A Minimal Model of Collective Behaviour Based on Non-reciprocal Interactions

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Abstract

The collective behaviour of individuals is widely observed in many natural and social systems. In these systems, Newton’s third law, or the law of action–reaction, is often violated. Hence, interaction between individuals is often non-reciprocal. Several previous studies focused on and partially elucidated the mechanism of the aforementioned systems. In this study, we aim to further deepen the understanding from a general perspective by proposing and analysing a simple mathematical model. The model is proposed by drawing inspiration from friendship formation in human society. It is demonstrated via simulations that various patterns emerge by changing the parameters. Further, these patterns are characterized by two macroscopic variables, based on which they are classified into six categories. Through this classification, we found that lifelike complex patterns emerge at the boundary between the parameter spaces where relative position among particles are fixed and where some particles move infinite distance from the others. Although this study is still in a rudimentary stage, we believe that our finding in which macroscopic patterns are related to local rules could move one step forward in understanding the core principle of collective behaviour.

Introduction

In many natural and social systems, collective behaviour emerges as a result of a local interaction between their constituting elements, such as the flocking of animals (Bellomo and Soler, 2012; Chen and Kolokolnikov, 2014; Cristiani et al., 2011; Couzin et al., 2002, 2005; Grégoire and Chaté, 2004; Hayakawa, 2010; Levine et al., 2000; Motsch and Tadmor, 2011; Olson et al., 2013; Pearce et al., 2014; Reynolds, 1987; Romanczuk et al., 2009; Vicsek et al., 1995; Zheng et al., 2005), cell migration (Szabó et al., 2006), chemical processes (Sayama, 2009; Tanaka, 2007), traffic and pedestrian flow (Bando et al., 1995; Helbing et al., 1995), and social networks (González et al., 2006; Kano et al., 2014; Nishimoto et al., 2013). A key insight into these systems is that Newton’s third law, or the law of action–reaction, is not necessarily hold; that is interactions between the components constituting the systems are often non-reciprocal. This non-reciprocal relationship occurs because components do not directly interact with each other; the components interact indirectly via the environment.

Understanding the core principle of the aforementioned systems is a challenging and formidable task. While several researchers addressed this grandiose issue and achieved partial success by proposing simple mathematical models (Bando et al., 1995; Calovi et al., 2014; Chen and Kolokolnikov, 2014; Gauthrais et al., 2012; Helbing et al., 1995; Moussaïd et al., 2011; Vicsek et al., 1995; Weitz et al., 2012), it is expected that the understanding is further deepened by proposing and analysing other simple models based on non-reciprocal interactions. Thus, in this study, an extremely simple and abstract model is proposed by drawing inspiration from the friendship formation process in human society (for example, a process in which several cliques are formed spontaneously in certain communities such as classes in schools).

This study has two contributions. First, it is demonstrated via simulations that various patterns emerge when the parameters of the proposed model are changed. Interestingly, several patterns are highly complex, lifelike, and non-trivial. Second, the patterns are classified into several categories and the classification is plotted in a parameter space. Consequently, it is found that the categories of the lifelike complex patterns exist only at the boundary between the categories of the other trivial patterns in the parameter space, which is similar to the ‘edge of chaos’ found in cellular automata (Langton, 1990). Although our model was just inspired by friendship formation process but not developed to reproduce real-world phenomena, our finding is expected to provide deep insight into one of the most important concerns in self-organized systems, that is, how macroscopic patterns emerge from local interactions.

Model

In this model, particles, each of which represents a person in a community, exist on a two-dimensional plane, and the position of the $r$th particle ($i = 1, 2, \cdots, N$) is denoted by $r_i$. The time evolution of $r_i$ is given by

\[ \dot{r}_i = \sum_{j \neq i} (k_{ij} |R_{ij}|^{-1} - |R_{ij}|^{-2}) \hat{R}_{ij}, \]

where $R_{ij} = r_j - r_i$, $\hat{R}_{ij} = R_{ij}/|R_{ij}|$, and $k_{ij}$ denotes a constant that represents “to what extent person $i$ prefers
by rewriting Eq. (1) as

\[ \dot{\mathbf{r}}_i = -\frac{\partial \Phi}{\partial \mathbf{r}_i} + \mathbf{u}_i, \]

where

\[ \Phi = \sum_{j \neq i} \left( k_{ij}^+ \log |\mathbf{R}_{ij}| + |\mathbf{R}_{ij}|^{-1} \right), \]

\[ \mathbf{u}_i = \sum_{j \neq i} k_{ij}^- |\mathbf{R}_{ij}|^{-1} \mathbf{R}_{ij}, \]

\[ \tilde{k}_{ij} = (k_{ij} + k_{ji})/2, \quad k_{ij}^- = (k_{ij} - k_{ji})/2. \]

The second term on the right-hand side of Eq. (2) cannot be described by any potential function. From an energetic viewpoint, the first and second terms in Eq. (2) denote the dissipation and injection of energy, respectively. Further, the momentum of the system is not conserved owing to the existence of the second term. Thus, Eq. (1) is a non-equilibrium open system in which both energy and momentum are non-conservative.

Figure 1: Phase diagram. The parameters \( k_p \) and \( k_m \) are varied with \( k_a = 0.8 \) and \( k_b = 0.4 \). The letters denote patterns whose photographs and movies are shown in Figs. 2–11 and Movie 1, respectively. The colour of the letters denotes the classification of the patterns (see main text). Patterns A’, D’, and F’ are qualitatively the same as patterns A, D, and F, respectively. However, the configurations of the red particles (\( i \leq 25 \)) and the blue particles (\( i > 25 \)) are opposite. Pattern C’ is similar to, but slightly different from, pattern C (see the caption in Fig. 4). Several boundaries between different patterns are ambiguous, and intermediate behaviours between different patterns are often observed. Further, several patterns are multi-stable for some parameter ranges. Thus, a representative behaviour is classified into a typical pattern in each parameter.

Results

We performed simulations of the proposed model. The initial position of the particles was set to be random within the range of \( x_i \in [-5, 5] \) and \( y_i \in [-5, 5] \), where \( x_i \) and \( y_i \) are the components of \( \mathbf{r}_i \). Because the number of parameters was large, we first investigated the case where \( N = 50 \) and

\[ k_{ij} = \begin{cases} k_a & (1 \leq i \leq 25, 1 \leq j \leq 25) \\ k_p + k_m & (1 \leq i \leq 25, 26 \leq j \leq 50) \\ k_p - k_m & (26 \leq i \leq 50, 1 \leq j \leq 25) \\ k_b & (26 \leq i \leq 50, 26 \leq j \leq 50) \end{cases} \]

with \( k_a \), \( k_b \), \( k_p \), and \( k_m \) being tunable parameters. The phase diagram, wherein \( k_p \) and \( k_m \) are varied with \( k_a = 0.8 \) and \( k_b = 0.4 \), is shown in Fig. 1. Various non-trivial patterns (Patterns A–K) emerge depending on the values of \( k_p \) and \( k_m \). Snapshots and movies for the obtained patterns are respectively provided in Figs. 2–11 and Movie 1 (which can be downloaded from http://www.riec.tohoku.ac.jp/~tkano/ECAL_Movie1.mp4).

and a detailed explanation for each pattern is provided in the figure captions.

We also found that other patterns emerged when \( k_a \) and \( k_b \) as well as \( k_p \) and \( k_m \) were varied (Patterns L–S in Movie 1), and when \( k_{ij} \) is defined in manners different from Eq. (5) (Patterns T–Z in Movie 1).

To understand the obtained patterns more systematically, we introduced the following macroscopic variables

\[ X = \left( N^{-1} \sum_{i=1}^{N} |\mathbf{r}_i - \mathbf{r}_g| \right)^{-1}, \]

\[ V = N^{-1} \sum_{i=1}^{N} |\mathbf{r}_i - \mathbf{r}_g|, \]

where \( \mathbf{r}_g \) denotes the position of the centre of gravity, that is,

\[ \mathbf{r}_g = N^{-1} \sum_{i=1}^{N} \mathbf{r}_i. \]

It is noted that \( X \) converges to zero when at least one of the particles moves an infinite distance from the centre of gravity, and that \( V \) converges to zero when the relative velocities of all particles with respect to the centre of gravity converge to zero.

Using \( X \) and \( V \), the obtained patterns can be classified into six categories (Table 1) (though the classification of the patterns might depend on the initial condition and the simulation time). Examples of the six classes are shown in Fig. 12. Importantly, patterns classified into classes III–VI never emerge if the interaction is reciprocal, i.e., \( k_{ij}^- = 0 \), because the system converges to the minimum of the potential \( \Phi \) (see Eqs. (2)–(4)), that is, \( V \) converges to zero. Thus, non-reciprocal interaction plays an important role in the emergence of versatile patterns.
<table>
<thead>
<tr>
<th>Class</th>
<th>Definition</th>
<th>Patterns</th>
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<tbody>
<tr>
<td>I</td>
<td>Both $X$ and $V$ converge to zero as the time passes.</td>
<td>A, A', B, J, U</td>
</tr>
<tr>
<td>II</td>
<td>$V$ converges to zero and $X$ converges to a non-zero fixed value.</td>
<td>C, F, F', K, S</td>
</tr>
<tr>
<td>III</td>
<td>$X$ converges to zero and $V$ converges to a non-zero fixed value.</td>
<td>D, D', E, L, M, Q, R, W, Z</td>
</tr>
<tr>
<td>IV</td>
<td>Both $X$ and $V$ converge to a non-zero fixed value.</td>
<td>I</td>
</tr>
<tr>
<td>V</td>
<td>A closed trajectory is drawn on the $X$–$V$ plane.</td>
<td>G, N, P, T</td>
</tr>
<tr>
<td>VI</td>
<td>An irregular trajectory is drawn on the $X$–$V$ plane.</td>
<td>H, O, V, X, Y</td>
</tr>
</tbody>
</table>

Table 1: Definition of classification. Patterns corresponding to each class, which were obtained through careful analysis, are described on the right.

Figure 2: (a) Pattern A ($k_a = 0.8$, $k_b = 0.4$, $k_p = -0.4$, and $k_m = -0.8$). The red particles are divided into several clusters, which scatter around the cluster formed by the blue particles. (b) Pattern A’ ($k_a = 0.8$, $k_b = 0.4$, $k_p = 0.4$, and $k_m = 0.6$). The configurations of the red and blue particles are opposite those in Pattern A. The following features are common to Figs 2–11: The $i$th particle is colored with red and blue for $i \leq 25$ and $i > 25$, respectively. Bars at the bottom denote the length scale, where the bar length corresponds to a length of 20. Insets show magnified views of the systems, and thin black arrows indicate the direction of movement of the particles.

The colour of the letters in Fig. 1 denotes the above classification. In general classes II and IV are found in the area of $k_p > |k_m|$, while classes I and III are found in the area of $k_p < |k_m|$. Interestingly, lifelike patterns, i.e., classes V and VI, are found in the area between the above two areas.

**Discussion**

In addition to the non-reciprocal property of the interaction, our model is also distinctive in that elements constituting the system have different characters. Systems similar to ours in this sense have been treated in several extant studies. For example, Sayama proposed a swarm chemistry model to explore self-organization of swarms made of kinetically distinct types of particles (Sayama, 2009). Nishimoto et al. proposed a model of self-driven particles incorporating the idea of the prisoner’s dilemma to analyse social dynamics (Nishimoto et al., 2013). Although these models exhibit various non-trivial patterns, our model is simpler, enabling the capture of the essence of collective behaviour. In contrast, Chen et al. proposed a simple model for the swarming behaviour of prey and predators (Chen et al., 2014). However, the number of patterns that emerge in this model is limited, because only cases, in which one or few predators and many prey exist, are considered. Our model is more general and can be used to treat other cases, e.g., Patterns T–Z in Movie.
Figure 3: Pattern B ($k_a = 0.8$, $k_b = 0.4$, $k_p = -0.2$, and $k_m = 0.0$). The red and blue particles form individual clusters, which repel each other.

Figure 4: Pattern C ($k_a = 0.8$, $k_b = 0.4$, $k_p = 0.4$, and $k_m = -0.2$). A cluster formed by the blue particles chases a cluster formed by the red particles. Note that the cluster formed by the blue particles does not chase the cluster formed by the red particles for pattern $C^*$ in Fig. 1.

1, by changing the $k_{ij}$ values.

Because our model is highly abstract, it is difficult to conclude that it mimics real social phenomena. Further, it is hard to describe the advantage of the proposed model over other existing models clearly because this study does not overcome a certain limitation of the previous ones. However, we believe that this study is meaningful because accumulating findings on how macroscopic patterns emerge from local interactions by using various types of mathematical models could move one step forward in elucidating the core principle of collective behaviour. Indeed, this study clarifies the relation between the macroscopic quantities $X$ and $V$ and the microscopic quantities $k_{ij}$ (though still the parameter range shown in Fig. 1), and it could impart a design principle of collective behaviour based on non-reciprocal interactions.

Conclusion and Future Work

A minimal model of collective behaviour based on non-reciprocal interaction was proposed. Simulations were performed to demonstrate that various patterns emerge depending on the parameters. These patterns were classified into six categories by introducing two macroscopic variables $X$ and $V$, and lifelike complex patterns classified into classes V and VI were found to exist at the boundary between the parameter space of classes II and IV (i.e., patterns in which relative positions of particles are fixed) and that of classes I and III (i.e., patterns in which at least one of the particles moves an infinite distance from the centre of gravity). This finding is expected to help understand how macroscopic patterns emerge from local interactions.

Finally, we would like to note that our finding bears similarly to the ‘edge of chaos’ found in cellular automata (Langton, 1990). Certainly, in the studies on cellular automata, complex patterns of localized structures are found in the boundary between an area in which simple patterns, e.g., static or periodic patterns, are observed and that in which aperiodic chaotic patterns are observed. This boundary is called the ‘edge of chaos,’ and it is apparently similar to the boundary found in our model. It is still unclear whether a common principle underlies these two or not, at the present stage, and its clarification will be a part of future work.

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Figure 5: (a) Pattern D ($k_a = 0.8$, $k_b = 0.4$, $k_p = 0.2$, and $k_m = -0.4$). The red particles are divided into two clusters. The cluster formed by the blue particles chases one of the red-particle clusters. (b) Pattern D’ ($k_a = 0.8$, $k_b = 0.4$, $k_p = 0.2$, and $k_m = 0.4$). The configurations of the red and blue particles are opposite those in Pattern D.

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References


Figure 6: Pattern E \((k_a = 0.8, k_b = 0.4, k_p = 0.6, \text{ and } k_m = -0.8)\). The red particles are divided into several clusters into which some of the blue particles are absorbed. The other blue particles form a cluster, and it chases, collides against, and divides a red particle cluster.

Figure 7: (a) Pattern F \((k_a = 0.8, k_b = 0.4, k_p = 0.6, \text{ and } k_m = -0.4)\). The blue particles are surrounded by a membrane formed by the red particles. (b) Pattern F' \((k_a = 0.8, k_b = 0.4, k_p = 0.4, \text{ and } k_m = 0.2)\). The configurations of the red and blue particles are opposite those in Pattern F.


Figure 8: Pattern G ($k_a = 0.8$, $k_b = 0.4$, $k_p = 0.4$, and $k_m = 0.4$). The red particles form a cluster, and it is surrounded by the blue particles (10000 time steps). However, the blue particles are divided into two clusters (50000 time steps), and the cluster of the red particles regularly oscillates between them (after 80000 time steps).

Figure 9: Pattern H ($k_a = 0.8$, $k_b = 0.4$, $k_p = 0.6$, and $k_m = -0.6$). The red particles are divided into several arc-like structures, and the blue particles irregularly move among them.

Figure 10: Patterns I and K. (a) Pattern I ($k_a = 0.8$, $k_b = 0.4$, $k_p = 0.6$, and $k_m = -0.2$). All particles aggregate and move in a circular manner (see Movie 1). (b) Pattern K ($k_a = 0.8$, $k_b = 0.4$, $k_p = 0.8$, and $k_m = -0.8$). The blue particles are surrounded by two arc-like structures formed by the red particles.
Figure 11: Pattern J ($k_a = 0.8$, $k_b = 0.4$, $k_p = 0.6$, and $k_m = 0.8$). The blue particles are divided into several clusters. The red particles form a cluster, and it chases, collides against, and divides one of the blue particle clusters.

Figure 12: Typical examples of trajectories on a $X$–$V$ plane for classes I–VI: (a) class I (pattern A, Fig. 2), (b) class II (pattern C, Fig. 4), (c) class III (pattern D, Fig. 5), (d) class IV (pattern I, Fig. 10), (e) class V (pattern G, Fig. 8), and (f) class VI (pattern H, Fig. 9). To enlarge the trajectory around the point of origin, $\tilde{X} = \log(X^3 + 1)$ and $\tilde{V} = \log(V^3 + 1)$ instead of $X$ and $V$ are taken as the horizontal and vertical axes, respectively. The arrows denote the time course of the trajectories. Since the simulation time is finite, the trajectory does not completely reach the point of origin in (a). However, the trajectory would be expected to reach there if the simulation was performed for infinite time. The expected trajectory is drawn by a dashed line.
