The Behavior of Number Concentration Tendencies for the Continuous Collection Growth Equation Using One- and Two-Moment Bulk Parameterization Schemes

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ABSTRACT

This paper presents a mathematical explanation for the nonconservation of total number concentration $N_t$ of hydrometeors for the continuous collection growth process, for which $N_t$ physically should be conserved for selected one- and two-moment bulk parameterization schemes. Where possible, physical explanations are proposed. The assumption of a constant $n_o$ in scheme A is physically inconsistent with the continuous collection growth process, as is the assumption of a constant $D_n$ for scheme B. Scheme E also is nonconservative, but it seems this result is not because of a physically inconsistent specification; rather the solution scheme’s equations simply do not satisfy $N_t$ conservation and $N_t$ does not come into the derivation. Even scheme F, which perfectly conserves $N_t$, does not preserve the distribution shape in comparison with a bin model.

1. Introduction

Straka et al. (2005, hereinafter S05) evaluated four commonly used parameterization schemes to determine whether they would conserve the total number concentration $N_i$ of hydrometeors for the continuous collection growth process, for which $N_i$ physically should be conserved for the processes of continuous collection growth and vapor diffusion growth. The four schemes evaluated were

1) scheme A, a one-moment scheme in which $q$ is predicted, $n_o$ is specified as a constant, and $N_i$ and $D_n$ are diagnosed;
2) scheme B, a one-moment scheme in which $q$ is predicted, $D_n$ is specified as a constant, and $N_i$ and $n_o$ are diagnosed;
3) scheme E, a two-moment scheme in which $q$ and $D_n$ are predicted and $N_i$ and $n_o$ are diagnosed; and
4) scheme F, a two-moment scheme in which $q$ and $N_t$ are predicted and $D_n$ and $n_o$ are diagnosed.

In these schemes, $q$ is mixing ratio, $n_o$ is the slope intercept, and $D_n$ is the characteristic diameter (inverse of the slope of the distribution). S05 showed that only scheme F conserved $N_i$ for vapor diffusion and the continuous collection growth processes for which $N_i$ should be conserved. In this paper, only the continuous collection growth process is considered, for which it is shown mathematically why $N_i$ is conserved for scheme F and why $N_i$ is not conserved for the other schemes. The results for vapor diffusion growth are omitted because they are qualitatively similar to those for continuous collection growth. A list of the schemes and the authors that have used them is presented in Table 1.

In section 2, the relevant equations are presented. In section 3, results are presented for the same set of parameters as those used in S05, except that a smaller initial graupel mixing ratio is used herein so as to have physically meaningful results at later times, such as 600 s. In section 4, it is shown mathematically why $N_i$ is
conserved for scheme F and is not conserved for the other schemes. In section 5, comparisons are made with bin-model results with number concentration as a function of size distributions of schemes A, B, E, and F for three selected times in the integrations. Where possible, the physical bases for these results are proposed. A summary is given in section 6.

2. Equations

The following equations presented below can be found in S05 and elsewhere in numerous references. However, it is deemed useful to repeat them so that the reader can have quick reference to them to complement the often complex explanations in section 3. Some of the following equations are definitions, some are diagnostic, and some are predictive.

The zeroth moment of a distribution is defined by \( N_i \) (m\(^{-3}\)) as

\[
N_i = \int_0^\infty n(D) \, dD,
\]

where \( D \) (m) is diameter. The number concentration per unit length is \( n(D) \) and is given by a form of the gamma distribution function, for example,

\[
n(D) = N_i \Gamma(\nu) D_n \left( \frac{D}{D_n} \right)^{\nu-1} \exp \left( -\frac{D}{D_n} \right),
\]

where \( D_n \) (m) is the characteristic diameter of the distribution (or the inverse of the slope of the distribution), \( \Gamma(x) \) is the gamma function, and \( \nu \) is the shape or breadth parameter of the distribution, which is assumed to be constant for the four parameterizations considered. For hydrometeors that may approximated by spheres, mass can be written as \( m(D) = \alpha D^6 \), where \( \alpha = \rho \pi/6 \), and \( \beta = 3 \). The particle density is denoted by \( \rho \) (kg m\(^{-3}\)). The third moment with respect to length, or mixing ratio \( q \) (kg kg\(^{-1}\)), can be defined from (2) and the definition of mass for a sphere, which results in an equation relating \( q \), \( D_n^3 \), and \( N_i \), and this diagnostic equation is used in all of the schemes analyzed herein:

\[
q = \frac{1}{\rho_o} \int_0^\infty m(D)n(D) \, dD = \frac{\alpha N_i \Gamma(\beta + \nu) D_n^6}{\rho_o \Gamma(\nu)}.
\]

Rearranging (3) and solving for \( D_n^6 \) results in the diagnostic equation,

\[
D_n^6 = \frac{\rho_o \Gamma(\nu) q}{\alpha N_i \Gamma(\beta + \nu)}.
\]

The slope intercept \( n_o [m^{-3}(\beta + \nu)] \) can be diagnosed from \( N_i \) and \( D_n \) with the equation from, for example, Ferrier (1994):

\[
n_o = N_i/D_n^\nu.
\]

The sixth moment, or the reflectivity factor \( Z \) (for Rayleigh scatterers), for spheres is defined as

\[
Z = \int_0^\infty D^3 n(D) \, dD = \frac{N_i \Gamma(2\beta + \nu)}{\Gamma(\nu) D_n^{2\beta}}.
\]

Relating the time rate of change of the third moment (3) with the rate of change of the sixth moment (6) results in an equation (Srivastava 1978; Passarelli 1978; Mitchell 1994) that can be used to derive a predictive equation for \( D_n \), which is only used in scheme E:

\[
\frac{dD_n}{dt} = \frac{D_n}{\beta q_x} \left( \frac{2\Gamma(\beta + \nu + \delta) \Gamma(\beta + \nu)}{\Gamma(\nu + \delta) \Gamma(2\beta + \nu)} - 1 \right) \frac{\partial q_x}{\partial t}.
\]

Now we seek an equation that is valid in predicting \( N_i \) for all schemes considered. Because all schemes use the same definition for \( q \) [that in (3)], a diagnostic expression for \( N_i \) can be derived. It can be found by by solving for \( N_i \) in (3), which results in

<table>
<thead>
<tr>
<th>Scheme</th>
<th>No. of predicted moments</th>
<th>Water content</th>
<th>Intercept</th>
<th>Slope</th>
<th>Number concentration</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>P</td>
<td>C</td>
<td></td>
<td></td>
<td>LFO; Rutledge and Hobbs (1983); Walko et al. (1995); Gilmore et al. (2004a,b)</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>P</td>
<td>C</td>
<td></td>
<td></td>
<td>Not used to our knowledge</td>
</tr>
<tr>
<td>D</td>
<td>2</td>
<td>P</td>
<td>P</td>
<td></td>
<td></td>
<td>Not used to our knowledge</td>
</tr>
<tr>
<td>E</td>
<td>2</td>
<td>P</td>
<td>P</td>
<td></td>
<td></td>
<td>Passarelli (1978); Srivastava (1979); Mitchell (1994)</td>
</tr>
<tr>
<td>F</td>
<td>2</td>
<td>P</td>
<td>P</td>
<td></td>
<td></td>
<td>Meyers et al. (1997); Ferrier (1994); Reisner et al. (1998); Ziegler (1985); Verlinde and Cotton(1993); Koenig and Murray (1976)</td>
</tr>
</tbody>
</table>

Table 1. Variables predicted and specified as constant in microphysical parameterizations and studies in which they were used; C indicates constant, P indicates predicted, and blank spaces are floating variables that are or can be diagnosed.
\[ N_i = \frac{q \rho_i \Gamma(v)}{\alpha \Gamma(\beta + v) D_n^b}, \]

which is the total number concentration of particles in a gamma distribution and has the same definition for all four schemes analyzed herein. To show the mathematical conservation of \( N_i \) or lack thereof, we do the following calculation. Because all schemes use the same definition of \( q \), differentiating (8) results in a predictive expression for \( N_i \) that is valid for all schemes. Thus, the partial derivative of (8) with respect to time \( t \) is taken, and the following predictive equation is found:

\[
\frac{\partial N_i}{\partial t} = \frac{\rho_i \Gamma(v)}{\alpha \Gamma(\beta + v)} \frac{\partial}{\partial t} \left( \frac{q}{D_n^b} \right) .
\]

Expanding the derivative and factoring \((D_n^b)/q\) on the right-hand side of (9) gives

\[
\frac{\partial N_i}{\partial t} = \frac{\rho_i \Gamma(v)}{\alpha \Gamma(\beta + v)} \frac{q}{D_n^b} \left( \frac{1}{q} \frac{\partial q}{\partial t} - \frac{\beta}{D_n} \frac{\partial D_n}{\partial t} \right) .
\]

Conservation of \( N_i \) requires the right-hand side of (10) to equal zero for any of the four schemes considered (and this requirement is equally valid for any other bulk scheme that predicts \( q \)). Thus, the terms in parentheses in (10) must cancel for mathematical conservation, because none of the terms in front of the parentheses are zero for continuous collection growth. By assuming finite and nonzero \( q \) and \( D_n \), which are consistent with continuous collection growth, a conservation condition for \( N_i \) may be written from (10) as

\[
\frac{1}{q} \frac{\partial q}{\partial t} = \frac{\beta}{D_n} \frac{\partial D_n}{\partial t}
\]

Term I = Term II. (11)

An alternative way of stating (11) is that the conservation of \( N_i \) for any bulk microphysics scheme that predicts \( q \) requires that the time rates of change of \( \ln(q) \) and \( \beta \ln(D_n) \) must be equal. Equations (10) and (11) allow the examination of the factors that control the \( N_i \) conservation as in scheme F (Cohard and Pinty 2000) or nonconservation as in schemes A, B, and E.

The following equation is valid for any bulk scheme in which a gamma distribution of solid, spherical ice particles (e.g., graupel) are accreting monodispersed cloud droplets (Lin et al. 1983, hereinafter LFO):

\[
\frac{\partial q_c}{\partial t} = \frac{\pi E_{\lambda\nu} N_{\nu} q_c D_n^{2.5} \Gamma(2.5 + \nu)}{4 \Gamma(\nu)} \left( \frac{4 \rho_i g}{3 \rho_i C_{d,\lambda\nu}} \right)^{0.5} .
\]

The last predictive equation for scheme F is

\[
dN_i/dt = 0.
\]

### Table 2. Variables used in each of the four schemes A–F with the associated equation in parentheses by which they are computed, where \( P \) = predicted, \( D \) = diagnosed, and \( S \) = specified as constant. The qualitative change in these terms with time is indicated by an arrow. In some cases there are horizontal arrowheads that indicate the values are constant in time. In addition to the equations in parentheses that are used to solve for the variables, in some cases these values are compared with the values obtained using either a simple finite difference (FD) between \( t = 0 \) s and \( t = 600 \) s or through the use of (10), and these are labeled E = examined. The finite difference is calculated using \( \Delta N_i/\Delta t \sim (D_{n+1}^b - D_n^b)/\Delta t \).

<table>
<thead>
<tr>
<th>Scheme</th>
<th>( q )</th>
<th>( N_i )</th>
<th>( D_n )</th>
<th>( n_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>P (12)↑</td>
<td>D (8)↑</td>
<td>D (4)↑</td>
<td>S tighten</td>
</tr>
<tr>
<td>E</td>
<td>P (12)↑</td>
<td>D (8)↑</td>
<td>E (10)</td>
<td>D (5)↓</td>
</tr>
<tr>
<td>F</td>
<td>P (12)↑</td>
<td>P (13)↑</td>
<td>D (4)↑</td>
<td>D (5)↓</td>
</tr>
</tbody>
</table>

For each of the four schemes, the time derivatives for the predictive equations in (7), (12), and (13) are evaluated with forward-in-time differences with a time step of 1 s to permit accurate solutions. The diagnostic equations or constant values are simply evaluated at a given time level. The following initial conditions are used for all of the schemes in the integrations of (12) and (13):

1) \( q_c = 1 \times 10^{-3} \) kg kg\(^{-1}\) as the initial value for graupel mixing ratio, and 2) \( N_i = 1000 \) m\(^{-3}\) as the initial value of total number concentration for graupel particles.

The following set of parameters is specified as constants in time and space for all the schemes:

1) \( \beta = 3 \) for a sphere \( \nu = 3 \) for the distribution shape parameter,
2) \( E_{\lambda\nu} = 0.55 \) is the collection efficiency,
3) \( \rho_i = 900 \) kg m\(^{-3}\) is the density of graupel,
4) \( C_d = 0.60 \) is the drag coefficient for graupel (12),
5) \( g = 9.8 \) m s\(^{-2}\) is gravity,
6) \( q_c = 1.0 \times 10^{-3} \) kg kg\(^{-1}\) is the constant value of mixing ratio for cloud water, and
7) \( \delta = 2.5 \) in (12) for continuous collection growth.

Using these initial conditions, results are now mathematically and graphically compared for each of the four schemes. Later, comparisons will be made with a bin model and discussed.

3. Results

Table 2 shows the variables predicted, diagnosed, and/or specified for each scheme and the corresponding
equation numbers in parentheses used to obtain the solutions. In addition, the nature of the changes of these variables over the integration period, 600 s, is denoted by an arrow. A horizontal arrow means the value is constant in time.

Figure 1 shows time series up to 600 s of integration for each of the four variables $N_t$, $q$, $D_n$, and $n_o$ for each of the four bulk schemes. The time series of $N_t$ for each of the schemes up to $t = 600$ s is given in Fig. 1a. From Fig. 1a, it is clear that scheme F is the only scheme that conserves $N_t$. The next best is scheme E, followed closely by scheme A, though both are nonconservative for $N_t$. The results of scheme B are very nonconservative for $N_t$. Thus schemes A, E, and, especially, B artificially introduce hydrometeor particles for a process in which $N_t$ should be conserved.

The time series of $q$ for each scheme are shown in Fig. 1b. In all cases the curves are of similar shape and $q$ increases by 5–6 times its original value. The similar shape of the curves is consistent with the fact that (12) has been used for all the schemes to obtain $q$ (Table 2). Time series of $D_n$ are shown in Fig. 1c, in which the characteristic diameter increases for schemes A, E, and F but does so most strongly for scheme F. For scheme B, $D_n$ is specified as constant and thus does not increase, despite the fact that the process is one of growth in size of graupel particles. Figure 1c thus demonstrates a physical inconsistency in the design of scheme B for continuous collection growth or breakup, because the mean diameter $vD_n$ cannot change in time.

Figure 1d shows the slope intercept $n_o$ versus time. Scheme A includes the specification of $n_o$ as constant (Fig. 1d), which again is a physical inconsistency, because all particles increase in size during growth and thus $n_o$ should decrease with time. For schemes E and F, $n_o$ does indeed decrease with time. However, for
scheme B, \( n_0 \) increases rapidly with time according to (4), because \( D_n \) is held constant (Fig. 1c and Table 2) and \( N_i \) increases rapidly (Fig. 1a). The greatest decrease of \( n_o \) with time occurs for scheme F (Fig. 1d). As the graupel particles get larger overall, they do so more rapidly than with any of the other schemes (Fig. 1c).

Using (4) and (5), this behavior of \( n_o \) is consistent mathematically, with constant \( N_i \) with time (Fig. 1a), increasing \( q \) with time (Fig. 1b), and a value of \( 1/D_n^o \) that is decreasing with time, as \( D_n \) gets larger with time (Fig. 1c).

In summary, Fig. 1a shows that scheme F is the only one to conserve \( N_i \) and that scheme B exhibits very unphysical results with regard to the conservation of \( N_i \). The performances of schemes A and E with regard to \( N_i \) conservation are intermediate to schemes B and F but are closer to F than to B.

The departure of \( N_i \) from conservation can be quantified for this set of parameters and initial conditions, by calculating the relative differences (RD) from scheme F of each variable \( (N_i, q, D_n, \text{and } n_o) \), denoted by \( y \), at 600 s using

\[
RD = 100 \left( \frac{y_{\text{scheme-X}} - y_{\text{scheme-F}}}{y_{\text{scheme-F}}} \right).
\]

where scheme X is any of schemes A, B, or E. The relative differences are shown in Table 3 for each of the four variables for each scheme. It is clear that scheme B has the most important differences from scheme F for all of the variables. Schemes A and E demonstrate non-conservation of \( N_i \), though also with significant relative differences from scheme F (Table 3), but perform better than scheme B.

### 4. Discussion

The conservation properties of \( N_i \) of all of the schemes are evaluated using (10). The value of \( N_i \) resulting from the integration of (10) holds, within truncation error, for all of the schemes, including scheme F, where \( N_i \) is conserved. This means that for the schemes in which \( N_i \) is diagnosed at each time step, the final \( N_i \) will give the same value as integrating (10). As far as the authors can tell, (10) can be applied to any one- or two-moment scheme and in that sense seems to be general.

To facilitate the mathematical demonstration of why scheme F conserves \( N_i \) and schemes A, B, and E do not conserve \( N_i \), the terms of (10) are decomposed into terms I and II as noted above for (11), and the conservation of \( N_i \) is evaluated using the difference of the terms I - II. If the difference is greater than zero, then the value of \( N_i \) will unrealistically increase with time, whereas if the difference is less than zero, then the value of \( N_i \) will unrealistically decrease with time. Only when the terms cancel each other is \( N_i \) conserved.

Next the detailed behavior of terms I and II (Figs. 2–5) for scheme F is examined to demonstrate why \( N_i \) is mathematically conserved. Then schemes A, B, and E are examined to see why \( N_i \) is not conserved mathematically (Figs. 2–5). Note that in the discussion below, each of the subparts \((1/q, \partial q/\partial t, \beta/D_n, \text{and } \partial D_n/\partial t)\) of terms in I and II are always positive.

#### a. Scheme F

Upon examination of term I and its subparts, it is found that, whereas \( 1/q \) gets smaller with time (Fig. 2a), \( \partial q/\partial t \) increases with time (Fig. 2b). This latter term increase is because of the increase in \( D_n \), because \( N_i \) is constant with time in accordance with (12) and (13). The term \( 1/q \) in scheme F is the largest of any of the other schemes for all times (Fig. 2a), which implies that \( q \) is the smallest for scheme F at all times, and this is consistent with Fig. 1b. This occurs even though \( D_n \) is the largest for scheme F. The reason for this behavior is that as \( q \) increases with time, \( N_i \) does not change with time; thus by (4), \( D_n \) increases the most rapidly with scheme F (Fig. 2c). In term II for scheme F, the subpart \( \beta/D_n \) (Fig. 3a) decreases with time and \( \partial D_n/\partial t \) (Fig. 3b) gets larger with time.

The net behavior for term I (Fig. 4a) is a decrease with time because the term \( 1/q \) dominates even though \( \partial q/\partial t \) increases with time. In addition, the net behavior for term II (Fig. 4b) is that it decreases with time because the subpart \( \beta/D_n \) dominates again over \( \partial D_n/\partial t \).

Note that the amplitudes and the rates of decrease with time of terms I and II are approximately equal (Figs. 4a,b). The result is such that terms I - II \( \approx 0 \), which approximately satisfies the necessary condition (11) for \( N_i \) conservation (Fig. 5). In the actual calculation of (10), the slight departure from conservation is less than 0.2\% and results from calculations using numbers with vast differences in order of magnitude.
b. Scheme A

Term I for scheme A behaves similarly to scheme F (Fig. 4a), though term I in scheme A decreases more slowly with time and thus is larger at 600 s. This is a result of $\frac{1}{q}$ being larger and increasing more quickly in scheme A than in scheme F (Fig. 2b), and that $\frac{1}{q}$ decreases at about the same rate and is about the same amplitude with both schemes A and F (Fig. 2a). Term II decreases with scheme A (Fig. 4b), like term II with scheme F, though it is smaller in magnitude than with scheme F. The reason for the increasing term $\frac{1}{q}$ is that the amplitude of the decreasing term $\beta D_n$ (Fig. 3a), which decreases with time but not as much as for scheme F. As a result of these calculations, it is found that $I - II > 0$ (Fig. 5), which can also be seen in (Fig. 1a) where $N_t$ nearly doubles its initial value in 600 s. Therefore, except for scheme B, scheme A performs most poorly with regard to the conservation of $N_t$ relative to scheme F. The relative difference (14) of scheme A relative to scheme F is 137% (Table 3). The large difference is due to the physically inconsistent specification of a constant $n_o$ for the continuous collection growth process.

c. Scheme B

Examining scheme B exposes some interesting differences that make this scheme perform very poorly with regard to the conservation of $N_t$ (Fig. 1a and Table...
2). First, $1/q$, in term I for scheme B is slightly smaller than in all the other schemes (Fig. 2a). This results from $q$ being a bit larger than with any other scheme (Fig. 1b). This does not result from the constant $D_n$ being large in (4); rather it is too small (Fig. 1c) and there are very large values of $N_I$ with scheme B. The values of $\partial q/\partial t$ in term I (Fig. 2b) are much larger than in any of the other schemes. Again, this is not a direct result of the smaller values of $D_n$ but rather of much larger values of $N_I$ which are indirectly forced by the small, specified constant $D_n$ (8). Term II (Fig. 4b) simply is zero because $D_n$ is constant with time, that is, $\partial D_n/\partial t = 0$ (Fig. 3b). This is the physical basis for nonconservation with scheme B. That is, the characteristic diameter is not allowed to increase during a growth process with increasing $q$, and this results in $N_I$ being forced to be very large according to (8) and (10), because term II (which is zero) cannot possibly offset term I (Fig. 4).

Thus, terms $I - II > 0$ (Fig. 5). Note, however, that $\partial q/\partial t$ is larger with scheme B than with any other scheme (Fig. 3a). This is due again to $D_n$ being too small and specified as a constant. Last, $n_o$ also is increasing (Fig. 1d) rapidly with time with this scheme, when it should be decreasing physically like scheme F. This is a result of rapidly increasing values of $N_I$ and a small but constant $D_n$ in (5). The result of a diagnosis using (8) and prediction using (10) gives an $N_I$ that has a relative difference of more than 500% from scheme F by 600 s (Table 3). Therefore, this scheme has the largest relative differences in comparison with scheme F of any of the schemes (Table 3) and is the most unphysical, because it forces unrealistic production of particles for a process in which there should not be any production of new particles as well as an unphysical behavior for $D_n$ and $n_o$ for this growth process.

d. Scheme E

Last, scheme E is the second-best-performing scheme with regard to $N_I$ conservation. Term $I$ is larger than in scheme F at 600 s (Fig. 4a), and at 600 s term II is smaller than in scheme F (Fig. 4b). In term I, $1/q$ decreases slightly more with scheme E than with scheme F (Fig. 2a). This means that $\partial q/\partial t$ must increase more than with scheme F (Fig. 2b), which is indeed the case. With scheme E, $D_n$ has the second largest value of all the schemes (Fig. 1c), and so $\partial D_n/\partial t$ is closer to scheme F than any other scheme (Fig. 3a). With scheme E, term II decreases with time as a result of $\partial q/\partial t$ decreasing more rapidly (Fig. 3a) than $\partial D_n/\partial t$ is increasing (Fig. 3b). The end result is then that terms $I - II > 0$ (Fig. 5), which makes the prediction of $N_I$ with (11) nonconservative, and the relative difference using (14) is $>100\%$ (Table 3).
In general, term II changes more with time than term I for all of the schemes, except for scheme B for which terms I and II are approximately constant (Fig. 4). The time series of term I − term II (Fig. 5), as calculated from (11) for each scheme, are consistent with the results in Tables 2 and 3 and Fig. 1, with regard to the conservation of $N_r$. Scheme B has the worst performance while schemes A and E have intermediate performance. The difference of term I − term II is positive for schemes A, B, and E, which indicates that particles are artificially added during this continuous collection growth process in which $N_r$ should physically be conserved.

5. Comparison of size distributions generated by each scheme with each other and with a continuous growth bin model

In this section the size distribution is compared for each scheme as a function of time to see its behavior and to check for physical consistency. Each scheme is also compared with a continuous growth bin model, and some unexpected results occur. Note that in the case of continuous collection growth, the bin model represents the approximate analytic solution.

First the evolution of the number concentration per unit length as a function of diameter (2) is plotted for each solution from $t = 0$, 300, and 600 s. Scheme A shows behavior consistent with that expected, with the peak sliding slightly to larger sizes and toward higher number concentrations (see direction of arrow in Fig. 6a; although it is admitted that in such schemes, individual particles are not tracked so that there are multiple possibilities for the direction of the arrows, depending on the scheme). Scheme B shows that the peak remains at a constant size but the number concentration of all sizes increases (Fig. 6b). Scheme E (Fig. 6c) shows particularly unusual behavior, with the sizes smaller than the peak generally getting even smaller with time. In addition, and correctly so, the number of larger particles increases. However, the peak number concentration and associated diameter increase with time as in schemes A and, in particular, B. Upon examination of scheme F (Fig. 6d), mostly correct behavior is found, with number of smaller particles decreasing with time and number of larger particles increasing with time. Unexpected, however, is that the peak in number concentration decreases as the associated diameter increases with time when it should be constant if a given size of particle is to maintain its concentration over the interval from $D$ to $D + dD$. It appears that what is happening is that there are far too many large particles with scheme F because of the constraint to conserve number concentration with time integration of the continuous collection equation. Perhaps the stringent condition of a constant shape parameter $\nu$ or lack of additional shape parameters in (2) is causing the error (and causing errors in the other schemes as well).

Next the distributions are compared with results from a continuous collection growth bin model, because some serious problems have been identified with all of the schemes and, most unexpected, with scheme F: the one that appeared to perform best in S05. The bin model should preserve the peak amplitude of number concentration in the size distribution as larger-sized particles are produced if number concentration is to be conserved for continuous collection growth. This is because all particles within a bin grow at the same rate, and therefore there are always the same number of particles at a later time (just with larger $D$). This is well documented in the simulation with the bin model (Fig. 7). In addition, small-sized particles grow slowly while the larger-sized particles grow more quickly.

In the comparison of the continuous collection growth bin model with the all of the other schemes (Fig. 8), it is seen that scheme B underpredicts the number of larger-sized particles, scheme A and E reasonably well predict the larger sizes, and scheme F overpredicts the size and number of larger-sized particles. Relative to the initial time, at smaller sizes ($< \sim 2$ mm), the diameter of the bin-model peak number concentration is associated with a growing diameter that increases faster than all the bulk parameterization schemes. All parameterized schemes have too many small-sized particles, which account for much of the mass in these schemes. For scheme F, it is found that there are too many small particles from sizes ranging from about 0.8 to 4 mm in diameter. It is apparent from these plots that it seems that the shape parameter $\nu$, representing what might be a gamma distribution for the bin model, changes not only with time but also with particle size. This is a very difficult problem to overcome. It also seems apparent that if the shape parameters for the other schemes were allowed to change with time the distributions might be more accurate. However, testing this is beyond the scope of this paper.

It is a limitation of this study that the process of collection growth and total number concentration conservation are the only processes considered. There are other processes that might be expected to conserve some of the other moments better with some of the other schemes, though these have not been examined yet. Also, it is worth mentioning that some authors have deliberately violated $N_r$ conservation in their budgets to preserve other moments such as reflectivity (e.g., Fier 1994) or to preserve the predominant diameter of a
distribution (Carrio and Nicolini 1999). Thus, recommendations herein are limited to $N_t$ conservation for collection growth. Last, the results cannot be used to advocate the use of scheme F for all processes.

6. Summary

By dissecting the terms in (10) or, equivalent, (11), which is a conservation condition for $N_t$ based on the partial time derivative of the definition of the third moment, it is possible to explain why only scheme F, the two-moment scheme that predicts $q$ and $N_t$, is conservative for $N_t$ when it should be for certain microphysical processes like continuous collection growth [and vapor diffusion growth (not shown; see S05)].

Equations (10) and (11) also explain mathematically why the other schemes are nonconservative. The as-

![Image](https://example.com/image.png)

Fig. 6. Log-log time sequences of $N$ per unit size ($m^{-4}$) interval as a function of diameter $D$ ($\mu m$) for continuous collection growth for schemes (a) A, (b) B, (c) E, and (d) F.
assumption of a constant \( n_a \) in scheme A is physically inconsistent with the continuous collection growth process, as is the assumption of a constant \( D_n \) for scheme B. Scheme E is also nonconservative, but it seems this result is not a result of a physically inconsistent specification, but rather the scheme’s equations simply do not satisfy \( N_t \) conservation because \( N_t \) does not come into the derivation. Even scheme F, which perfectly conserves \( N_t \), does not preserve the size distribution shape in comparison with a bin model.

Comparison of a continuous collection growth bin-model solution with all four schemes complicates the problem. It is found that none of the schemes agree well with the bin model at smaller-sized particles (<2 mm), and the bin model’s peak number concentration occurs at a greater size diameter than in any of the schemes. However, it does unexpectedly agree very well with schemes A and E at larger-sized diameters (2–7 mm). A weakness was discovered with scheme F in that it does not preserve the diameter at which there is peak number concentration as a function of time. This is an interesting and not previously revealed result of this scheme (in the literature, to our knowledge). In addition, scheme F overproduces the number of the smallest particles (because of the constant \( D_n \)).

In summary, we feel it is crucial that numerical model users be aware of the physical behavior of these schemes so that they may properly interpret their model output when simulating these and other similar microphysical processes using bulk parameterizations.

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