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Radial Deflection of Ring-Stiffened Cylinders Under Hydrostatic Pressure

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Ring-stiffened cylinders under hydrostatic pressure have been analyzed for more than a century. The contribution of Salerno and Pulos in the fifties of the previous century can be considered as a standard that nowadays still can be found in the rules and guidelines of classification societies. Their work comprises stresses and deformation in a perfectly circular cylindrical shell with ring frames, and solutions are presented midbay between the frames and at frame position. It is remarkable that the formula for the radial deflection midbay is much simpler than the one for frame position. This brief note shows a significant simplification of the latter one. [DOI: 10.1115/1.4066325]

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1 Introduction

Salerno and Pulos in their seminal paper [1] solve the problem of the axisymmetric ring-stiffened cylinder under hydrostatic load by first solving the fourth-order differential equation for the radial deflections. They were the first to include the beam-column effect. However, in the derivation [1], (Appendix A) they seem to introduce the effect twice but still arrive at the correct differential equation. This is rather confusing.

They rightly assume that normally their readers are only interested in the stresses, so while they present their expression for radial deflections [1], (Eqs. (62) and (63)), the emphasis is to arrive at explicit expressions for the critical axial and hoop membrane stresses [1], (Eqs. (64) and (65)). However, when working backward from the latter, to arrive at the radial deflection, one discovers an anomaly in the radial deflection at the frame location in that the equation found is much simpler than shown in Ref. [1], (Eq. (62)), raising the question whether both results are equal or not.

In order to avoid future confusion both points will be cleared up in this brief technical note. In particular, the simple equation for radial deflection at frame location could find its way to the rules and guidelines of classification societies and probably the software code of designers.

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2 Governing Equation

In the derivation of the fourth-order differential equation, Salerno and Pulos introduced the beam-column effect directly in the radial force equilibrium [1], (Eq. (A1)). However, when deriving moment equilibrium, they introduced the beam-column effect again [1], (Eq. (A3)) and here also made a sign error.

Figure 1(*a*) is a derivative of Fig. 15 in Ref. [1] and shows the forces per unit length acting on the shell with length dx.

Essential for this approach is the rotation of the shell. The sides of the shell part remain perpendicular to the deformed shell. The small difference in angle between the two sides of the element dx causes a small resultant of the normal force N_x in the radial direction.

The situation in Fig. 1(*a*) has an alternative by considering the element dx with sides in the radial direction. This is illustrated in Fig. 1(*b*). In this situation, the axial force N_x does not contribute to the equilibrium of forces in the radial direction [1], (Eq. (A1)). However, the force N_x has a lever and contributes to the equilibrium of moments [1], (Eq. (A3)), but with a sign error.

Both options are valid, but not simultaneously. For a straight beam column, the difference is explained by Timoshenko and Gere [3]. In their final result [1], Eqs. (A5) and (A12), Salerno and Pulos seem to have ignored the beam-column effect in the moment equilibrium and therefore arrived at the correct result.

3 Radial Deflection as Published

The radial deflection of the ring-stiffened shell is presented by Salerno and Pulos [1], (Eq. (61)) as a function over the length $w(x)$. The origin is located between the frames midbay $x=0$, and at frame position $x=\pm L/2$.

3.1 Radial Deflection Midbay. With $x=0$ follows from Ref. [1], Eq. (61) the radial deflection midbay

$$w_m = -\frac{p \cdot R^2}{E \cdot h} \cdot \left(1 - \frac{\nu}{2}\right) \cdot \left\{1 - \frac{\alpha \cdot F_2}{\alpha + \beta + (1-\beta) \cdot F_1}\right\} \tag{1}$$

The load and geometry functions F_1 and F_2 are defined by

$$F_1 = \frac{4}{\theta} \cdot \frac{\cosh^2(\eta_1 \cdot \theta) - \cos^2(\eta_2 \cdot \theta)}{\dfrac{\cosh(\eta_1 \cdot \theta) \cdot \sinh(\eta_1 \cdot \theta)}{\eta_1} + \dfrac{\cos(\eta_2 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2}} \tag{2}$$

$$F_2 = \frac{\dfrac{\cosh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2} + \dfrac{\sinh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_1}}{\dfrac{\cosh(\eta_1 \cdot \theta) \cdot \sinh(\eta_1 \cdot \theta)}{\eta_1} + \dfrac{\cos(\eta_2 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2}} \tag{3}$$

Salerno and Pulos presented this deflection in Ref. [1], (Eq. (62)), however, it must be noted that their formula contains a typing error: the {1 −} is missing in the equation.

Pulos introduced a short-hand notation [2]

$$a \equiv \left(1 - \frac{\nu}{2}\right) \cdot \frac{\alpha}{\alpha + \beta + (1-\beta) \cdot F_1} \tag{4}$$

This simplifies the deflection in Eq. (1) to

$$w_m = -\frac{p \cdot R^2}{E \cdot h} \cdot \left\{\left(1 - \frac{\nu}{2}\right) - a \cdot F_2\right\} \tag{5}$$

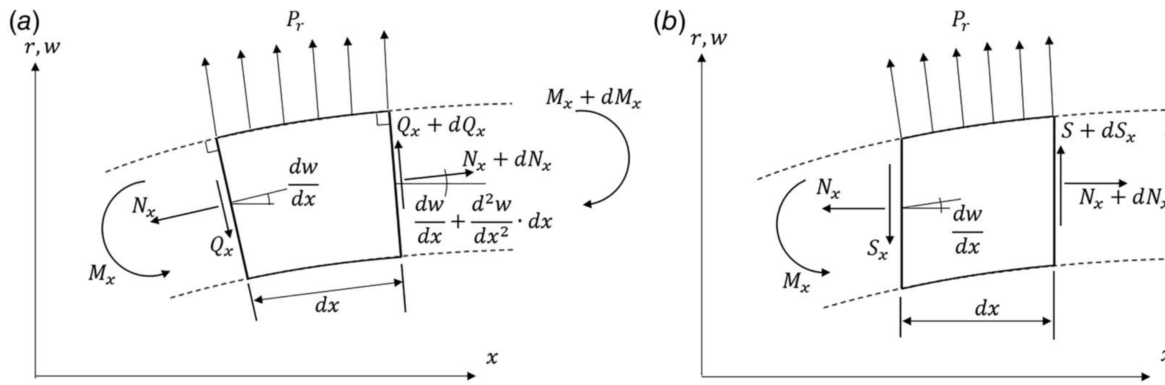

Fig. 1 The effect of axial pressure: (*a*) sides perpendicular to the shell and (*b*) sides in the radial direction

3.2 Radial Deflection at Frame Location. The radial deflection at frame location $x = \pm L/2$ is given, with the short-hand notation in Eq. (4), by

$$w_f = -\frac{p \cdot R^2}{E \cdot h} \cdot \left\{ \left(1 - \frac{\nu}{2}\right) - a \cdot F_2 \cdot C_{ADD} \right\} \tag{6}$$

The additional coefficient C_{ADD} is defined by

$$C_{ADD} = \cosh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta) + \frac{\sqrt{\frac{0.91}{3}} \cdot \frac{F_4}{F_2} + \gamma}{4 \cdot \eta_1 \cdot \eta_2} \cdot \sinh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta) \tag{7}$$

and the function F_4 in C_{ADD} is given by

$$F_4 = \sqrt{\frac{3}{0.91}} \cdot \frac{\frac{\cosh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2} - \frac{\sinh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_1}}{\frac{\cosh(\eta_1 \cdot \theta) \cdot \sinh(\eta_1 \cdot \theta)}{\eta_1} + \frac{\cos(\eta_2 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2}} \tag{8}$$

The radial deflection in Eq. (6) was presented in Ref. [1], (Eq. (63)), and it must be noticed that the radial deflection at frame location is far more complicated than the one midbay (5).

An alternative approach to obtain the radial deflection at frame location is based on the stresses. The membrane hoop stress was presented by Salerno and Pulos in Ref. [1], (Eqs. (67) and (69))

$$\frac{\sigma_\theta}{\sigma_u} = 1 - \left\{ \left(1 - \frac{\nu}{2}\right) \cdot \frac{\alpha}{\alpha + \beta + (1 - \beta) \cdot F_1} \right\} \tag{9}$$

The basic hoop stress σ_u is given by the stress for unstiffened cylinders

$$\sigma_u = -\frac{p \cdot R}{h} \tag{10}$$

With the short-hand notation presented in Eq. (4), the hoop stress is presented by

$$\frac{\sigma_\theta}{\sigma_u} = 1 - a \ \rightarrow \ \sigma_\theta = -\frac{p \cdot R}{h} \cdot (1 - a) \tag{11}$$

The shell has a biaxial stress state with an axial membrane stress given by the basic pressure vessel formula

$$\sigma_{ax} = -\frac{p \cdot R}{2 \cdot h} \tag{12}$$

The hoop strain in combination with Hooke's law gives

$$\varepsilon_\theta = \frac{w_f}{R} = \frac{1}{E} \cdot (\sigma_\theta - \nu \cdot \sigma_{ax}) \tag{13}$$

With Eqs. (11) and (12) follows for the radial deflection:

$$w_f = \frac{R}{E} \cdot (\sigma_\theta - \nu \cdot \sigma_{ax}) = -\left\{ \frac{p \cdot R^2}{E \cdot h} \cdot (1 - a) - \nu \cdot \frac{p \cdot R^2}{2 \cdot E \cdot h} \right\}$$
$$= -\frac{p \cdot R^2}{E \cdot h} \cdot \left(\left(1 - \frac{\nu}{2}\right) - a \right) \tag{14}$$

Comparing this radial deflection to the one presented in Eq. (6) reveals that the latter is unnecessarily complicated since

$$F_2 \cdot C_{ADD} = 1 \tag{15}$$

The mathematical proof of this finding is presented in the next section.

4 Some Algebraic Processing

For reasons of simplification the parameters F_2 [1], ((Eq. (73)) and F_4 [1], ((Eq. (75)) are rewritten to

$$F_2 = \frac{\frac{\cosh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2} + \frac{\sinh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_1}}{\frac{\cosh(\eta_1 \cdot \theta) \cdot \sinh(\eta_1 \cdot \theta)}{\eta_1} + \frac{\cos(\eta_2 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2}}$$
$$\equiv \frac{A_{24} + B_{24}}{Denom} \tag{16}$$

$$F_4 = \sqrt{\frac{3}{0.91}} \cdot \frac{\frac{\cosh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2} - \frac{\sinh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_1}}{\frac{\cosh(\eta_1 \cdot \theta) \cdot \sinh(\eta_1 \cdot \theta)}{\eta_1} + \frac{\cos(\eta_2 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2}}$$
$$\equiv \sqrt{\frac{3}{0.91}} \cdot \frac{A_{24} - B_{24}}{Denom} \tag{17}$$

Note: Salerno and Pulos seem to have assumed a Poisson's ratio $\nu = 0.3$ to arrive at $\sqrt{1 - \nu^2} = 0.91$.

With

$$A_{24} \equiv \frac{\cosh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2}, \quad B_{24} \equiv \frac{\sinh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_1} \tag{18}$$

All the load and geometry functions have a common denominator

$$Denom \equiv \frac{\cosh(\eta_1 \cdot \theta) \cdot \sinh(\eta_1 \cdot \theta)}{\eta_1} + \frac{\cos(\eta_2 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2} \tag{19}$$

It is worthwhile to consider the product in Ref. [1], (Eq. (63)) first

$$F_2 \cdot \frac{\sqrt{\frac{0.91}{3} \cdot \frac{F_4}{F_2}} + \gamma}{4 \cdot \eta_1 \cdot \eta_2} = \frac{\sqrt{\frac{0.91}{3}} F_4 + \gamma \cdot F_2}{4 \cdot \eta_1 \cdot \eta_2} = \frac{\sqrt{\frac{0.91}{3}} \cdot \sqrt{\frac{3}{0.91}} \cdot (A_{24} - B_{24}) + \gamma \cdot (A_{24} + B_{24})}{4 \cdot \eta_1 \cdot \eta_2 \cdot Denom} = \frac{A_{24} \cdot (1+\gamma) - B_{24} \cdot (1-\gamma)}{4 \cdot \eta_1 \cdot \eta_2 \cdot Denom} \quad (20)$$

The parameters for the beam-column effect, η_1 and η_2 are rewritten to

$$\eta_1 = \frac{1}{2}\sqrt{1-\gamma} \;\rightarrow\; (1-\gamma) = 4 \cdot \eta_1^2$$
$$\eta_2 = \frac{1}{2}\sqrt{1+\gamma} \;\rightarrow\; (1+\gamma) = 4 \cdot \eta_2^2$$

Application in Eq. (20) gives

$$F_2 \cdot \frac{\sqrt{\frac{0.91}{3} \cdot \frac{F_4}{F_2}} + \gamma}{4 \cdot \eta_1 \cdot \eta_2} = \frac{A_{24} \cdot 4 \cdot \eta_2^2 - B_{24} \cdot 4 \cdot \eta_1^2}{4 \cdot \eta_1 \cdot \eta_2 \cdot Denom} = \frac{1}{Denom} \cdot \left(A_{24} \cdot \frac{\eta_2}{\eta_1} - B_{24} \cdot \frac{\eta_1}{\eta_2}\right) \quad (21)$$

With the simplifications presented in the Eqs. (16), (17), and (21) follows:

$$F_2 \cdot C_{ADD} = \frac{(A_{24} + B_{24}) \cdot \cosh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta) + \left(A_{24} \cdot \frac{\eta_2}{\eta_1} - B_{24} \cdot \frac{\eta_1}{\eta_2}\right) \cdot \sinh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{Denom} \quad (22)$$

The numerator in Eq. (22) consists of four parts and application of the definition for A_{24} and B_{24} gives for

The first part containing A_{24}

$$\frac{\cosh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2} \cdot \cosh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta) = \frac{\cosh^2(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_2} \quad (23)$$

The second part containing B_{24}

$$\frac{\sinh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_1} \cdot \cosh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta) = \frac{\sinh(\eta_1 \cdot \theta) \cdot \cosh(\eta_1 \cdot \theta) \cdot \cos^2(\eta_2 \cdot \theta)}{\eta_1} \quad (24)$$

The third part containing A_{24}

$$\frac{\cosh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2} \cdot \frac{\eta_2}{\eta_1} \cdot \sinh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta) = \frac{\cosh(\eta_1 \cdot \theta) \cdot \sinh(\eta_1 \cdot \theta) \cdot \sin^2(\eta_2 \cdot \theta)}{\eta_1} \quad (25)$$

The fourth part containing B_{24}

$$-\frac{\sinh(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_1} \cdot \frac{\eta_1}{\eta_2} \cdot \sinh(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta) = -\frac{\sinh^2(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2} \quad (26)$$

Further simplification is possible when the basic equation $\sin^2(x) + \cos^2(x) = 1$ is applied to the summation of the second (24) and third part (25)

$$\frac{\sinh(\eta_1 \cdot \theta) \cdot \cosh(\eta_1 \cdot \theta) \cdot \cos^2(\eta_2 \cdot \theta)}{\eta_1} + \frac{\cosh(\eta_1 \cdot \theta) \cdot \sinh(\eta_1 \cdot \theta) \cdot \sin^2(\eta_2 \cdot \theta)}{\eta_1} = \frac{\sinh(\eta_1 \cdot \theta) \cdot \cosh(\eta_1 \cdot \theta)}{\eta_1} \quad (27)$$

With the basic equation $\cosh^2(x) - \sinh^2(x) = 1$ follows for the summation of the first (23) and the fourth part (26)

$$\frac{\cosh^2(\eta_1 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_2} - \frac{\sinh^2(\eta_1 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2} = \frac{\sin(\eta_2 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_2} \quad (28)$$

All these simplifications and the definition of the denominator (19) give for Eq. (22)

$$F_2 \cdot C_{ADD} = \frac{\frac{\sinh(\eta_1 \cdot \theta) \cdot \cosh(\eta_1 \cdot \theta)}{\eta_1} + \frac{\sin(\eta_2 \cdot \theta) \cdot \cos(\eta_2 \cdot \theta)}{\eta_2}}{\frac{\cosh(\eta_1 \cdot \theta) \cdot \sinh(\eta_1 \cdot \theta)}{\eta_1} + \frac{\cos(\eta_2 \cdot \theta) \cdot \sin(\eta_2 \cdot \theta)}{\eta_2}} = 1 \quad (29)$$

5 Revised Deflections and Conclusion

With the modifications presented above follows for the deflections:

Midbay

$$w_m = -\frac{p \cdot R^2}{E \cdot h} \cdot \left\{\left(1 - \frac{\nu}{2}\right) - a \cdot F_2\right\} \quad (30)$$

At frame location

$$w_f = -\frac{p \cdot R^2}{E \cdot h} \cdot \left\{\left(1 - \frac{\nu}{2}\right) - a\right\} \quad (31)$$

In particular, the latter expression is much simpler than the one originally published and repeated amongst others the Det Norske Veritas rules [4].

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Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

No data, models, or code were generated or used for this paper.

Nomenclature

- b = faying width of the ring frame in contact with the shell
- h = shell thickness
- p = hydrostatic pressure
- E = Young's modulus of elasticity
- R = radius to the shell mid-fiber
- A_f = actual cross-sectional area of the frame
- A_{eff} = effective cross-sectional area of the frame
- L_f = frame distance
- $L = L_f - b$ = unsupported length of the shell
- $F_1,\ F_2,\ F_4$ = load and geometry functions for the shell
- $\alpha = \frac{A_{eff}}{L_f h}$ = ratio of effective frame area to shell area
- $\beta = \frac{b}{L_f}$ = ratio of faying width to frame distance
- $\gamma \equiv \frac{p}{p^*} = \frac{p\sqrt{3(1-\nu^2)}}{2E(h/R)^2}$ = measure of the beam-column effect
- ε_φ = strain in the circumferential direction
- $\eta_1 = \frac{1}{2}\sqrt{1-\gamma}$ = parameter beam-column effect
- $\eta_2 = \frac{1}{2}\sqrt{1+\gamma}$ = parameter beam-column effect
- $\theta = \frac{\sqrt[4]{3(1-\nu^2)} \cdot L}{\sqrt{Rh}}$ = shell flexibility parameter
- ν = Poisson's ratio
- $\sigma_{\varphi M}$ = membrane stress in the circumferential direction

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