Limiting dynamics for a rotating system

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The aim of this paper is to explore the asymptotic properties of the velocity of motion described by a second-order differential equation with periodic damping, \( \alpha + Q(\omega t) \). The asymptotic behavior of the time-averaged velocity is found as \( \omega \) goes to zero and as \( \omega \) goes to infinity. These limits represent slow and fast rotation limits of the average velocity of a rotating object whose motion satisfies such a differential equation. It is shown that the time average converges to an asymptotic ensemble average over orientations. So, the result can be considered an ergodic-type theorem.

**Keywords:** ergodic; periodic damping; differential equations; rotating system.

1. Introduction

The results in this paper pertain to motion with two degrees of freedom rotational and translational described by the differential equation

\[
\frac{d^2 y}{dt^2} + \left( \alpha + Q(\omega t) \right) \frac{dy}{dt} = F.
\]

Here \( Q(t) \) represents a continuous periodic function assumed to have period \( 2\pi \), \( \omega \) represents the angular frequency of the motion, \( \alpha \) a constant and \( F \) represents the magnitude of an external force acting on the system. The term in brackets has a damping effect on the motion. Lacking a restoring term, rotations modeled by (1.1) are free and non-spring-like, such as, settling/rotation motion (see below), Brownian rotations and vortex-induced vibrations; see Govardhan & Williamson (2002) and Aref & Balachandar (2017). Agarwal et al. (2002) is good reference on oscillatory equations.

Let \( v_\omega(t) = \frac{dy}{dt} \), so that

\[
\frac{dv_\omega}{dt} + \left[ \alpha + Q(\omega t) \right] v_\omega = F.
\]

Our main result is as follows:
THEOREM 1.1  For the differential equation (1.2), let \( Q(t) \) be a continuous periodic function of period \( 2\pi \) on \([0, \infty)\). For some constant \( M, 0 < M < \alpha \), let \( Q(t) \) satisfy the assumptions

A-1  \( \int_s^t Q(\omega \xi) d\xi < M(t - s), \omega \geq 0, 0 \leq s < t \)

A-2  \( \alpha + Q(\omega t) > 0, \omega \geq 0, 0 \leq t < \infty \).

Let the time average of \( v_\omega(t) \) be denoted by

\[
\langle v_\omega \rangle_T = \frac{1}{T} \int_0^T v_\omega(t) dt
\]

(1.3)

then \( \langle v_\omega \rangle_T \) satisfies

\[
\lim_{\omega \to 0} \left( \lim_{T \to \infty} \langle v_\omega \rangle_T \right) = F \int_0^{2\pi} \frac{1}{\alpha + Q(\theta)} \frac{d\theta}{2\pi} = F \int_0^{2\pi} \frac{1}{\alpha + Q(\theta)} \frac{d\theta}{2\pi}^{-1} = \frac{F}{Q_h},
\]

(1.4)

\[
\lim_{\omega \to \infty} \left( \lim_{T \to \infty} \langle v_\omega \rangle_T \right) = F \int_0^{2\pi} (\alpha + Q(\theta)) \frac{d\theta}{2\pi} = \frac{F}{Q_{av}}.
\]

(1.5)

Interestingly, two different ensemble averages occur, \( Q_h \) and \( Q_{av} \). In (1.4), \( F \) is divided by the phase averaged harmonic mean, and in (1.5), \( F \) is divided by the phase averaged arithmetic mean. The physical representation of the phase in this case is the orientation angle \( \theta \) for non-spherical particles. Since the friction coefficient depends on the orientation, when the rotation is slow the system has time to relax between rotations and one gets one limit and when the rotation is very fast the system doesn’t relax and another limit results. This an ergodic like result, because time averages on the left are related to ensemble averages on the right. Setting the first term in (1.2) to zero, we obtain

\[
v_\omega(t) = \frac{F}{\alpha + Q(\omega t)}.
\]

(1.6)

If the rotation velocity is small compared to the exponential damping term, the instantaneous velocity of the object is approximately equal to (1.6).

The application that led us to this differential equation is the study of the mobility of non-spherical aerosol particles. A widely used method for characterizing the size of aerosol particles is based on measurement of their drift velocity under an external force such as an electrical field or gravitational force. For spherical particles, the Stokes–Einstein expression together with the Cunningham slip correction (Friedlander & Smoke, 2000; Hinds, 1999) allows one to predict the drift velocity from knowing the particle diameter. There is an increasing interest in studying the properties of non-spherical particles such as nanorods and clusters of spherical particles both for commercial application and health effects. In the case of non-spherical non-skew particles such as cylinders and prolate spheroids, Happel & Brenner (1983) derived an expression for the orientation averaged drift velocity in terms of the harmonic mean of the three eigenvalues of the friction tensor for the particle.

\[
\langle \vec{v} \rangle = \frac{1}{K} \hat{F} = \frac{1}{3} \left( \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3} \right) F \hat{k}.
\]

(1.7)
\( \hat{k} \) is the unit vector in the \( z \) direction and \( F \) is the magnitude of \( \vec{F} \). For a non-skew particle the coupling tensor between the translational and rotational motions is zero (Happel & Brenner, 1983). As pointed out by Li et al. (2014), (1.7) is not valid if the particle is rotating rapidly compared to the aerosol relaxation time. For this case the drift velocity is nearly independent of the orientation because there is not time for the drift velocity to adjust to the orientation dependent friction coefficient. Li et al. (2014) derived the following expression for the drift velocity in the limit that the rotation time is much smaller than the aerosol relaxation time:

\[
\langle \vec{v} \rangle = \frac{\vec{F}}{(K)} = \frac{1}{(1/3)(K_1 + K_2 + K_3)} F\hat{k}. \tag{1.8}
\]

Li et al. (2014) also showed that for nanorods in the free molecular limit, the expression for \( \langle \vec{v} \rangle \) in (1.8) is equivalent to the expression computed using the orientation averaged collision integral based on hard sphere collisions for randomly oriented ions. The analysis of Happel and Brenner does not include the inertia term, the first term in (1.2). When the Happel–Brenner expression is used, there is an implied assumption that the rotation time scale is much slower than the aerosol relaxation time given by \( \tau = 1/(\alpha + Q(\omega t)) \) in (1.2). The inclusion of the inertia term in (1.2) allows us to use one model equation to explain both the Happel and Brenner and Liu methods, and in the process study the effect of the rotation frequency on the drift velocity. Mulholland et al. (2016) proposed a specific version of this model equation including an inertial term and a rotation term with the periodic function \( Q(\omega t) = K \cos^2(\omega t) \) for a constant \( K \). The numerical solutions of the average velocity at high frequency and low frequency approached the arithmetic average and the inverse of the harmonic mean given by

\[
\langle v \rangle_{\text{slow}} = F / \left[ \int_0^{2\pi} \frac{1}{\alpha + Q(\theta)} \frac{d\theta}{2\pi} \right] = \frac{F}{Q_h}, \tag{1.9}
\]

\[
\langle v \rangle_{\text{fast}} = F / \left[ \int_0^{2\pi} \alpha + Q(\theta) \frac{d\theta}{2\pi} \right] = \frac{F}{Q_a}. \tag{1.10}
\]

Theorem 1.1 states that these limits hold not only for \( Q(t) = K \cos^2(t) \) but also more generally for any periodic function satisfying the conditions of the theorem. The remainder of the paper is focused on proving Theorem 1.1. In the next section the asymptotic time average is determined and in Section 3 the asymptotics in \( \omega \) are proved.

2. Asymptotic time average

Several results about time averages from the theory of \((C, 1)\) summability of improper integrals are needed, see Zygmund (2002). Let \( f \) be a continuous real-valued function

**P-1** If \( \lim_{t \to \infty} f(t) = c \), then \( \lim_{t \to \infty} \frac{1}{t} \int_0^t f(s) ds = c \).

**P-2** If \( f \) is periodic of period \( 2\pi \), then \( \lim_{t \to \infty} \frac{1}{t} \int_0^t f(s) ds = \frac{1}{2\pi} \int_0^{2\pi} f(s) ds \).

It is straightforward to show that the solution to the first-order differential equation (1.2) is

\[
v_\omega(t) = v_0 e^{-\alpha t - \int_0^t Q(\omega \xi) d\xi} + Fe^{-\alpha t - \int_0^t Q(\omega \xi) d\xi} \int_0^t e^{\alpha s + \int_0^s Q(\omega \xi) d\xi} ds = v_0 v_{\omega,1}(t) + F v_{\omega,2}(t) \tag{2.1}
\]
where \( \exp\{\int_0^t (\alpha + Q(\omega \xi)) d\xi \} \) is an integrating factor and \( v_0 = v_\omega(0) \). The time average of \( v_\omega(t) \) is

\[
\langle v_\omega \rangle_T = \frac{v_0}{T} \int_0^T e^{-\alpha t} - \int_0^t Q(\omega \xi) d\xi \, dt + \frac{F}{T} \int_0^T e^{-\alpha t} - \int_0^t Q(\omega \xi) d\xi \int_0^t e^{\alpha s} + \int_0^t Q(\omega \xi) d\xi \, ds \, dt. \tag{2.2}
\]

In view of assumption A-1, \( e^{-\alpha t} - \int_0^t Q(\omega \xi) d\xi \) converges to zero as \( t \to \infty \), and by property P-1, this results in the first term, \( v_0 \langle v_\omega,1 \rangle_T \), in (2.2) converging to zero as \( T \to \infty \).

By application of A-1, the time average in the second term in (2.2)

\[
\langle v_{\omega,2} \rangle_T = \frac{1}{T} \int_0^T v_{\omega,2}(t) \, dt = \frac{1}{T} \int_0^T \int_0^t e^{-\alpha (t-s)} - \int_s^t Q(\omega \xi) \, d\xi \, ds \, dt \tag{2.3}
\]

satisfies the inequality

\[
\langle v_{\omega,2} \rangle_T \leq \frac{1}{T} \int_0^T \int_0^t e^{-\alpha (t-s)} \, ds \, dt = \frac{1}{\alpha - M} \frac{1 - e^{-(\alpha - M)T}}{(\alpha - M)^2} \frac{1}{T} \tag{2.4}
\]

and by the comparison test converges as \( T \) approaches infinity. It converges to this same limit if \( T \) approaches infinity along any subsequence going to infinity. Take the subsequence \( T_N = 2\pi N/\omega, N = 1, 2, \ldots \), and consider

\[
\langle v_{\omega,2} \rangle_{T_N} = \frac{1}{2\pi N/\omega} \int_0^{2\pi N/\omega} v_{\omega,2}(t) \, dt = \frac{\omega}{2\pi N} \sum_{k=0}^{N-1} \int_{2\pi k/\omega}^{2\pi (k+1)/\omega} v_{\omega,2}(t) \, dt
\]

\[
= \frac{\omega}{2\pi N} \sum_{k=0}^{N-1} \int_0^{2\pi/\omega} v_{\omega,2}(t + 2k\pi/\omega) \, dt. \tag{2.5}
\]

The integrand in (2.5) can be expanded as follows. Substituting the expression for \( v_{\omega,2}(t + 2k\pi/\omega) \) and by a change of variables \( \mu = s - 2\pi k/\omega \), for \( k \geq 0, t \geq 0 \),

\[
v_{\omega,2}(t + 2k\pi/\omega) = \int_0^{t+2\pi k/\omega} e^{-\alpha (t+2\pi k/\omega - s)} - \int_s^{t+2\pi k/\omega} Q(\omega \xi) \, d\xi \, ds \tag{2.6}
\]

\[
= \int_{-2\pi k/\omega}^t e^{-\alpha (t+2\pi k/\omega - u)} - \int_u^{t+2\pi k/\omega} Q(\omega \xi) \, d\xi \, du \tag{2.7}
\]

\[
= \int_{-2\pi k/\omega}^t e^{-\alpha (t-u)} - \int_u^{t+2\pi k/\omega} Q(\omega (\xi + 2\pi k/\omega)) \, d\xi \, du = \int_{-2\pi k/\omega}^t e^{-\alpha (t-u)} - \int_u^{t+2\pi k/\omega} Q(\omega (\xi)) \, d\xi \, du \tag{2.8}
\]

\[
= \int_0^t e^{-\alpha (t-u)} - \int_u^{t+2\pi k/\omega} Q(\omega \xi) \, d\xi \, du + \int_{-2\pi k/\omega}^0 e^{-\alpha (t-u)} - \int_u^{t+2\pi k/\omega} Q(\omega \xi) \, d\xi \, du \tag{2.9}
\]

\[
= v_{\omega,2}(t) + f_k(t) \tag{2.10}
\]
where \( f_k(t) = e^{-\alpha t} \int_0^t e^{\alpha s} Q(\omega \xi) \, d\xi \) ds. These calculations provide additional insight into the properties of the solution of (1.2). They show, for example, the solution \( v_{\omega,1}(t) \) is quasi-periodic. Here we are calling a function \( f \) quasi-periodic if \( f(z + \theta) = g(z, f(z)) \) where \( g \) has a simple expression.

Substituting this expression for \( v_{\omega,2}(t + 2\pi k/\omega) \) into (2.5) gives

\[
(v_{\omega,2})_{TN} = \frac{\omega}{2\pi N} \sum_{k=0}^{N-1} \left( \int_0^{2\pi/\omega} v_{\omega,2}(t) \, dt + \frac{\omega}{2\pi N} \int_0^{2\pi/\omega} \sum_{k=0}^{N-1} f_k(t) \, dt \right)
\]

\[
= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} v_{\omega,2}(t) \, dt + \frac{\omega}{2\pi N} \sum_{k=0}^{N-1} e^{-\alpha t} \int_0^{2\pi/\omega} e^{-\alpha t} Q(\omega \xi) \, d\xi \, ds \, dt.
\]

Applying P-1 to the term in (2.13), as \( N \to \infty \), we get

\[
\lim_{N \to \infty} (v_{\omega,2})_{TN} = (v_{\omega,2})_{\infty}
\]

where

\[
(v_{\omega,2})_{\infty} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} v_{\omega,2}(t) \, dt + \frac{\omega}{2\pi} \left( \int_0^{2\pi/\omega} e^{-\alpha t} Q(\omega \xi) \, d\xi \, ds \, dt \right)
\]

\[
= (v_{\omega,2,1})_{\infty} + (v_{\omega,2,2})_{\infty}.
\]

Above it was shown that \( v_{\omega,1}(t) \) in (2.1) satisfies \( \langle v_{\omega,1} \rangle_{\infty} = 0 \); therefore,

\[
\langle v_{\omega} \rangle_{\infty} = F \lim_{T \to \infty} (v_{\omega})_{T} = F(v_{\omega,2,1})_{\infty} + F(v_{\omega,2,2})_{\infty}.
\]

With this partition of \( \langle v_{\omega} \rangle_{\infty} \), the asymptotics in \( \omega \) can be derived. The asymptotic value of the first component as \( \omega \to 0 \) gives the harmonic mean part of the theorem in (1.4), and the asymptotic value of second component as \( \omega \to \infty \) gives the arithmetic mean part of the theorem in (1.5). Equation (2.15) can also be used to compute the long time average velocity as a function of \( \omega \).

3. Asymptotics in \( \omega \)

Consider first the asymptotics of \( \langle v_{\omega} \rangle_{\infty} = F(v_{\omega,2,1})_{\infty} + F(v_{\omega,2,2})_{\infty} \) as \( \omega \) approaches zero. In the limit as \( \omega \) approaches zero, \( \langle v_{\omega,2,2} \rangle_{\infty} = 0 \) because

\[
\langle v_{\omega,2,2} \rangle_{\infty} = \frac{\omega}{2\pi} \left( \int_0^{2\pi/\omega} e^{-\alpha t} - \int_0^t e^{-\alpha(s-t)} Q(\omega \xi) \, d\xi \right) \, ds \, dt \leq \frac{\omega}{2\pi} \int_0^{2\pi/\omega} e^{-(\alpha M)(t-s)} \, ds \, dt
\]

\[
= \frac{\omega}{2\pi} \frac{1 - e^{-(\alpha M) 2\pi/\omega}}{(\alpha - M)^2} \to 0 \quad \omega \to 0.
\]

By several variable changes (four) all of the form \( (z = \omega u) \) we get

\[
\langle v_{\omega,2,1} \rangle_{\infty} = \frac{1}{2\pi \omega} \int_0^{2\pi/\omega} \int_0^t e^{-\frac{1}{\omega}(\alpha t + \int_0^t Q(\xi) \, d\xi) - \alpha s + \int_0^s Q(\xi) \, d\xi} \, ds \, dt = \frac{1}{2\pi \omega} \int_\Omega e^{-\frac{1}{\omega} f(s,t)} \, ds \, dt
\]
where $\Omega$ is the region bounded by triangle $\Gamma$ with vertices $(0, 0)$, $(0, 2\pi)$ and $(2\pi, 2\pi)$, see Fig. 1. The asymptotic value of

$$I_\lambda = \lambda \int_\Omega e^{-\lambda f(s,t)} \, ds \, dt \quad \left( \lambda = \frac{1}{\omega} \right)$$

is determined using a 2D integration by parts. This technique is common in the theory of computing asymptotic approximations of integrals, see Wong (2014). The underlying principle is the divergence theorem. The steps are added below for clarity.

By A-2, $\nabla f \neq 0$, therefore define a vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$F(s,t) = \left( \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right).$$

Take $\varphi(s,t) = e^{-\lambda f(s,t)}$ and apply the product rule for divergence operators to get

$$\nabla \cdot (e^{-\lambda f(s,t)} F) = (\nabla e^{-\lambda f(s,t)}) \cdot F + e^{-\lambda f(s,t)} \nabla \cdot F$$

$$= -\left( \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right) \lambda e^{-\lambda f(s,t)} \cdot F + e^{-\lambda f(s,t)} \nabla \cdot F.$$ (3.7)

Substituting the value of $F$ from (3.5) into (3.7),

$$\nabla \cdot (e^{-\lambda f(x,y)} F) = -\left( \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right) \lambda e^{-\lambda f(s,t)} \cdot \left( \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right) + e^{-\lambda f(x,y)} \nabla \cdot F$$

$$= -\lambda e^{-\lambda f(s,t)} + e^{-\lambda f(s,t)} \nabla \cdot F.$$ (3.8)

Or equivalently,

$$\lambda e^{-\lambda f(s,t)} = e^{-\lambda f(s,t)} \nabla \cdot F - \nabla \cdot (e^{-\lambda f(s,t)} F).$$ (3.10)

Integrating both sides gives

$$I_\lambda = \iint_\Omega e^{-\lambda f(s,t)} \nabla \cdot F \, ds \, dt - \iint_\Omega \nabla \cdot (e^{-\lambda f(s,t)} F) \, ds \, dt.$$ (3.11)

Applying the divergence theorem to the second term in (3.11)

$$\iint_\Omega \nabla \cdot (e^{-\lambda f(s,t)} F) \, ds \, dt = \int_\Gamma (e^{-\lambda f(s,t)} F) \cdot n \, d\sigma = \int_\Gamma e^{-\lambda f(s,t)} (F \cdot n) \, d\sigma.$$ (3.12)
So

\[ I_\lambda = - \int_\Gamma e^{-\lambda f(s,t)} (F \cdot n) \, ds \, dt + \frac{1}{\lambda} \int_\Omega e^{-\lambda f(s,t)} \nabla \cdot F \, ds \, dt \]  \tag{3.13}

where \( \Gamma \) is oriented in the counterclockwise direction, \( \sigma \) represents arc length along \( \Gamma \) and \( n \) is the unit outward normal to \( \Gamma \).

Repeat these steps on the second term in (3.13) to get the second-order term in the expansion of \( I_\lambda \). Let \( J_\lambda \) represent the second term

\[ J_\lambda = \int_\Gamma e^{-\lambda f(s,t)} (F_1 \cdot n) \, ds \, dt \]  \tag{3.14}

and let the vector field \( F \) in (3.5) be replaced by the vector field

\[ F_1(s,t) = \left( \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right) \frac{\nabla f}{|\nabla f|^2} g(s,t) \]  \tag{3.15}

where \( g(s,t) = \nabla \cdot F \). Repeating steps (3.6)–(3.13) gives

\[ J_\lambda = -\frac{1}{\lambda} \int_\Gamma e^{-\lambda f(s,t)} (F_1 \cdot n) \, ds \, dt + \frac{1}{\lambda} \int_\Omega e^{-\lambda f(s,t)} \nabla \cdot F \, ds \, dt. \]  \tag{3.16}

The following second-order asymptotic expansion results

\[ I_\lambda = -\int_\Gamma e^{-\lambda f(s,t)} (F \cdot n) \, ds \, dt - \frac{1}{\lambda} \int_\Gamma e^{-\lambda f(s,t)} (F_1 \cdot n) \, ds \, dt + \frac{1}{\lambda} \int_\Omega e^{-\lambda f(s,t)} \nabla \cdot F_1 \, ds \, dt. \]  \tag{3.17}

The first term on the right side of (3.17) is the limit of \( \langle v_\omega,2,1 \rangle_\infty \) as \( \omega \) goes to zero or equivalently as \( \lambda = 1/\omega \) goes to infinity. To evaluate this term, partition \( \Gamma \) into \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) as shown in Fig. 1.

It is evaluated as follows

\[ \int_\Gamma e^{-\lambda f(s,t)} (F \cdot n) \, ds = \sum_{i=1}^{3} \int_{\Gamma_i} e^{-\lambda f(s,t)} (F_1 \cdot n) \, ds = (I_1 + I_2 + I_3) \]  \tag{3.18}

where \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \). Each component represents a parameterized contour of the form, \( \sigma(\gamma) = (s(\gamma), t(\gamma)), 0 \leq \gamma \leq 2\pi \). Parametric expression for the components are

\[ \begin{align*}
\Gamma_1 & : s(\gamma) = \gamma & t(\gamma) = \gamma & n = (1, -1)/\sqrt{2} \\
\Gamma_2 & : s(\gamma) = 2\pi - \gamma & t(\gamma) = 2\pi & n = (0, 1) \\
\Gamma_3 & : s(\gamma) = 0 & t(\gamma) = 2\pi - \gamma & n = (-1, 0).
\end{align*} \]
Recall
\[ f(s, t) = \alpha t + \int_0^t Q(\xi) \, d\xi - \alpha s - \int_0^s Q(\xi) \, d\xi. \]  \hspace{1cm} (3.19)

Along \( \Gamma_1 \)
\[ f(s, t) = 0 \]  \hspace{1cm} (3.20)
\[ \nabla f = (-\alpha - Q(\gamma), \alpha + Q(\gamma)) = (\alpha + Q(\gamma))(1, -1)/\sqrt{2} \]  \hspace{1cm} (3.21)
\[ |\nabla f|^2 = 2(\alpha + Q(\gamma))^2 \]  \hspace{1cm} (3.22)
\[ F = \frac{\nabla f}{|\nabla f|^2} = \frac{(-1, 1)/\sqrt{2}}{2(\alpha + Q(\gamma))} \]  \hspace{1cm} (3.23)
\[ F \cdot n = \frac{(-1, 1)/\sqrt{2}}{2(\alpha + Q(\gamma))^2} \cdot (1, -1)/\sqrt{2} = -\frac{1}{\alpha + Q(\gamma)}. \]  \hspace{1cm} (3.24)

and
\[ I_1 = \int_{\Gamma_1} e^{-\lambda f(s, t)} F \cdot n \, d\sigma = -\int_0^{2\pi} \frac{dy}{\alpha + Q(\gamma)}. \]  \hspace{1cm} (3.25)

Along \( \Gamma_2, s = 2\pi - \gamma \)
\[ f(s, t) = 2\pi \alpha + \int_0^{2\pi} Q(s) \, ds - \alpha(2\pi - \gamma) - \int_0^{2\pi - \gamma} Q(\xi) \, d\xi \]  \hspace{1cm} (3.26)
\[ \nabla f(s, t) = (Q(2\pi - \gamma), 0) = Q(2\pi - \gamma)(1, 0) \]  \hspace{1cm} (3.27)
\[ |\nabla f|^2 = Q^2(2\pi - \gamma). \]  \hspace{1cm} (3.28)
One can show along $\Gamma_2$, $F \cdot n = 0$. Similarly along $\Gamma_3$, $F \cdot n = 0$. Therefore,

$$\lim_{\omega \to 0} \langle v_{\omega,2,1} \rangle_\infty = \int_0^{2\pi} \frac{1}{\alpha + Q(\theta)} \frac{d\theta}{2\pi}.$$  \hfill (3.29)

This expression with (3.2) gives

$$\lim_{\omega \to 0} \langle v_{\omega} \rangle_\infty = F \int_0^{2\pi} \frac{1}{\alpha + Q(\theta)} \frac{d\theta}{2\pi}. \hfill (3.30)\]$$

Consider the asymptotics as $\omega$ approaches infinity. Using the expression in (3.2) for $\langle v_{\omega,2,1} \rangle_\infty$ it is easily seen that $\langle v_{\omega,2,1} \rangle_\infty$ approaches zero as $\omega$ goes to infinity. Using (2.14)

$$\langle v_{\omega,2,2} \rangle_\infty = \left( \frac{\omega}{2\pi} \int_0^{2\pi/\omega} e^{-\alpha t - \int_0^\xi Q(\omega \xi) \, d\xi} \, dt \right) \left( \int_{-\infty}^0 e^{\alpha s + \int_0^s Q(\omega \xi) \, d\xi} \, ds \right) \hfill (3.31)$$

The first factor converges to 1 by L'Hospitals rule. The second factor may be written as

$$\int_{-\infty}^0 e^{\alpha s + \int_0^s Q(\omega \xi) \, d\xi} \, ds = \int_{-\infty}^0 e^{(\alpha + (1/\omega s)) \int_0^s Q(\xi) \, d\xi} \, ds. \hfill (3.32)$$

Since $Q(\xi)$ is periodic, it follows from P-2 that for fixed $s$

$$\frac{1}{\omega s} \int_0^{\omega s} Q(\xi) \, d\xi \to \frac{1}{2\pi} \int_0^{2\pi} Q(\xi) \, d\xi$$  \hfill (3.33)

as $\omega \to \infty$. Interchanging limit and integral

$$\lim_{\omega \to \infty} \int_{-\infty}^0 e^{\alpha s + \int_0^s Q(\omega \xi) \, d\xi} \, ds = \int_{-\infty}^0 \lim_{\omega \to \infty} e^{(\alpha + (1/\omega s)) \int_0^s Q(\xi) \, d\xi} \, ds = \frac{1}{\int_0^{2\pi} (\alpha + Q(\xi)) \frac{d\xi}{2\pi}}. \hfill (3.34)$$

Therefore,

$$\lim_{\omega \to \infty} \langle v_{\omega,2,2} \rangle_\infty = \int_{-\infty}^0 e^{(\alpha + (1/2\pi)) \int_0^{2\pi} Q(\xi) \, d\xi} \, ds = \frac{1}{\int_0^{2\pi} (\alpha + Q(\xi)) \frac{d\xi}{2\pi}} \hfill (3.35)$$

and thus

$$\lim_{\omega \to \infty} \langle v_{\omega} \rangle_\infty = \frac{F}{\int_0^{2\pi} (\alpha + Q(\xi)) \frac{d\xi}{2\pi}} \hfill (3.36)$$

completing the proof of the Theorem.
References


