau}d/\mu$  and indicate that a quantity is dimensionless by placing a tilde over it. Too, in the previously discussed thermal problems, we nondimensionalize temperature by  $\bar{q}''_d/k$ . We define the dimensionless spreading resistance as

$$\tilde{R}''_{sp} = \frac{R''_{sp}k}{d} \quad (9)$$

It then follows from Eq. (7) that

$$R''_{tc} = \frac{\tilde{R}''_{sp}d(k_1 + k_2)}{k_1k_2} \quad (10)$$

Thus, by defining the dimensionless contact resistance as

$$\tilde{R}''_{tc} = \frac{R''_{tc}k_1k_2}{d(k_1 + k_2)} \quad (11)$$

we establish that

$$\tilde{R}''_{tc} = \tilde{R}''_{sp} \quad (12)$$

The dimensionless analogs of the thermal problem shown in Fig. 2(c) and the hydrodynamic one shown in Fig. 3 are shown in Fig. 4 and, correspondingly, those of Eqs. (4) and (8) are

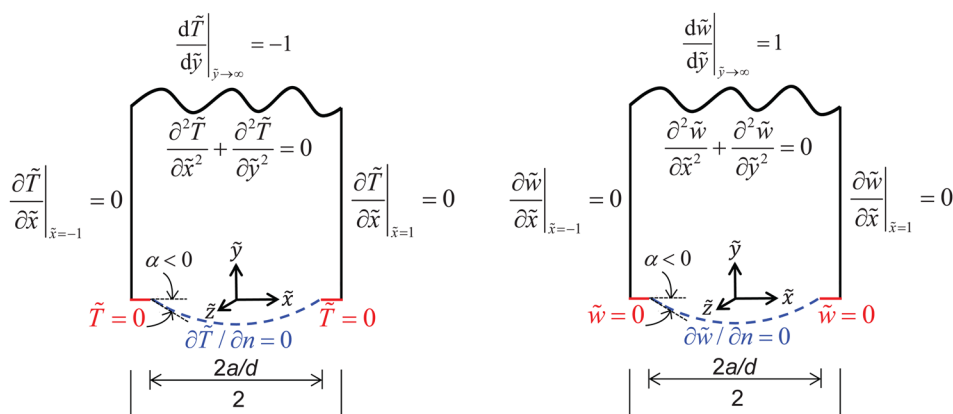
$$\tilde{R}''_{sp} = \lim_{\tilde{y} \rightarrow \infty} (-\tilde{T} - \tilde{y}) \quad (13)$$

$$\tilde{b} = \lim_{\tilde{y} \rightarrow \infty} (\tilde{w} - \tilde{y}) \quad (14)$$

Hence, as noted by Enright et al. [8] in the case of flat, adiabatic regions, since  $-\tilde{T} = \tilde{w}$ , the dimensionless spreading resistance equals the dimensionless hydrodynamic slip length. Therefore, we have established our key result that

$$\tilde{R}''_{sp} = \tilde{R}''_{tc} = \tilde{b} \quad (15)$$

We proceed to document relevant expressions from the literature on apparent slip which may be used to quantify



**Fig. 4** Dimensionless analogs of thermal problem in Fig. 2(c) and hydrodynamic problem in Fig. 3

spreading and contact resistances in domains with curved, adiabatic boundaries.

#### 4 Spreading Resistances With Curved Adiabatic Boundaries

Here, we provide a suite of expressions for  $\tilde{R}_{sp}''$  (and thus  $\tilde{R}_{tc}''$ ) based upon those for  $\tilde{b}$  compiled from the apparent slip literature. Except when solid fraction is small but  $\alpha < 0$  and curvature is not small, they essentially span the full range of  $\phi$  and  $\alpha$ . We also discuss and extend results from the apparent slip literature relevant to some additional spreading resistance problems.

**4.1 Flat Boundary Limit:  $\phi$  Arbitrary,  $\alpha = 0$ .** Utilizing a conformal map, Veziroğlu and Chandra [9] showed that, for the case of a flat isothermal contact surrounded by a flat adiabatic boundary

$$\tilde{R}_{sp}'' = \frac{2}{\pi} \ln \left\{ \sec \left[ \frac{\pi(1-\phi)}{2} \right] \right\} \quad (16)$$

The equivalent result was obtained by Philip [10] in the context of flows through porous media and cast into the form of a slip length by Lauga and Stone [11].

**4.2 Small Contact Fraction:  $\phi \ll 1$ ,  $\alpha \geq 0$ .** A small parameter in many relevant studies is the solid fraction of the ridges ( $\phi$ ) in the hydrodynamic problem or, equivalently, the contact fraction in the thermal one. Using the method of matched asymptotic expansions, Schnitzer [12] provides an expression for the slip length valid over the range  $0 \leq \alpha \leq \pi/2$  of protrusion angles in the  $\phi \rightarrow 0$  limit. The corresponding expression for the dimensionless spreading resistance is

$$\tilde{R}_{sp}'' \approx (1-\phi) \left\{ \frac{2 \cosh^{-1}[(\pi/2 - \alpha)/\sqrt{2\phi}]}{[(\pi/2 - \alpha)^2 - 2\phi]^{1/2}} - \frac{1}{\pi/2 - \alpha} \ln \left[ 2(\pi/2 - \alpha)^2 \right] + \mathcal{B}_i(\alpha) + \mathcal{B}_o(\alpha) \right\} + o(1) \quad (17)$$

where

$$\mathcal{B}_i(\alpha) = \frac{1}{\pi/2 - \alpha} \ln \frac{2^{1-2\alpha/\pi} (1/2 - \alpha/\pi) \sqrt{\pi}}{\Gamma(3/2 - \alpha/\pi) \Gamma(1 - \alpha/\pi)} \quad (18)$$

$$\mathcal{B}_o(\alpha) = \frac{1}{\pi/2 - \alpha} \ln \frac{2 \cot(\alpha) \Gamma(1 - \alpha/\pi) \Gamma(1/2 + \alpha/\pi)}{\Gamma(\alpha/\pi) \Gamma(3/2 - \alpha/\pi)} + 2 \cot(\alpha) - \frac{2}{\pi} \ln 4 + \frac{2}{\pi} \gamma_E + \frac{2}{\pi} \psi(\alpha/\pi) \quad (19)$$

where  $\Gamma$  and  $\psi$  are the gamma and digamma functions, respectively, and  $\gamma_E \approx 0.5772$  is the Euler–Mascheroni constant. As  $\phi \rightarrow 0$ , the singularity in the spreading resistance scales as  $\log(1/\phi)$  for  $0 \leq \alpha < \pi/2$  and as  $1/\sqrt{\phi}$  for  $0 < \pi/2 - \alpha = O(\sqrt{\phi})$ . There is an excellent agreement between Eq. (17) and the seminumerical results by Luca et al. [13], even for contact fractions up to 10%.

**4.3 Small Curvature:  $\phi$  Arbitrary,  $\alpha \ll 1$ .** Another relevant choice of the small parameter is the curvature of the adiabatic boundary nondimensionalized by half its planform length ( $a$ ), i.e.,

$$\tilde{\epsilon} = -\frac{\sin(\alpha)}{2(1-\phi)} \quad (20)$$

which is small when the protrusion angle  $\alpha$  is small. Then, the expression developed by Sbragaglia and Prosperetti [14] yields the first-order correction to the slip length for  $-\pi/2 \leq \alpha \leq \pi/2$  relative to that for a flat boundary as per Eq. (16) based on a study by Philip [10]. Then

$$\tilde{R}_{sp}'' \approx \frac{2}{\pi} \ln \left\{ \sec \left[ \frac{\pi(1-\phi)}{2} \right] \right\} - \tilde{\epsilon} (1-\phi)^3 \int_0^1 \frac{\{1 - \cos[(1-\phi)\pi s]\} (1-s^2) ds}{\cos[(1-\phi)\pi s] - \cos[(1-\phi)\pi]} + O(\tilde{\epsilon}^2) \quad (21)$$

Teo and Khoo [15] showed that this is accurate to within 10% for  $-40 \text{ deg} \leq \alpha \leq 30 \text{ deg}$ . Also, as discussed in more detail later, Crowdy [16] has rederived this result using very different ideas based on reciprocity arguments.

**4.4 Contact Fraction  $\geq 10\%$ ,  $\alpha$  Arbitrary.** Crowdy [1,17] provides a series of 3 (explicit) expressions in the limit of high contact fraction ( $\phi \rightarrow 1$ ) at various orders of accuracy, all valid for arbitrary protrusion angle. Equation (4.13) in Crowdy [1], an explicit result accurate to  $O((1-\phi)^7)$ , has maximum relative errors of 1–2% across the full range of protrusion angles as per a comparison with the numerical data of Teo and Khoo [15] for contact fractions as low as 25% and remains accurate to within 8–9% down to contact fractions as low as 10%. An approximation of this result which ignores certain terms of eighth order in no-shear fraction is

$$\tilde{R}_{sp}'' \approx \frac{\frac{\pi(1-\phi)^2 \Phi(\alpha)}{2} \left[ 1 + \frac{(1-\phi)^4 (\pi/2)^4 \beta(\alpha)}{15} \right]}{1 - \frac{(1-\phi)^2 (\pi/2)^2 \Phi(\alpha)}{3} - \frac{(1-\phi)^4 (\pi/2)^4 \beta(\alpha)}{15}} \quad (22)$$

where

$$\Phi(\alpha) = \frac{3\pi^2 - 4\pi\alpha + 2\alpha^2}{6(\pi - \alpha)^2} \quad (23)$$

$$\beta(\alpha) = \frac{32\alpha^4 - 128\pi\alpha^3 + 212\pi^2\alpha^2 - 168\pi^3\alpha + 45\pi^4}{360(\pi - \alpha)^4} \quad (24)$$

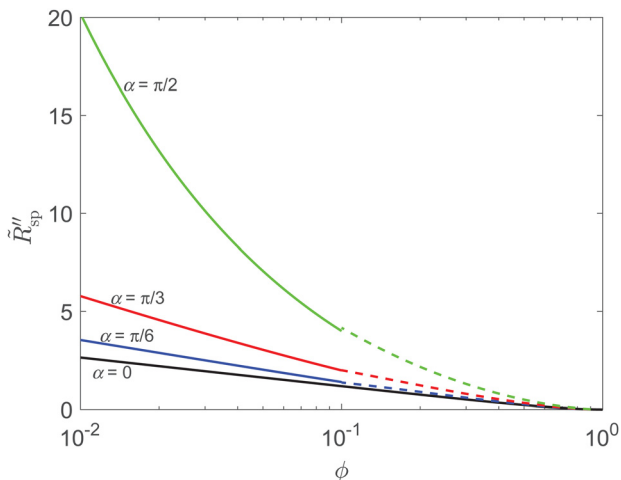
This has maximum relative error of 2% and 12% down to contact fractions as low as 25% and 10%, respectively. Thus, for engineering purposes, Eqs. (17) and (22) span nearly the full range of contact fraction and protrusion angle, and numerical calculations need not be performed. The exception is when contact fraction is small and protrusion angle is negative, which is only relevant to spreading resistance.

**4.5 Illustrative Results.** We plot  $\tilde{R}_{sp}''$  versus  $\phi$  in Fig. 5. The  $\alpha = 0$  curve is from the result by Veziroğlu and Chandra [9], Eq. (16). Those for  $\alpha = \pi/6$ ,  $\pi/3$ , and  $\pi/2$  are from the result by Schnitzer [12], Eq. (17), for  $\phi \leq 0.1$  and from Eq. (4.13) in Crowdy [1] for  $0.1 \leq \phi \leq 1$ . At  $\phi = 0.1$ ,  $\tilde{R}_{sp}''$  calculated from Eq. (4.13) in Crowdy [1] exceeds that calculated from Eq. (17) by 1.84%, 0.47%, and 5.61% at  $\alpha = \pi/6$ ,  $\pi/3$ , and  $\pi/2$ , respectively. At  $\phi = 0.01$ , the dimensionless spreading resistance is 1.40, 2.19, and 7.34 times that for a flat surface when  $\alpha = \pi/6$ ,  $\pi/3$ , and  $\pi/2$ , respectively.

#### 5 Additional Spreading Resistance Expressions

**5.1 Flat Adiabatic Regions With Arbitrary Pattern in a Period Window.** In an extension of the result for a single no-shear slot per period window discussed in Sec. 4.1, Crowdy [18] has derived analogous formulas for the case of any (finite) number of flat, no-shear slots per period. These formulas are expressed in terms of a classical special function called the Schottky–Klein prime function. They may be used to account for defects within the contact regions, where adiabatic boundary conditions apply. The two slot case is studied in detail in Ref. [18] and, in light of the analogy discussed in this paper, it is useful to record here how those results translate into explicit formulas for the spreading





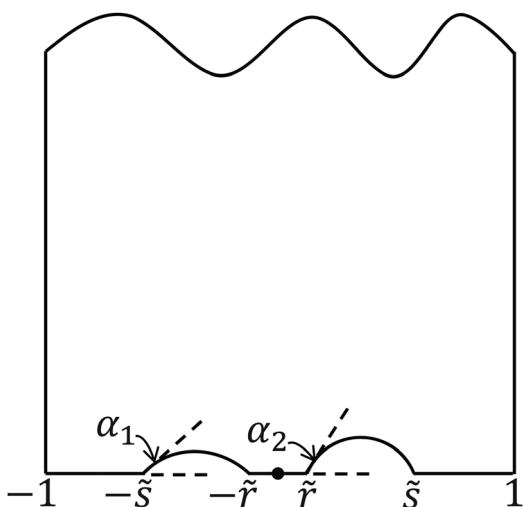
**Fig. 5**  $\tilde{R}''_{sp}$  versus  $\phi$ . The  $\alpha = 0$  curve is from the result by Veziröglu and Chandra [9], Eq. (16). Those for  $\alpha = \pi/6, \pi/3$ , and  $\pi/2$  are from the results by Schnitzer [12], Eq. (17), for  $\phi \leq 0.1$  and from Eq. (4.13) in Crowdy [1] for  $0.1 \leq \phi \leq 1$ .

resistance of a periodic surface comprising two flat adiabats per period window. We then show how to extend these results to weakly curved gaps thereby directly generalizing the results of Sec. 4.3.

It was shown in Ref. [18] that, on introduction of the special function

$$P(\zeta, \rho) \equiv (1 - \zeta) \prod_{k=1}^{\infty} (1 - \rho^{2k} \zeta)(1 - \rho^{2k} / \zeta), \quad 0 < \rho < 1 \quad (25)$$

which is easy to compute by simple truncation of this infinite product definition (note that  $\rho < 1$  so that later terms in this product for large values of  $k$  tend to unity), the problem of a linear shear flow over two flat shear-free regions may be resolved. As per Fig. 6, the shear-free regions occupy the intervals  $[-\tilde{s}, -\tilde{r}]$  and  $[\tilde{r}, \tilde{s}]$  on  $0 < \tilde{r} < \tilde{s}$ , where lengths have been nondimensionalized by half the period window. The result is that  $\tilde{w}_0(\tilde{x}, \tilde{y}) = \text{Im}[\tilde{h}(\tilde{z})]$ , where  $\tilde{w}_0$  is the streamwise velocity nondimensionalized by the shear rate times half the period window and  $\tilde{h}(\tilde{z})$ , where  $\tilde{z} = \tilde{x} + i\tilde{y}$ , is the complex potential specified henceforth.



**Fig. 6** Single-period window of surface comprising gaps between  $[-\tilde{s}, -\tilde{r}]$  and  $[\tilde{r}, \tilde{s}]$  protruding with (positive) angles  $\alpha_1$  and  $\alpha_2$ , respectively. Under the assumptions that the angles are small, the contact resistance is given, to leading order in these angles, by the explicit formulas (30)–(33).

Crowdy [18] defines

$$t = \sqrt{\rho} e^{i\theta_t} \quad (26)$$

with the two real parameters  $\rho$  and  $\theta_t$  chosen to satisfy the two nonlinear equations

$$\begin{aligned} \tilde{r} &= -\frac{i}{\pi} \log \left[ \frac{P(-\rho/t, \rho)P(-\rho t, \rho)}{P(-\rho/\bar{t}, \rho)P(-\rho\bar{t}, \rho)} \right] + \frac{\theta_t}{\pi}, \\ \tilde{s} &= -\frac{i}{\pi} \log \left[ \frac{P(\rho/t, \rho)P(\rho t, \rho)}{P(\rho/\bar{t}, \rho)P(\rho\bar{t}, \rho)} \right] + \frac{\theta_t}{\pi} \end{aligned} \quad (27)$$

These two equations, which relate the mathematical parameters  $\rho$  and  $\theta_t$  to the physical geometry of the surface, are readily solved for using Newton's method. Then, the complex potential  $\tilde{h}(\tilde{z})$  can be written parametrically in terms of the intermediate complex variable  $\zeta$  as

$$\begin{aligned} \tilde{z} &= \tilde{Z}(\zeta) \equiv -\frac{i}{\pi} \log \left[ \frac{P(\zeta/t, \rho)P(\zeta t, \rho)}{P(\zeta/\bar{t}, \rho)P(\zeta\bar{t}, \rho)} \right] + \frac{\theta_t}{\pi}, \\ \tilde{h} &= \tilde{H}(\zeta) \equiv -\frac{i}{\pi} \log \left[ \frac{P(\zeta/t, \rho)P(\zeta t, \rho)}{P(\zeta/\bar{t}, \rho)P(\zeta\bar{t}, \rho)} \right] \end{aligned} \quad (28)$$

Too, with  $\rho$  and  $\theta_t$  determined as above, the associated spreading resistance is given explicitly by the formula

$$\tilde{R}''_{sp} = \frac{2}{\pi} \log \left| \frac{P(t^2, \rho)}{P(|t|^2, \rho)} \right| \quad (29)$$

There is no need to restrict the problem to adiabats of equal length as was done here for convenience. The treatment of Ref. [18] extends to two flat adiabats of any length in each period window of the surface, as well as to any number of flat adiabats per period. A wealth of new formulas for a diverse array of surface geometries is therefore available.

Extension of these formulas to incorporate weakly curved gaps can also be achieved. Crowdy [16] has shown how the formula (21) for the first-order correction in the spreading resistance and for small protrusion angles of the gaps (when there is just one per period window) can be derived by means of reciprocal theorem arguments based on the use of Green's second integral identity; the original derivation of the result by Sbragaglia and Prosperetti used more direct arguments where those authors solved for the first-order correction to the full field. When just the correction to the slip length is of interest, the reciprocity arguments of Ref. [16] bypass any need to compute the associated field correction. It is straightforward to adapt the latter arguments to the case of two (or more) adiabats per period window, and thus, deduce integral expressions for the first-order correction to the spreading resistance when the curvature of each of the gap is assumed to be small. Those integral formulas require knowledge of the solution to the flat-adiabat problem but, as just discussed, these are available from the results of Ref. [18].

To illustrate the case of two weakly protruding gaps per period window as illustrated schematically in Fig. 6, on introduction of the two independent protrusion angles  $\alpha_1, \alpha_2 \ll 1$ , an expression for the modified spreading resistance, denoted by  $\tilde{R}''_{sp}$ , is

$$\tilde{R}''_{sp} = \tilde{R}''_{sp} + \alpha_1 \tilde{R}_1^{(1)} + \alpha_2 \tilde{R}_1^{(2)} + o(\alpha_1, \alpha_2) \quad (30)$$

where

$$\tilde{R}_1^{(1)} = \frac{1}{2} \int_{-\tilde{s}}^{-\tilde{r}} \tilde{\eta}_1(\tilde{x}) \left( \frac{\partial \tilde{w}_0}{\partial \tilde{x}} \right)^2 d\tilde{x}, \quad \tilde{R}_1^{(2)} = \frac{1}{2} \int_{\tilde{r}}^{\tilde{s}} \tilde{\eta}_2(\tilde{x}) \left( \frac{\partial \tilde{w}_0}{\partial \tilde{x}} \right)^2 d\tilde{x} \quad (31)$$

where

$$\begin{aligned}\tilde{\eta}_1(\tilde{x}) &= \frac{1}{\tilde{s} - \tilde{r}} \left( \frac{(\tilde{s} - \tilde{r})^2}{4} - \left( \tilde{x} + \frac{\tilde{r} + \tilde{s}}{2} \right)^2 \right), \\ \tilde{\eta}_2(\tilde{x}) &= \frac{1}{\tilde{s} - \tilde{r}} \left( \frac{(\tilde{s} - \tilde{r})^2}{4} - \left( \tilde{x} - \frac{\tilde{r} + \tilde{s}}{2} \right)^2 \right)\end{aligned}\quad (32)$$

and

$$\frac{\partial \tilde{w}_0}{\partial \tilde{x}} = \text{Im} \left[ \frac{d\tilde{h}}{d\tilde{z}} \right] = \text{Im} \left[ \frac{H'(\zeta)}{Z'(\zeta)} \right] \quad (33)$$

Expressions for the latter quantity are available from Eq. (28) meaning that the two integrands in Eq. (31) are completely known and simple quadrature produces the first-order modifications to the spreading resistance (if required, it is convenient to compute these integrals in the parametric  $\zeta$ -plane).

**5.2 Isoflux Heat Source.** Lam et al. [19] provide an analytical expression, given by their Eq. (36), for the dimensionless apparent thermal slip length, and therefore, spreading resistance, in the case of an isoflux ridge surrounded by an adiabatic, arc-shaped boundary. It is based on the same boundary perturbation used by Sbragaglia and Prosperetti [14] and readily converted to a dimensionless spreading resistance.

## 6 Discussion

The results for single- or periodic-idealized contacts provided here are relevant because the prediction of contact resistance between engineering surfaces, where contacts are of nonuniform size and spacing, is based upon them together with the roughness profiles of the surfaces (prior to contact) and deformation theory. Generally, the single idealized contacts are assumed to be circular, with radii specified such that they have the same area as the actual contacts and placed centrally in a cylindrical region. The effects of adiabatic, arc-shaped boundaries surrounding the contacts in this geometry may not be extracted from the existing slip literature, but we resolve them in a companion paper [20]. Key parameters appearing in theoretical expressions for  $\tilde{R}_{\text{tc}}''$  include the mean of the absolute slope of a surface as measured by, say, a stylus-type profilometer and spreading resistance. We expect them to be more accurate when the spreading resistance accounts for arc-shaped adiabatic boundaries rather than flat ones. Finally, we note that the protrusion angle in the foregoing expressions for  $\tilde{R}_{\text{sp}}''$  may be approximated as the inverse tangent of the mean of the absolute slope of a surface.

Analogies between the notion of the slip length associated with superhydrophobic surfaces and related concepts arising in other areas of mathematical physics have been pointed out. It has been shown that the longitudinal slip length associated with unidirectional patterning on a superhydrophobic surface is mathematically the same object as the so-called blockage coefficient which measures the degree to which an array of obstacles occludes an oncoming uniform potential flow [21]. (It is interesting to point out that such analogies can afford valuable theoretical insights: for example, Schnitzer [22] found this analogy with blockage coefficients useful in his asymptotic analysis of slip over a superhydrophobic surface with very small solid fraction.) Other analogies exist; for example, to the added mass of translating objects, and to the notion of impedance of an aperture in acoustics and diffraction problems [21]. The present paper shows that such analogies extend to heat transfer and thermal resistance contexts and that this cross-disciplinary realization provides new insights and immediate advantages.

## 7 Conclusions

Expressions for spreading resistance are embedded in those which predict the thermal contact resistance between mating

surfaces. In practice, outside the contacting regions of such mating surfaces, their topography is usually assumed to be flat. We enable the relaxation of this assumption over the complete range of geometrical parameters for the Cartesian problem by adapting results from the recent literature on apparent hydrodynamic slip to enable one to model such regions as adiabatic arcs of constant radius of curvature, which better resemble the actual topography. At a realistic contact fraction of 1%, the increase in resistance ranges from a factor of 1.40 to a factor of 7.34.

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## Nomenclature

- $a$  = half-width of gap (cavity) (m)
- $b$  = apparent hydrodynamic slip length (m)
- $d$  = pitch (m)
- $k$  = thermal conductivity (W/(m K))
- $L$  = length of material (m)
- $n$  = direction normal to arc (meniscus)
- $P$  = special function defined by Eq. (25)
- $q''$  = heat flux (W/m<sup>2</sup>)
- $\tilde{r}$  = dimensionless distance defined in Fig. 6
- $R''$  = flux-based thermal resistance (m<sup>2</sup> K/W)
- $\tilde{R}_{\text{sp}}'' = \tilde{R}_{\text{sp}}''$  capturing effects of 2 weakly curved gaps
- $\tilde{s}$  = dimensionless distance defined in Fig. 6
- $t$  = complex parameter defined by Eq. (26)
- $T$  = temperature (K)
- $w$  = streamwise velocity (m)
- $x$  = direction parallel to flat contacts
- $y$  = direction perpendicular to flat contacts

## Greek Symbols

- $\alpha$  = protrusion angle
- $\phi$  = solid or contact fraction
- $\theta_t$  = real parameter defined by Eq. (27)
- $\mu$  = viscosity (kg/(m s))
- $\rho$  = real parameter defined by Eq. (27)
- $\bar{\tau}$  = mean shear stress
- $\zeta$  = intermediate complex parameter

## Subscripts

- p = perturbative
- sp = spreading
- t = total
- tc = thermal contact
- 1 = material 1
- 2 = material 2

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