

# Random Networks with Quantum Boolean Functions

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## Abstract

We propose quantum Boolean networks, which can be classified as deterministic reversible asynchronous Boolean networks. This model is based on the concept of quantum Boolean functions. A quantum Boolean network is a Boolean network where the functions associated with the nodes are quantum Boolean functions. We study some properties of this novel model and, using a quantum simulator, we study how the dynamics change in function of the connectivity of the network and the set of operators we allow. For some configurations, the behavior of this model resembles that of reversible Boolean networks (RevBN), while for other configurations a more complex dynamics can emerge. For example, cycles larger than  $2^N$  were observed. Additionally, using a scheme akin to one used previously with random Boolean networks, we computed the average entropy and complexity of the networks. As opposed to classic random Boolean networks, where “complex” dynamics are restricted mainly to a connectivity close to a phase transition, quantum Boolean networks can exhibit stable, complex, and unstable dynamics independently of their connectivity.

## Introduction

Boolean networks were originally proposed in the late 1960s to model the behavior of regulatory networks by Kauffman (Kauffman, 1969). Boolean networks have been used since then to model the operation of discrete dynamical systems. This model is suitable for the study of complex systems as it offers a simple and flexible way to control the interaction of the elements of a system and observe the effects of structure change in the network dynamics. At the same time, quantum algorithms are still not well understood and it is as yet unclear for which kind of problem quantum computers will provide us with some sort of advantage. Quantum computation may prove beneficial to enrich the model of Boolean networks, considering that the dynamics described by quantum mechanics is fundamentally different from that of classical dynamics. In Franco et al. (2021), we introduced a model similar to Boolean networks whose behaviour is ruled by quantum mechanics.

## Quantum Boolean Functions

To define an analogous model of Boolean networks within the quantum realm, we use the notion of quantum Boolean function from Montanaro and Osborne (2010). A *quantum Boolean function*, QBF, on  $k$  qubits is a  $2^k \times 2^k$  unitary matrix  $U$  such that  $UU = I$ . The definition of QBF is rather general and it may not be useful for us without some simple restrictions since we want functions affecting a single node, not the whole network. This problem can be easily solved if we restrict ourselves to the subset of QBFs that can only affect one qubit at a time. In short, this can be done using diagonal matrices whose entries are single qubit QBF; additionally, we require a change of basis to target any possible qubit in the network. For more details about this construction we refer the reader to the Franco et al. (2021).

A *quantum Boolean network* (QBN) is a tuple  $(\Sigma, \mathcal{F}, \Pi)$ , where  $\Sigma$  is a set of  $N$  qubits,  $\mathcal{F} = \{U_1, \dots, U_N\}$  is a set of quantum Boolean operators and  $\Pi = \{\pi_1, \dots, \pi_N\}$  is a permutation of the set of indices  $\{1, \dots, N\}$  that serves as the update order of the network. Each  $U_i$  is a QBF on  $1 \leq k_i \leq N$  qubits that only modifies the state of the  $i$ th qubit. The set  $\Pi$  is required to specify the behavior of the network without any ambiguity since, in general, the Boolean operators do not commute. Specifically, the transition matrix  $T$  of the network is the product:

$$T = U_{\pi_1} \cdots U_{\pi_N} \quad (1)$$

Quantum Boolean networks are reversible and have an update scheme that is asynchronous and deterministic.

## Dynamics

The state of the network is a linear combination of all the vectors spanned by the tensor product of the qubits. In other words, the state may be any vector of norm one in  $\mathbb{C}^{2^N}$ . In QBNs every state is part of an attractor because the evolution is reversible, i.e., every state comes from a particular state and goes to another one.

In order to shed light to what a QBN is, we can visualize the state of a network, classical or quantum, at timestep  $t$

as a probability distribution over all possible strings of zeros and ones of length  $N$ . For a classical Boolean network the only distributions allowed are those where all strings but one have probability zero of being observed. Meanwhile, a quantum Boolean network may assign a non-trivial probability distribution to this set of strings (states). Nonetheless, if we measure the qubits of the QBN we will end up with a particular string. Specifically, the probability of observing a string is given by the distribution associated with the state of the QBN.

When we compute the survival probability of cycles in the network (the probability that a cycle does not return to the initial state at the next timestep  $t + 1$  given that its current size is  $l$ ), see Figure 1, it becomes clear that we can now produce cycles larger than  $2^N$ . Just by including operators that induce a phase ( $Y$  and  $Z$ ) to the state it is possible to obtain larger cycles. Special considerations were taken to prevent the possibility of network inducing an irrational phase factor upon the state; strictly, an irrational phase factor will produce cycles of infinite length which may be not desirable in a model like this. Furthermore, an explosion in the length of the cycles can be observed when we introduce the possibility to create superposition of states, via the the  $H$  operator.

We computed the Shannon entropy and complexity for different operator sets and network sizes (see Figure 2). The entropy of a QBN is calculated as the average entropy of its cycles. The entropy of a cycle is just the average Shannon entropy of the nodes, using the node measurement probability of being 0 or 1. The complexity of a network is given by:

$$C(S) = 4S(1 - S) \quad (2)$$

with  $S$  the entropy of the network. Intuitively, complexity aims to measure the balance between change and stability (Fernández et al., 2014). It is minimal for  $S = 0$  (no change, only one state possible) and also for  $S = 1$  (constant change, all states equally possible). Complexity is maximal  $C = 1$  when  $S = 0.5$ , which coincides with phase transitions in classical random Boolean networks and the Ising model, as well as with class IV cellular automata (Gershenson and Fernández, 2012).

## Conclusion

QBNs exhibit a deterministic, asynchronous, and reversible dynamics. If we consider a non-degenerated version of the state space that a RevBNs spans, for a restricted set of operators, the unitary dynamics of our model behaves similarly to the dynamics of reversible Boolean networks. Moreover, QBNs can create networks with an odd number of nodes in contrast with RevBNs.

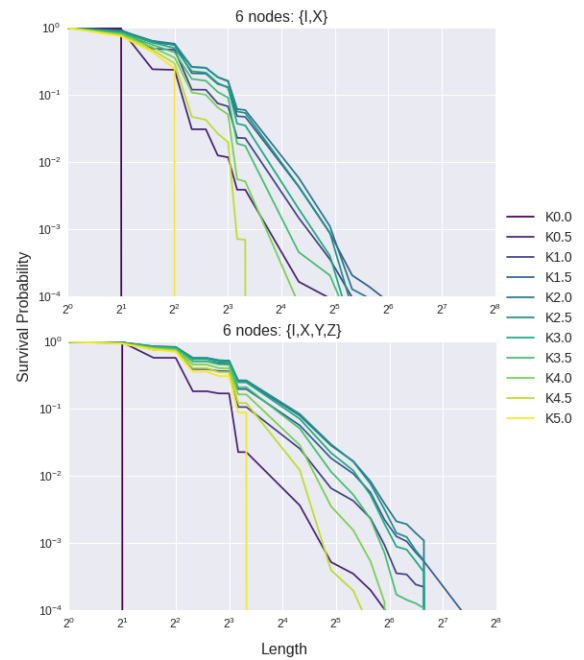


Figure 1: Survival probability of a QBN with 6 nodes for two different operator set configurations and different average connectivities ( $K$ ).

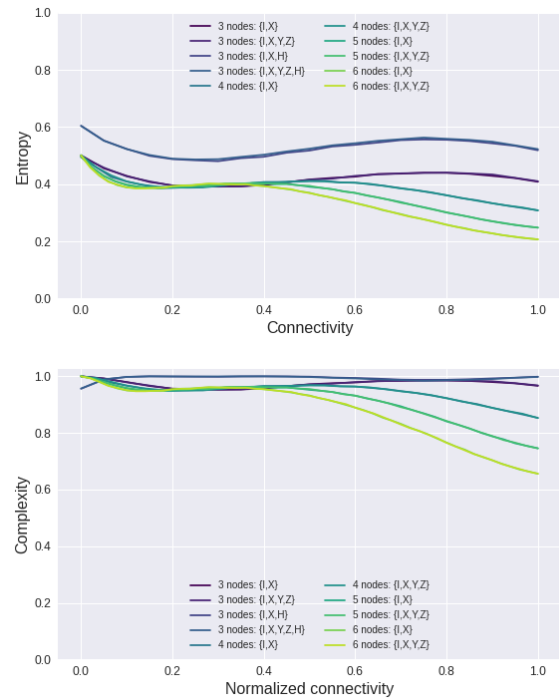


Figure 2: Shannon entropy (up) and complexity (bottom) for different QBN configurations. Single qubit operator set is indicated in brackets. Connectivity was normalized to allow a direct comparison between networks of different sizes.

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