SYMPOSIUM

Exchange of Tears under a Contact Lens Is Driven by Distortions of the Contact Lens

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Synopsis
We studied the flow of the post-lens tear film under a soft contact lens to understand how the design parameters of contact lenses can affect ocular health. When a soft contact lens is inserted, the blinking eyelid causes the lens to stretch in order to conform to the shape of the eye. The deformed contact lens acts to assume its un-deformed shape and thus generates a suction pressure in the post-lens tear film. In consequence, the post-lens tear fluid moves; it responds to the suction pressure. The suction pressure may draw in fresh fluid from the edge of the lens, or it may eject fluid there, as the lens reassumes its un-deformed shape. In this article, we develop a mathematical model of the flow of the post-lens tear fluid in response to the mechanical suction pressure of a deformed contact lens. We predict the amount of exchange of fluid exchange under a contact lens and we explore the influence of the eye’s shape on the rate of exchange of fluid.

Introduction
The development of soft contact lenses revolutionized the contact lens industry. Today approximately 35 million Americans wear contact lenses daily. The introduction of contact lenses made of sophisticated soft hydrogel material improved corneal oxygenation, overcoming hypoxia and allowing for extended, comfortable wear. Even with these improvements, complications remain. For example, those who wear contact lenses have more ocular problems than those who don’t (Craig et al. 2013). The most severe complication from wearing contact lenses is corneal infection, with the most visually devastating being bacterial keratitis (Evans et al. 2007; Robertson and Cavanagh 2008).

In situ, a contact lens is completely immersed in tears (Fig. 1). The contact lens is separated from the eye’s surface by a film of tears 2–3 μm thick called the post-lens film (Nichols and King-Smith 2003), and is covered by a film of tears a few microns thick and referred to as the pre-lens film (King-Smith et al. 2004). Over the years, much research effort has been exerted toward understanding the movement of the pre-lens film with an emphasis on understanding dry eye syndrome (Braun 2012). When the contact lens is inserted the tear film is split into two parts, the pre-lens and the post-lens films, and this affects the biophysical and biochemical properties of the film and causes additional complications (Craig et al. 2013). For example, one biophysical concern is the change in the flow of fluid over the cornea. Most clinicians regard it as necessary to have considerable interchange of tears under the contact lens to preserve ocular health (McNamara et al. 1999). Specifically, the lack of exchange of fluid underneath the lens has been shown to cause the accumulation of debris and cellular material, and has been linked to inflammation (McNamara et al. 1999; Paugh et al. 2001).

The movement of the film under the contact lens has been observed clinically with slit-lamp fluorophotometry (Pulse 1979; McNamara et al. 1999; Paugh et al. 2001), and interestingly, with observations of red blood cell movement (the cells serve as tracer particles) added to the film (Carter 1972). Various studies have found that under rigid gas-permeable lenses there is a substantially higher rate of exchange of fluid than there is under soft contact
lenses, because of the motion of the rigid gas-permeable lenses during a blink (Polse 1979; McNamara et al. 1999; Paugh et al. 2001). In addition, they have reported a fractional replenishment rate of tears under soft lenses of less than 3% per blink (Polse 1979; McNamara et al. 1999; Paugh et al. 2001). The studies hypothesize that the material characteristics of the contact lens, and its fit, affect the exchange of fluid. For example, McNamara et al. (1999) found that soft lenses 12-mm in diameter had a 0.6% per blink greater exchange of tears than lenses 13.5-mm in diameter. Carter (1972) found that red blood cells moved less under contact lenses judged to fit tightly than they did under contact lenses judged to fit properly.

Both the vertical motion of a contact lens during a blink and the action of the lens returning to its undeformed shape can cause an exchange of fluid under the lens. When a contact lens is placed on an eye, the forces of the blink stretch the lens, pushing fluid out at the edges, and deforming the lens to conform to the eye. Once the eye opens, the lens slowly relaxes back toward its original shape. The suction pressure under the lens generated by the deformed contact lens causes the post-lens fluid to flow back under the lens. By design, the relaxation time of the contact lens is longer than a typical time of 4–5 s between blinks.

Hayashi and Fatt (1976) developed a theoretical model of the exchange of film under a contact lens caused by the squeezing of the soft contact lens onto the cornea. Using lubrication theory, they calculated a 10–20% rate of tear exchange per blink from the action of the upper lid pushing on the contact lens. Because they modeled the contact lens as a flat plate, their simple model did not account for the dissipation of energy from the blink caused by the deformation of the soft contact lens. Chauhan and Radke (2002) significantly extended the theoretical modeling of the post-lens flow of tears by including the mechanics of the soft contact lens during blinking. They considered a flat eye and a spherical contact lens, and they found the volume of post-lens fluid squeezed out during a blink to be greater than the volume of post-lens fluid drawn back in by mechanical suction during the interblink period. Therefore, they predicted that the contact lens would eventually contact the cornea. Finally, there have been other efforts to understand the exchange of film caused by the vertical motion of the contact lens during a blink (Kamiyama and Khonsari 2000; Chauhan and Radke 2001; Creech et al. 2001).

In this work, we focus on understanding the post-lens flow of fluid driven by the mechanics of the contact lens, that is, the suction pressure, in an effort to quantify the amount of exchange of fluid during the interblink period; that is, the period during which the eyes are open. We recently developed a new mathematical model to compute the suction pressure under an axisymmetric contact lens conformed to an eye (Maki and Ross 2014). The model captures the basic physics—the elastic tensions—needed to understand a working contact lens. Using the shape of an eye, the shape of a lens, and the elastic properties of the lens, the model determines the suction pressure generated when the given contact lens conforms to the given shape of eye.

In this article, we study the flow of the post-lens film in response to suction pressure. We predict the fractional post-lens exchange of fluid and we explore the influence of the shape of the eye and the thickness of the contact lens on this flow. In what follows, we first present the mathematical model, then discuss results, and finally present conclusions.

**Formulation**

We begin by summarizing our model of the suction pressure under a contact lens.

**Suction pressure model**

We (Maki and Ross 2014) derived a mathematical model for the distribution of pressure in the post-lens film behind a radially-symmetric contact lens on a radially-symmetric eye. We used scaling arguments to show that bending stresses in the lens and shear stresses at the lens-fluid interface are negligible in comparison with the stresses induced by the stretching of the lens. The lens stretches in two directions: the radial direction and the angular direction. We refer to the stress in the radial direction as the *radial stress*, and that in the angular direction as the *hoop stress*.
When the lens is placed on the eye, the tangential components of the induced stress equilibrate instantaneously, and the pressure in the post-lens film balances the normal component. The suction pressure, thus, is determined by the stretching stresses in the lens, which in turn are determined by the shapes of the lens and the eye, and by the elastic properties of the lens.

With the model, we found that a lens whose undeformed shape is a spherical cap produces a negative pressure at its center when it is conformed to a flatter spherical eye and tends to draw fluid under the lens. A pointed eye creates a positive pressure under the center of the lens and tends to push tear fluid out from under the lens. For a realistically shaped eye with a change in curvature that mimics the change in corneal and scleral curvature, and with spherical un-deformed contact lenses, the suction pressure under the center of the lens increases in magnitude as the radius of curvature of the lens increases, whereas the positive pressure at the edge decreases. In addition, the negative suction pressure generated in the transition region also increases with increasing radius of curvature.

The tear-film model

We model the post-lens film as an incompressible, Newtonian thin film. Specifically, the characteristic 2 µm of thickness of the post-lens film is much smaller than the 0.7 cm radius of the un-deformed contact lens. Therefore, it is appropriate to approximate the dynamics of the post-lens film by thin-film lubrication theory (Oron et al. 1997).

We let the coordinate $z$ denote distance from a plane normal to the axis of the radially-symmetric eye. The location of the bottom of the contact lens is the graph of a function $z = h(R,t)$, and therefore the thickness of the post-lens film is given by $z = h(R,t) - C(R)$, where the graph of the function $z = C(R)$ is the shape of the eye. The evolution of the thickness of the post-lens film is governed by the Reynolds lubrication equation (Oron et al. 1997):

$$\frac{\partial h}{\partial t} - \frac{1}{R} \frac{\partial}{\partial R} \left( RH^3 \frac{\partial p_{\text{suction}}}{\partial R} \right) = 0,$$

where

$$\tilde{h}(R,t) = h(R,t) - C(R)$$

and $p_{\text{suction}}$ denotes the suction pressure generated by the deformed contact lens.

Because the function $\frac{\partial p_{\text{suction}}}{\partial R}$ is presumed known, having been determined, for example, from our model (Maki and Ross 2014), this equation is a first-order partial differential equation for $\tilde{h}$; it has the form of a hyperbolic conservation law (John 1991). We can solve it with the method of characteristics (John 1991; LeVeque 1999). We introduce a new variable $z = \frac{R^6}{12\mu} \frac{\partial p_{\text{suction}}}{\partial R}$, so that our partial differential equations becomes

$$\frac{\partial z}{\partial t} - \frac{h^2}{4\mu} \frac{\partial p_{\text{suction}}}{\partial R} \frac{\partial z}{\partial R} = 0. \quad (3)$$

We can rewrite this as

$$\frac{\partial z}{\partial t} - z^{2/3} \phi(R) \frac{\partial z}{\partial R} = 0, \quad (4)$$

where $\phi(R) = \left( \frac{9}{4\mu} \frac{\partial p_{\text{suction}}}{\partial R} \right)^{1/3}$. On any interval on which $\frac{\partial p_{\text{suction}}}{\partial R}$ is not zero, we can introduce the change of spatial variable

$$\frac{d\eta}{dR} = \frac{1}{\phi(R)} \quad (5)$$

and we can write the conservation law as

$$\frac{\partial z}{\partial t} - z^{2/3} \frac{\partial z}{\partial \eta} = 0. \quad (6)$$

In the $(t,\eta)$ coordinates, we define curves by the equation $\frac{d\eta}{dt} = -z^{2/3}$. These are called characteristic curves, or characteristics (John 1991). It follows directly from Equation (6) that $z$ is constant on such curves. This, in turn, implies that the characteristics are lines. These observations form the foundation of the method of characteristics that we use. As long as characteristics do not cross, we can track the values of $z$ across the entire $\eta$ interval; given a point $(t, \eta)$, we identify the characteristic passing through this point, and the value of $z$ at this point is the value of $z$ at which the characteristic intersects the initial interval.

This method allows us to solve Equation (1) in the $(t,R)$ plane between points at which $\frac{\partial p_{\text{suction}}}{\partial R} = 0$. At points at which $\frac{\partial p_{\text{suction}}}{\partial R} = 0$, Equation (1) has the form

$$\frac{\partial \tilde{h}}{\partial t} = \frac{h^3}{12\mu} \frac{\partial^2 p_{\text{suction}}}{\partial R^2}. \quad (7)$$

This is an ordinary differential equation that can be integrated in closed form to yield $\tilde{h}$ as a function of time at such points.

The point at which $R = 0$ is one at which $\frac{\partial p_{\text{suction}}}{\partial R} = 0$, but there is a special consideration at this point because the ratio $\frac{\partial p_{\text{suction}}}{R}$ appears in Equation (1). In the limit as $R$ approaches 0, $\frac{\partial p_{\text{suction}}}{R} \rightarrow \frac{\partial^2 p_{\text{suction}}}{\partial R^2}$. Thus, the ordinary differential equation that defines $\tilde{h}$ as a function of time at $R = 0$ is

$$\frac{\partial \tilde{h}}{\partial t} = \frac{h^3}{6\mu} \frac{\partial^2 p_{\text{suction}}}{\partial R^2}. \quad (8)$$
Results

We first examine the flow of the post-lens film under a lens of constant thickness. Specifically, we consider a lens with a Young’s modulus of 10^7 dynes/cm^2, a characteristic thickness of 0.005 cm and a Poisson’s ratio of 0.5 (i.e., the lens is incompressible). In all simulations, we assume that the post-lens film initially has a uniform thickness of 0.0002 cm (Nichols et al. 2003). We simulate the evolution of the post-lens film’s thickness for 5 s, the average time between blinks (Cruz 2011). Finally, we calculate the total exchange of fluid. In what follows, we explore how the thickness of the lens, the shape of the lens, and the shape of the surface of the eye influence the exchange of fluid. To build our understanding, we first study the flow on spherical caps.

Spherical eye

We first consider a case in which the un-deformed shape of the contact lens, the graph of \( z = g(r) \), and the shape of the eye, the graph of \( z = C(r) \), are spherical caps. For example, the latter has an equation of the form

\[
C(r) = \frac{h_0^2 + a_0^2}{2h_0} - r^2 - \frac{a_0^2 - h_0^2}{2h_0},
\]

where \( h_0 \) is the height of the cap, \( a_0 \) is the radius of the base of the cap, and the expression \( R = (h_0^2 + a_0^2)/(2h_0) \) is the radius of curvature of the sphere, all in centimeters. In general, the shape of the eye (sclera region) has a smaller mean curvature than does the un-deformed contact lens. Therefore, we take \( a_0 = 3/2, h_0 = 1/2, R = 5/4 \) for the shape of the eye and we take the graph of \( g(r) = \sqrt{(3/2)^2 - r^2} - \frac{3}{2} \) as the shape of the contact lens. We previously found that when a contact lens conforms to an eye with less curvature, it produces a negative suction pressure in the center of the lens that draws fluid under the lens (Maki and Ross 2014).

Figure 2 shows the evolution of the thickness of the post-lens film under the contact lens during the 5-s-interblink period. The largest increase in thickness, a 0.2 \( \mu \)m increase, occurs at the edge of the lens. Therefore, our assumption that the flow of the post-lens film is negligible in the derivation of the suction pressure model is self-consistent. The bold lines plotted on the plane \( z = 1.9 \mu \)m are the characteristics of Equation (6). They indicate the propagation of information. The post-lens fluid is drawn into the center of the lens to lift up the deformed (flattened) contact lens. To determine the thickness of the post-lens film at the edge of the contact lens (the dashed line along \( z = 1.9 \mu \)m), we need another boundary condition. The choice for the boundary condition is not obvious, as the contact lens is immersed in tear fluid. We don’t think there is a physical mechanism to keep the film’s thickness at the edge of the contact lens constant (and therefore the flux of fluid is constant) or the slope is zero. We do anticipate that away from the edge of the contact lens, the pressure will be atmospheric because the film is basically flat (so the Laplace pressure is negligible). For this study, we require that the slope of the tear’s thickness be zero at the edge of the contact lens. The characteristics that enter the domain at the edge of the lens carry with them information specified by the boundary condition. (These are red dashed lines shown online in Fig. 2.) The fractional replenishment of the volume of the post-lens tear under the lens is 0.04.

Realistic eye shape

We now consider a realistic eye shape. The surface of the eye is composed of three regions: the cornea (contains the pupil); the limbus (which is a transitional region); and the sclera (the white part of the eye). Specifically, cross-sectional measurements indicate that a good approximation of the cornea is the end of an ellipse (Braun et al. 2012). We use the ellipse whose equation is

\[
\left( \frac{R}{0.87} \right)^2 + \left( \frac{z}{0.96} \right)^2 = 1
\]

where \( R \) and \( z \) are measured in centimeters (Braun et al. 2012). The diameter of the cornea is approximately 1.2 cm (Young et al. 2010; Hall et al. 2011). We approximate the sclera, which has less mean curvature than the cornea, by a sphere with radius of curvature \( R = 1.2 \) cm (Hall et al. 2011). We model the limbus region by fitting a fifth degree polynomial to connect the cornea to the sclera smoothly; that is, the fit for the realistic eye-shape function has two continuous derivatives. Interestingly, little has been known about the limbal region until recently. Measurements made by videokeratoscopy, a non-invasive imaging technique that is used regularly in ophthalmology for mapping the surface curvature of the cornea, cannot resolve the cornea–sclera junction because the technology is limited to a diameter of about 0.9 cm (Klein et al. 2002). Recent advances in optical coherence tomography have allowed for the characterization of the limbal region (Hall et al. 2011; Shen et al. 2014; Cruz 2011). We first study the flow on spherical caps.
Hall et al. (2011) found the limbal region to provide a gradual transition in topography between the cornea and sclera.

To begin, we explore the flow under a spherical contact lens whose radius of curvature is 0.87 cm and whose thickness is 0.005 cm. The suction pressure under this contact lens on the realistically shaped eye is shown in Fig. 3. In the limbal region, the suction pressure is negative and changes from decreasing to increasing. We attribute this change to the fact that the un-deformed contact lens goes from having less mean curvature (pointy cornea) to more mean curvature than the eye’s normal shape (flat sclera). At the limbus, the transition between the cornea and the sclera, the pressure gradient is zero and the pressure is concave down. As we explained above, we can solve the evolution equation for the post-lens thickness exactly at this location. Specifically, the evolution equation, Equation (1), is a separable ordinary differential equation. The post-lens film thickness is governed by the equation

$$\tilde{h}(t) = \frac{h_0}{\sqrt{1 - \frac{h_0^2}{\mu} \frac{\partial p_{suction}}{\partial R} t}}, \quad (11)$$

where $h_0$ is the initial thickness of the film. The post-lens film has a finite singularity at $t = 6\mu / (h_0 \frac{\partial p_{suction}}{\partial R})$ meaning the thickness of the post-lens film will blow up to positive infinity in finite time. If we’re to avoid the singularity over the average 5-s-interblink period, the second derivative of the pressure has to be less than $2.1 \times 10^5$ dynes/cm$^4$.

For the configuration of a contact lens of constant thickness and whose suction profile is shown in Fig. 3, the second derivative of the pressure is larger than $2.1 \times 10^5$ dynes/cm$^4$. Thus, fluid would rush into the limbal region and lift up the contact lens to relieve the large negative suction pressure. Clinically, such pooling of fluid in the limbal region under a contact lens is regularly observed (Shen et al. 2011). Moreover, it has been suggested that the shape of the limbal region should be used to assess the “fit” of a contact lens (Hall et al. 2011). Of course, the presence of a finite singularity is unrealistic. That said, such a distribution of suction pressure under a contact lens is extremely effective in facilitating exchange of fluid under the contact lens.
Figure 4 shows the evolution of the post-lens film 1 s after a blink. It is interesting to note that we cannot use the standard method of characteristics to simulate the post-lens film’s thickness much past 1 s because the characteristics cross and the mathematical theory breaks down. To estimate the film-exchange, as we did above, we assume that the tear’s thickness has a slope of zero at the edge of the contact lens, to resolve the film’s profile (dashed line at 1 s). We find the fractional replenishment of the post-lens tear-volume under the lens to be 0.04. In 1 s, 4% of the fluid is replenished underneath the contact lens (greater than the 3% per blink rate of replenishment of the film quoted experimentally (Polse 1979; McNamara et al. 1999; Paugh et al. 2001)).

The finite singularity can be removed by considering a contact lens whose thickness varies like \( \tau(R) = \tau_0(1.1 - (R/R_{end})^2) \) where \( \tau_0 = 0.005 \) cm and \( R_{end} \) denote the edge of the contact lens. The contact lens’s thickness is used as a design parameter in creating comfortable-fitting lens (Jones et al. 2013). For example, different manufacturers have different lens-edge profiles, including so-called “rounded,” “knife,” and “chisel” edges. Studies have been done to correlate contact lens comfort with the shape of the lens’s edge profile (Jones et al. 2013). Figure 5 shows the suction pressure of the varying thicknesses of contact lenses on the realistically shaped eye. In the limbal region, the pressure gradient is still zero, but now the second derivative of the pressure is smaller. Thus, the film’s thickness will increase but the finite singularity occurs after the 5-s-interblink period. At the edge, where the pressure gradient is again zero, the film will become thinner because the pressure is concave down.

Figure 6 shows the evolution of the thickness of the post-lens film during the interblink period. The film under the center of the contact lens slowly moves into the low-pressure limbal region. Near the limbal region, the fluid collects at the location of the low-pressure or zero-pressure gradient (the dashed line connects these points). Over the 5-s-interblink period, the thickness of the post-lens tear film increases by 4 \( \mu \)m. As before, the bold lines plotted on the plane \( z = 0.5 \) \( \mu \)m are the characteristics.

Fig. 4 The evolution of the post-lens film on a realistically shaped eye under a contact lens whose thickness is 0.005 cm and undeformed shape is a spherical cap.

Fig. 5 The suction pressure under a contact lens, with 0.87 cm radius of curvature and varying thickness of \( \tau(R) = 0.005(1.1 - (R/R_{end})^2) \) cm, on the realistically shaped eye.
of Equation (6) and indicate the propagation of information. Near the edge of the contact lens, the characteristics are oriented such that the post-lens fluid is pushed out from under the contact lens. That is, no fluid from the edge of the contact lens is drawn under the contact lens. We find that 0.3% of the volume of tears is pushed out from under the contact lens.

Conclusions

We have developed a mathematical model that determines the flow of the post-lens fluid in response to the contact lens’s distortion. Using our model, we have quantified the amount of exchange of fluid takes place under the contact lens during the interblink period.

Our results indicate a sensitivity of the post-lens flow to the contact lens’s thickness profile. Specifically, we found that a contact lens must stretch significantly to accommodate the changes in the ocular curvature. Contact lenses whose changes in thickness do not facilitate the stretching over the sclera (no thinning at the edge) are effective at exchanging fluid. That is, the post-lens film rushes in under the contact lens to lift the stiffer contact lens. We note this is true as long as the contact lens does not bury itself into the corneal surface and prevent post-lens flow of tears. In contrast, we found that if the thickness of the contact lens thins as it approaches the edge of the contact lens then fluid is ejected at the edge of the lens. The amount of fluid ejected over the 5-s-interblink period is only 0.3% of the volume. We conclude that the dependence of the post-lens flow of fluid on the thickness-distribution of the contact lens is not trivial.

Our theoretical study provides insight into contact lens performance, and its dependence on design. More work is needed, especially with experimental data, to better understand how the post-lens flow responds to distortions of the contact lens and to understand how these relate to comfort and ocular health. In future work, we plan to couple evolution of the suction pressure and the post-lens film.

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