The effect of mutation on equilibrium properties of deterministic and random evolutionary games

Manh Hong Duong[1] and The Anh Han[2]

[1] School of Mathematics, University of Birmingham, B15 2TT, UK. Email: h.duong@bham.ac.uk
[2] School of Computing, Media and Art, Teeside University, TS1 3BX, UK. Email: T.Han@tees.ac.uk

The replicator-mutator equation is a set of differential equations describing the evolution of frequencies of different strategies in a population that takes into account both selection and mutation mechanisms. It is a fundamental mathematical framework for the modelling, analysis and simulation of complex biological, economical and social systems and has been utilized in the study of, just to name a few, population genetics (Hadeler, 1981), autocatalytic reaction networks (Stadler and Schuster, 1992), language evolution (Nowak et al., 2001) and the evolution of cooperation (Imhof et al., 2005). Consider an infinite population consisting of n types/strategies $S_1, \cdots, S_n$ whose frequencies are, respectively, $x_1, \cdots, x_n$. The reproduction rate of each type, S_i, is determined by its fitness or average payoff, f_i, which is obtained from interacting with other individuals in the population. The interaction of the individuals in the population take place within randomly selected groups of multiple participants. That is, they play and obtain their payoffs from a multi-player game, defined by a payoff matrix. We consider here symmetric games where the payoffs do not depend on the ordering of the players in a group. Due to mutation, individuals spontaneously change from one strategy to another, which is modeled via a row-stochastic matrix (called the mutation matrix), $Q = (q_{ji}), j, i \in \{1, \cdots, n\}$. The entry q_{ji} characterizes the probability that a player of type S_j changes its type or strategy to S_i. The replicator-mutator is then given by, see e.g. (Komarova, 2004)

$$\dot{x}_i = \sum_{j=1}^{n} x_j f_j(\mathbf{x}) q_{ji} - x_i \bar{f}(\mathbf{x}) =: g_i(\mathbf{x}), \quad i = 1, \ldots, n, \quad (1)$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\bar{f}(\mathbf{x}) = \sum_{i=1}^{n} x_i f_i(\mathbf{x})$ denotes the average fitness of the whole population. The replicator equation is a special case of (1) when the mutation matrix is the identity matrix. In reality, individuals' interactions are often affected by the constantly changing environments making it impossible to assign deterministic payoffs that correctly characterize these interactions. To capture uncertainty, we consider random multi-player multi-strategy games, in which the payoff entries are random variables.

In this extended abstract, we report our recent works on the statistics of the equilibria of the replicator-mutator equation (Duong et al., 2019; Duong and Han, 2019). According to (1), these equilbrium points satisfy the following equation

$$\begin{cases} g_i(\mathbf{x}) = 0, \quad i = 1, \ldots, n-1, \\ \sum_{i=1}^{n} x_i = 1, \quad 0 \le x_i \le 1. \end{cases} \quad (2)$$

In a general d-player n-strategy game, each g_i is a random multivariate polynomial of n variables and degree $d+1$, thus (2) is a system of multivariate polynomial equations, which is not analytically solvable in general for large d and n according to Abel's impossibility theorem. Therefore, we focus on multi-player two-strategy random games, particularly the statistics of the number of equilibria. This enables one to predict the levels of social and biological diversity as well as the overall complexity in a dynamical system. As in (Komarova, 2004), we consider an independent mutation model that corresponds to a uniform random probability of mutating to alternative strategies, for which the transition matrix reduces to $Q = (q_{ij})_{i,j=1}^2$: $q_{12} = q_{21} = q$, $q_{11} = q_{22} = 1 - q$. The parameter q represents the strength of mutation and ranges from 0 to $1 - \frac{1}{n}$. In the context of dynamics of learning (Komarova, 2004), the case $q = 0$ corresponds to perfect learning where learners always end up speaking the grammar of their teachers, while the case $q = \frac{n-1}{n}$ represents random learning where the chance for the learner to pick any grammar is the same for all grammars and is independent of the teacher's grammar.

Probability distributions of the number of equilibria in two-player social dilemma games. We first consider two-player social dilemma games adopting the parameterized payoff matrix as in (Santos et al., 2006), $a_{11} = 1$; $a_{22} = 0$; $0 \le a_{21} = T \le 2$ and $-1 \le a_{12} = S \le 1$. The four popular social dilemmas are: the Prisoner's Dilemma (PD): $2 \ge T > 1 > 0 > S \ge -1$, the Snow-Drift (SD) game: $2 \ge T > 1 > S > 0$, the Stag Hunt (SH) game: $1 > T > 0 > S \ge -1$, and the Harmony (H) game: $1 > T \ge 0, 1 \ge S > 0$. Equilibria of a social dilemma game are roots in the interval $[0, 1]$ of the following cubic equation

$$\Big(T+S-1\Big)x^3+\Big(1-T-2S+q(S-1-T)\Big)x^2+\Big(S+q(T-S)\Big)x = 0.$$

By analyzing this equation, where T and S are uniformly distributed in the corresponding intervals, we compute the

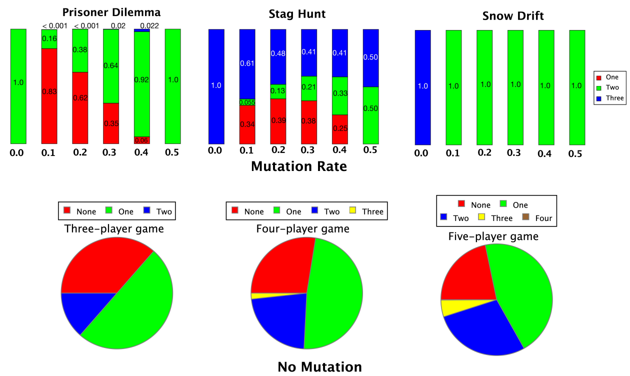

Figure 1: *(Upper row)*: impact of mutation rate on the probability of having a certain number of internal equilibrium points for three popular social dilemmas. *(Lower row)* probability of having a certain number of internal equilibrium points for two-strategy games with three, four and five players without mutation.

probabilities that H-games and SD-games have $1, 2$ or 3 equilibria: $p_1 = p_3 = 0, \quad p_2 = 1$; while for SH-games and SD-games, we obtain exact formula for p_2:

$$p_2^{SH} = \frac{q}{2(1-q)}, \quad p_2^{PD} = \begin{cases} \frac{3q}{2(1-q)} & \text{if} \quad 0 < q \leq 1/3, \\ 3 - \frac{1}{2q(1-q)} & \text{if} \quad 1/3 \leq q \leq 1/2. \end{cases}$$

We also numerically compute these probabilities for two-strategy games with three, four and five players (without mutation) to demonstrate the impact of mutation rate, see Figure 1. From our analysis and simulations, we observe that, except for the SD game, introducing mutation leads to the probability of gaining an additional internal equilibrium (thus increasing behavioural diversity) in all social dilemmas. This probability is 100% in the H-game, increases with q in the PD-game (reaching 100% when $q = 0.5$) and is roughly 40-60% in the SH-game.

Expected number of equilibria of multi-player two strategy games. We next consider d-player two-strategy games. In this case, using techniques from the theory of random polynomials, we can compute the expected number of equilibria. Suppose that the payoff entries are independent standard normally distributed random variables with mean zero. Then the expected number of equilibria of a d-player two-strategy replicator-mutator dynamics is given by

$$E = \frac{1}{\pi} \int_0^\infty \left(\frac{\partial^2}{\partial x \partial y} \left(\log H(x,y) \right) \Big|_{y=x=t} \right)^{\frac{1}{2}} dt,$$

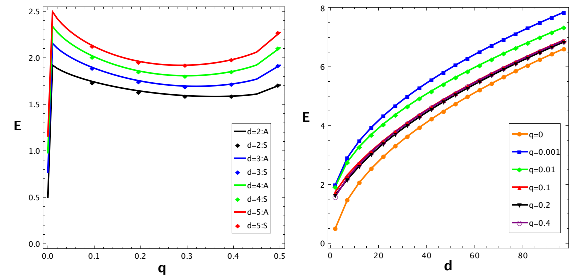

Figure 2: Analytical vs simulation results of the average number of internal equilibrium points (E) for varying mutation rate q and different group sizes d. The solid lines are generated from analytical (A) formulas of E. The solid diamonds capture simulation (S) results obtained by averaging over 10^6 samples of the payoff entries (normal distribution).

where

$$H(x,y) = \sum_{k=0}^{d+1} C_{kk} x^k y^k + \sum_{k=0}^{d} C_{kk+1} (x^k y^{k+1} + x^{k+1} y^k),$$

$$C_{kk} = q^2 \binom{d-1}{k-2}^2 + 2(q-1)^2 \binom{d-1}{k-1}^2 + q^2 \binom{d-1}{k}^2,$$

$$\text{for} \quad k = 0, \ldots, d+1,$$

$$C_{kk+1} = q(q-1) \binom{d-1}{k-1}^2 + q(q-1) \binom{d-1}{k}^2,$$

$$\text{for} \quad k = 0, \ldots, d,$$

$$C_{ij} = 0 \quad \text{for} \quad 0 \leq i < j \leq d+1 \; : j - i \geq 2.$$

The results are in accordance with simulations as demonstrated in Figure 2. In general, we observe that E is always larger in the presence of mutation (i.e. when $q > 0$) than when mutation is absent (i.e. when $q = 0$), implying that mutation leads to larger behavioural diversity in a dynamical system. Interestingly, E is largest when q is close to 0 (i.e. rare mutation), rather than when it is very large. These results fundamentally extend previous works on random evolutionary game theory which have focused on replicator equations where mutation is not accounted for (Gokhale and Traulsen, 2010; Han et al., 2012; Duong and Han, 2015, 2016). In biological, social and artificial life systems, mutation or randomness are ubiquitous and unavoidable (e.g. genetic mutation, random exploration of new behaviours), which is important to introduce variation to the systems (Stadler and Schuster, 1992; McNamara, 2013). Therefore, our works have provided new insights into these important scenarios.

In short, we have studied statistical properties of the number of equilibria for the replicator-mutator equation, thereby providing new insights into the overall complexity of dynamical systems, including biological, social and Artificial Life ones. We expect that our approach can be extended to other more complex models in population dynamics such as evolutionary dynamics of complex multiple games and evolutionary games with environmental feedback.

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