Assessment of Hydrometeor Collection Rates from Exact and Approximate Equations. Part I: A New Approximate Scheme

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ABSTRACT

The collection equation is analyzed for the case of two spherical hydrometeors with collection efficiency unity and exponential size distributions. When the fall velocities are significantly different a more general form of the conventional Wisner approximation can be formulated. The accuracy of the new formula exceeds that of the Wisner approximation for all cases considered, except for the collection of a faster species by a slower species if the amount of the faster species is relatively small compared with that of the slower species. The exact solution of the collection equation is then rederived and cast into the form of a power series involving the ratio of the two characteristic fall velocities. It is shown that the new formulation is a first-order correction to the continuous collection equation for hydrometeors with finite diameters and fall velocities. Based on the analysis, the implications for the behavior of both the exact collection equation and its representation in numerical models are discussed.

1. Introduction

The treatment of liquid water and ice microphysics by numerical models is quite complex, as the number of potential interactions increases nearly with the square of the number of species [see, e.g., the flow chart of Lin et al. (1983)]. Many of these interactions depend on computing the rate at which two different species collide, which is essentially a kinematic problem. The collection rate is the rate at which species both collide and interact, so rebounding collisions are excluded.

The rate at which a member of one species, the collected, collides with a member of another species, the collector, depends in part on the volume swept out by the collector species per unit time, moving at a velocity relative to a member of the collected species. The radius of the collector species can vary substantially at a particular location; the result is a spectrum of sweep-out volume, due to the $r^2$ dependence in the cross-sectional area of a hydrometeor and the dependence of its fall velocity on radius. Generally operational numerical models use bulk microphysics schemes, which assumes that hydrometeor radii obey a particular class of size distribution, allowing the net interaction between two species to be expressed as an integral (Lin et al. 1983; Rutledge and Hobbs 1983, 1984; Walko et al. 1995).

The form of the integral representing the collision between two hydrometeor species is unfortunately not very tractable. It has been solved exactly (Verlinde et al. 1990), but the form involves general hypergeometric functions, which, to our knowledge, have not been used in any operational numerical model due to computational expense. Such models either make physical simplifications to produce a more tractable integral (e.g., Lin et al. 1983; Rutledge and Hobbs 1983, 1984; Hodur 1997; Reisner et al. 1998) or perform the integration numerically for a range of values before the start of the simulation (Ferrier 1994; Walko et al. 1995). It is known that the particular approximation used at least in part for the first set of models (Wisner et al. 1972), hereafter referred to as the conventional Wisner approximation (CWA), has a number of biases that makes the solution inaccurate at times (Lin et al. 1983; Murakami 1990; Verlinde et al. 1990), though many details about the bias have not yet been presented in the literature. The numerical integration method is of course more accurate than any analytic approximation given a sufficiently fine resolution in diameter space. However, it is more difficult to gain physical and mathematical insight into the behavior of a numerical integration over all possible parameter space than into an appropriate analytic representation, so we feel that there is utility in seeking the latter. Furthermore, as long as physical approximations continue to be used operationally, it is worthwhile to present explanations of their behavior.

This paper will present another derivation of the exact solution to the collection equation in the form of a
power series of the characteristic fall velocities of the hydrometeors. An approximation, referred to as the general Wisner approximation (GWA), will be derived as a first-order correction to the continuous collection equation for finite-size hydrometeors. The exact solution will be compared to the two approximations (the CWA and the GWA) to show where the GWA is appropriate. Finally, implications for the numerical modeling of hydrometeor collection processes will be discussed.

2. Conventional Wisner approximation

The rate of collection of species $x$ by species $y$ is given by

$$
\frac{dr_x}{dt} = -\frac{1}{\rho_0} \int_0^\infty \int_0^\infty E_{xy} \frac{\pi(D_x + D_y)^2}{4} |v_x(D_x) - v_y(D_y)|
$$

$$
m_x(D_x)n_x(D_x)n_y(D_y)dD_x dD_y.
$$

(1)

Above, $r$ is the mixing ratio of each species, $D$ is the major diameter of an individual hydrometeor, $m(D)$ is the mass of a hydrometeor of size $D$, $v(D)$ is its terminal velocity, and $n(D)$ is its average number concentration per size interval $dD$. Finally, $E_{xy}$ is the collection efficiency.

For simplicity and because we are not considering cloud droplets here, $E_{xy}$ will be assumed to be unity. We will further assume throughout this paper that $n_x(D_x)$ and $n_y(D_y)$ have the form of the exponential distribution function

$$
n(D) = n_0 e^{-\lambda D},
$$

(2)

which has usually been observed for such hydrometeors (Gunn and Marshall 1958; Federer and Waldvogel 1975; Houze et al. 1979). The parameter $n_0$ will be treated as a constant (Marshall and Palmer 1948), though other studies have questioned the value of this assumption (Manton and Cotton 1977). In this case $\lambda$ is a variable parameter. It can be shown that $\lambda$ is the inverse of the number-weighted mean diameter, which will henceforth be known as the characteristic diameter, $D_c$. For nonspherical hydrometeors $D$ will be assumed to be the maximum diameter.\(^1\)

As previously stated, the exact solution to this equation has already been presented in the literature, using general hypergeometric functions\(^2\) (Verlinde et al. 1990; Čurić and Jane 1997). The solution is complicated by the absolute value of $(v_x - v_y)$, a quantity that changes sign within the double integration. One way to avoid this complication is to replace this term by the absolute value of the difference in mass-weighted velocities. The velocity difference can be taken outside the integral, leaving

$$
\frac{dr_x}{dt} = -\frac{n_0 \rho_0 |v_x - v_y|}{\rho_0} \int_0^\infty \int_0^\infty \pi(D_x + D_y)^2
$$

$$
m_x(D_x)e^{-\lambda_x D_x}e^{-\lambda_y D_y}dD_x dD_y.
$$

(3)

This approximation was first made by Wisner et al. (1972) and is that which we will refer to as the conventional Wisner approximation, or CWA. We will assume that $v(D) = aD^b$ and $m(D) = cD^d$, where $a$, $b$, $c$, and $k$ are constants. Then both $|v_x - v_y|$ and the integral can easily be evaluated using

$$
\int_0^\infty D^e e^{-\lambda D} dD = \frac{\Gamma(\eta + 1)}{\lambda^{n+1}}.
$$

(4)

We first define the $k$th-moment average velocity, $\bar{v}_k$, as (Srivastava 1978)

$$
\bar{v}_k = \frac{\int_0^\infty D^k n_x e^{-\lambda D} dD}{\int_0^\infty D^k n_x e^{-\lambda D} dD} = \frac{\Gamma(k + b + 1)}{\Gamma(k + 1)} v_c,
$$

(5)

where $v_c = a/\lambda^b$ is the characteristic velocity. For spherical hydrometeors the mass-weighted velocity uses $k = 3$, while $c = \pi \rho_v / 6$, where $\rho_v$ is the density of hydrometeor $x$. Integrating yields

$$
\frac{dr_x}{dt} = -\frac{n_0 \rho_0 \rho_v |v_x - v_y|}{24 \rho_v} \left[ \frac{1}{\lambda_x^3 \lambda_y^4} \left[ \Gamma(3) \Gamma(1) + 2 \Gamma(2) \Gamma(5) \left( \frac{\lambda_y}{\lambda_x} \right)^4 + \Gamma(1) \Gamma(6) \left( \frac{\lambda_y}{\lambda_x} \right)^2 \right] \right].
$$

(6)

Finally, we can use the fact that the mixing ratio is

$$
r = \frac{1}{\rho_0} \int_0^\infty cD^b n_x e^{-\lambda D} dD = \frac{n_0 \rho \Gamma(k + 1)}{\rho_0 \lambda^{k+1}}
$$

(7)

to express the collection rate as

$$
\frac{dr_x}{dt} = -\frac{r_0 \lambda_x \rho_x |v_x - v_y|}{24 \rho_v} \left[ \Gamma(3) \Gamma(1) + 2 \Gamma(2) \Gamma(5) \left( \frac{\lambda_y}{\lambda_x} \right)^4 + \Gamma(1) \Gamma(6) \left( \frac{\lambda_y}{\lambda_x} \right)^2 \right].
$$

(8)
where \( v_y \), \( v_x \) denote the third-moment average \( y \) and \( x \) velocities. We see that the fractional collection of species \( x \), \( (1/r_x) dr_x/dt \), contains only two terms dependent on the amount of species \( x \): the absolute velocity difference and \( \Delta A/\Delta y \), which approaches zero as \( r_x/r_y \) approaches zero (Fig. 1).

A major difficulty with the CWA lies with the fact that when the mass-weighted fall velocities of the two species are nearly equal, the collection rate goes to zero unrealistically (Verlinde et al. 1990; Murakami 1990; Mizuno 1990). For two species with nearly the same velocity–diameter relationship, Verlinde et al. (1990) found that the CWA was most accurate for \( D_y/D_x \approx 4 \), but overpredicts collection for larger values of this ratio and underpredicts for smaller values. For species with different velocity–diameter relationships, however, no such simple trend was apparent. This is the case that we would like to investigate here.

3. General Wisner equation

From the fall velocity curves in Fig. 2, we can see that, for hydrometeors larger than a certain threshold, the fall velocities for different species are quite distinct. Hence, in the interaction between two different species it is reasonable to designate one as the “fast” species and the other as the “slow” species, based on the behavior of their fall velocities at larger sizes. We wish to emphasize at the outset that, although it may be natural to assume that the collector species \( y \) is the fast species while the collected species \( x \) is the slow species, this is not necessarily the case, neither mathematically nor in numerical model applications. So we will introduce a distinct classification, using a subscript \( F \) to denote the fast species and \( S \) to denote the slow species. For the moment the relationship between \( F, S \) and \( x, y \) is left indeterminate. However, it can be seen that the integral within the collection equation, (1), is invariant to the interchange of \( x \) and \( y \) except for the factor \( m_x(D_x) \). Thus, for the other factors we need not know whether \( F \) corresponds to \( x \) or \( y \), and we may simply write

\[
\frac{dr_x}{dt} = -\frac{1}{\rho_0} \int_0^\infty \int_0^\infty \int_0^\infty \left( \frac{\pi(D_k + D_s)^2}{4} [v_F(D_k) - v_S(D_s)] \right) m_x(D_x)n_x(D_x)n_s(D_s)dD_k dD_s.
\]

In the manner of Verlinde et al. (1990), we begin by noting that for each \( D_S \), there is a \( \tilde{D}(D_S) \) defined such that, when \( D_F = D_S \), the corresponding member of species \( F \) has the same terminal velocity as the member of species \( S \) of size \( D_S \). Because of the nature of the fall velocity curves for different species, \( \tilde{D}(D_S) \) increases much more slowly than \( D_S \) at larger sizes. Under the reasonable assumption that the fall velocities are increasing functions of diameter, the velocity difference in (9) changes from negative to positive during the \( D_F \) integration when \( D_F \) reaches \( \tilde{D}(D_S) \). Thus, we may write

\[
\frac{dr_x}{dt} = -\frac{\pi m_x \tilde{D} n_S}{4 \rho_0} \int_{D_S = 0}^{D_S = \infty} \left[ \int_{D_F = 0}^{D_F = \tilde{D}} G(D_S, D_F) e^{-\lambda F D_F} dD_F + \int_{D_F = \tilde{D}}^{D_F = \infty} G(D_S, D_F) e^{-\lambda F D_F} dD_F \right] e^{-\lambda S D_S} dD_S.
\]
The function $G(D_S, D_F)$ is defined as the functional form of the integrand in (9) with the absolute value of the velocity difference replaced by $v_F(D_F) - v_S(D_S)$ itself. Thus, for $D_F < \tilde{D}(D_S)$, the integrand is $-G(D_S, D_F)$, whereas for $D_F > D(D_S)$, the integrand is $+G(D_S, D_F)$. A repartitioning of the inner integrals leads to the form

$$\frac{d r}{dt} = - \frac{\alpha_{DS} \eta_0}{4 p_0} \int_{D_S = 0}^{D_S = \infty} \left[ -2 \int_{D_F = 0}^{D_F = \infty} G(D_S, D_F) e^{-\lambda_F D_F} dD_F + \int_{D_F = 0}^{D_F = \infty} G(D_S, D_F) e^{-\lambda_F D_F} dD_F \right] e^{-\lambda_S D_S} dD_S. $$

(11)

We now point out that neglecting the first inner integral in (11) is equivalent to replacing the integrand in (9) with $G(D_S, D_F)$ throughout the range of integration, which is equivalent to disregarding the absolute value brackets in (9). We will henceforth refer to $|v_F(D_F) - v_S(D_S)| = v_F(D_F) - v_S(D_S)$ as the general Wisner approximation (GWA), for a purpose that will become more apparent later. This is an appealing approximation to make for two reasons. The first reason is that, if we anticipate that $G(D_S, D_F)$ can be written as a finite series of terms involving powers of $D_S$ multiplied by powers of $D_F$, the second inner integral in (11) can easily be evaluated as products of gamma functions. However, the first inner integral evaluates as incomplete gamma functions involving the parameter $\lambda_F \tilde{D}(D_S)$, whose subsequent integration over $D_S$ creates the hypergeometric functions found in the exact solution of Verlindt et al. (1990). Therefore, neglecting the first integral leads to a great simplification of algebra, and makes a closed-form solution possible. The second reason is that, whenever $|v_F(D_F) - v_S(D_S)|$ is large, it is, in fact, true that $|v_F(D_F) - v_S(D_S)| = v_F(D_F) - v_S(D_S)$, given the assumed distinction between fall velocities at larger sizes. In the only regions of diameter space where it is not true, both $v_F(D_F)$ and $v_S(D_S)$ are relatively small, and the contribution to the total integral is small anyway.

Therefore, we assert that the GWA can be a useful approximation to evaluate the behavior of the collection equation between different hydrometeors. However, before establishing more rigorously its range of validity, we first derive the GWA collection equation for a concrete situation.

### 4. Fast-falling collector, slow-falling collected

We will now explicitly assume that the fast-falling species is the collector, so we can identify $m_s(D_s)$ as $m_F(D_F)$. We also now assume that the quantities $v_F(D_F)$, $v_S(D_S)$, and $m_S(D_S)$ are given by the power-law relations $a_F D_F^{b_F}$, $a_S D_S^{b_S}$, and $\pi D_F^2 p_S / 6$, respectively. In this case, $G(D_S, D_F)$ becomes a six-term sum:

$$G(D_S, D_F) = \frac{\pi p_S}{6} \left[ a_F D_F^{b_F} + 2a_F D_F^{b_F} - a_F D_F^{b_F} - 2a_F D_S^{b_S} - 2a_F D_S^{b_S} D_F - a_S D_S^{b_S} D_F^2. \right]$$

(12)

Substitution into (11), followed by neglect of the first inner integral, produces

$$\frac{d r}{dt} = - \frac{\pi^2 n_{DS} \eta_0 p_S}{24 p_0} \left[ a_F \Gamma(1 + b_F) \Gamma(6) \frac{\lambda_F^{1 + b_F} \lambda_S^6}{\lambda_F^{1 + b_F} \lambda_S^6} + 2a_F \Gamma(2 + b_F) \Gamma(5) \frac{\lambda_F^{2 + b_F} \lambda_S^5}{\lambda_F^{2 + b_F} \lambda_S^5} + a_F \Gamma(3 + b_F) \Gamma(4) \frac{\lambda_F^{3 + b_F} \lambda_S^4}{\lambda_F^{3 + b_F} \lambda_S^4} + a_F \Gamma(1) \Gamma(6 + b_S) \frac{\lambda_F^{6 + b_S}}{\lambda_F^{6 + b_S}} - 2 \frac{a_F \Gamma(2) \Gamma(5 + b_S)}{\lambda_F^{5 + b_S}} - \frac{a_F \Gamma(3) \Gamma(4 + b_S)}{\lambda_F^{4 + b_S}} \right]. \tag{13}$$

Using (7), algebraic manipulation eventually yields

$$\frac{d r}{dt} = - \frac{r_F r_S \lambda_F p_0}{24 p_0} \left[ v_F \left( \Gamma(3 + b_F) \Gamma(4) + 2 \Gamma(2 + b_F) \Gamma(5) \left( \frac{\lambda_F}{\lambda_S} \right) + \Gamma(1 + b_F) \Gamma(6) \left( \frac{\lambda_F}{\lambda_S} \right)^2 \right) \right]$$

$$- v_S \left[ \Gamma(3) \Gamma(4 + b_S) + 2 \Gamma(2) \Gamma(5 + b_S) \left( \frac{\lambda_F}{\lambda_S} \right) + \Gamma(1) \Gamma(6 + b_S) \left( \frac{\lambda_F}{\lambda_S} \right)^2 \right]. \tag{14}$$

We can see that (14) is a more general form of (8), produced by the conventional Wisner approximation. The two expressions are identical for the special case $b_F = b_S = 0$. Otherwise, the GWA differs in that the two characteristic velocities are multiplied by
different terms and that the terms are dependent on the ratio of the characteristic sizes of the categories. Furthermore, as will be seen later, the terms for the collection of a fast hydrometeor by a slower one differ.

The range of validity of the GWA can be found by analyzing the term derived from the first inner integral in (11), which is the difference between the GWA and the exact solution. As stated previously, we cannot obtain a convenient closed-form solution for the double integration, but in the current case we can obtain an exact solution in the form of a set of infinite power series of \((v_{sc}/v_{fc})\). The solution method is outlined in appendix A. The result is that the exact solution consists of the GWA expression plus a correction term of

\[
\frac{dr_S}{dt} = -\frac{r_s r_F \lambda_s \rho_0}{24 p_F} v_{sc} \times \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1+j)} \left( \frac{v_{sc}}{v_{fc}} \right)^{(3+j)b_F} \Gamma \left[ 4 + b_S + \frac{b_S}{b_F} (3+j) \right] \left( \frac{b_F}{b_F + 3+j} \right) \right\}
\]

It is also shown in appendix A that the necessary and sufficient conditions for convergence of the series are \(b_S < b_F\) when \(b_S \neq b_F\), and \(v_{sc} < v_{fc}\), when \(b_S = b_F\). A consequence is that these are necessary conditions for the applicability of the GWA. Usually this condition is satisfied since hydrometeors with large fall velocity exponents will, in fact, have larger terminal velocities than hydrometeors with smaller exponents at sufficiently large sizes.

A scaling analysis of the terms in (14) and (15) is carried out in detail in appendix B. The main result is that the terms of (14) are all of either zeroth or first order in \(v_{sc}/v_{fc}\), whereas the terms in (15) are all of higher order. Therefore, for \(v_{sc}/v_{fc}\) sufficiently small and \(b_S \leq b_F\), the GWA equation will increasingly approach the exact solution. (How small depends on the case; see appendix B.) When \(v_{sc}/v_{fc}\) is not sufficiently small, the reason, based on Fig. 2, is generally that the amount of the fast hydrometeor is too small rather than that the amount of slow hydrometeor is too large. For \(b_S < b_F\) but \(v_{sc}/v_{fc}\) not small, the GWA is not valid, but we may approximate the exact solution to any desired accuracy by adding a finite amount of terms from (15) to the GWA equation (14). It can be seen that the lowest order term in (15) is negative; that is, it represents a further loss of collected species. Therefore, when the difference between the GWA and the exact solution first becomes apparent, the GWA will be an underestimate of the true collection rate.

We now assume that the above conditions for the applicability of the GWA are satisfied, so the difference between the GWA and the exact solution is negligible. We wish to assess the prediction of the CWA under these circumstances. To do this we use (8), with the understanding that \(y, x\) corresponds to \(F, S\). If the GWA is valid, \(b_F > b_S\) and \(v_{fc} > v_{sc}\), which means that \(v_{sc}/v_{fc}\) and the velocity difference within the absolute value bars in (8) is positive. Hence, the absolute value function in (8) is not necessary, and the CWA equation for this case is

\[
\frac{dr_S}{dt} = -\frac{r_s r_F \lambda_s \rho_0}{24 p_F} \times \left[ \Gamma(3)\Gamma(4)(\tilde{v}_{2F} - \tilde{v}_{3S}) + 2\Gamma(2)\Gamma(5)(\tilde{v}_{1F} - \tilde{v}_{2S})(\frac{\lambda_F}{\lambda_S}) + \Gamma(1)\Gamma(6)(\tilde{v}_{1F} - \tilde{v}_{3S})(\frac{\lambda_F}{\lambda_S})^2 \right].
\]

In comparison, we can use the formula for the kth-moment average velocity [(5)] to rewrite the GWA equation, (14), as

\[
\frac{dr_S}{dt} = -\frac{r_s r_F \lambda_s \rho_0}{24 p_F} \left[ \Gamma(3)\Gamma(4)(\tilde{v}_{2F} - \tilde{v}_{3S}) + 2\Gamma(2)\Gamma(5)(\tilde{v}_{1F} - \tilde{v}_{2S})(\frac{\lambda_F}{\lambda_S}) + \Gamma(1)\Gamma(6)(\tilde{v}_{1F} - \tilde{v}_{3S})(\frac{\lambda_F}{\lambda_S})^2 \right].
\]

For velocities that increase with diameter, the kth-moment average velocity increases with \(k\). If the general Wisner approximation is valid, then the average velocity moments for the slow species can be neglected compared to the average velocity moments for the fast species. Then the relative behaviors of the GWA and CWA equations are determined by the \(\tilde{v}_F\) terms. We see that the third-order moment used by the CWA
equation (i.e., the mass-weighted velocity) is of higher order than any of the \( F \) moments used in (17), which are determined by a cross-sectional area weighting. Thus, the velocity differential in the CWA equation (16) is an overestimate, and the CWA must overestimate the true collection rate in this regime.

We now show some comparisons of the various formulations of the collection rate. The values of the microphysical parameters used are given in Table 1 and are taken from the Coupled Ocean/Atmosphere Mesoscale Prediction System (COAMPS) described in Hodur (1997). (COAMPS is a registered trademark of the Naval Research Laboratory.) The microphysics scheme in this model is based on that of Rutledge and Hobbs (1983, hereafter RH83). We first examine the case of snow being collected by “power rain.” The fall velocity of rain in COAMPS, as in RH83, is treated as a cubic polynomial fit to the data of Gunn and Kinzer (1949); power rain uses the formula of Lin et al. (1983), which is a power-law fit to the same data, and, while not quite as accurate, is close enough (Fig. 3) to demonstrate the general behavior of rain hydrometeor interactions using various methods. Figure 4 shows the fractional collection rate of power rain collecting snow as a function of rain mixing ratio for the case of \( r_s = 0.5 \text{ g kg}^{-1} \). Rain collecting snow is often a sink for snow in numerical models (e.g., Rutledge and Hobbs 1984, hereafter RH84; Ferrier 1994; Walko et al. 1995; Reisner et al. 1998). Because the snow fall velocity stays nearly constant and much less than the rain fall velocity, the GWA should perform well, which is borne out by the figure. By contrast, the CWA overestimates the true rate by nearly a factor of 3, causing a fractional depletion of unity in a 5-s time step when \( 1 \text{ g kg}^{-1} \) of rain is present.

In practice, most bulk microphysics models employ other criteria (e.g., mixing ratio thresholds or heat storage variables) to regulate the amount of mass transfer in mixed-phase processes rather than use the unmodified CWA rate. Though the GWA is always quite close to the exact solution, it does have the problem of becoming negative when less than 0.1 g kg\(^{-1}\) of rain is present, which is a small but nonnegligible amount.

We should note that the solid horizontal line in the figures shows the fractional depletion rate that would represent a complete loss of the collected hydrometeor in a 5-s time step. We see that this is a possibility when the CWA is used to represent the collection of snow by large amounts of power rain. For mesoscale models, where time steps are on the order of tens of seconds, the potential for complete hydrometeor depletion within a time step is even more pronounced.

Figure 5 shows the collection of 0.5 g kg\(^{-1}\) of graupel by power rain. This does not correspond to a sink term in RH84 but can be a sink for graupel in models that have a hail category (e.g., Ferrier 1994; Walko et al. 1995). In this case the Wisner approximation still overestimates the true rate. The performance of the GWA is much worse than in the previous example because the fall velocity of graupel is more significant than snow compared to power rain. When less than 3 g kg\(^{-1}\) of rain is present, the GWA becomes negative, indicating

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**Table 1.** Microphysical parameters for various hydrometeors used in study. Parameters are used to determine velocity, number concentration, and density of hydrometeors in formulas of section 2.

<table>
<thead>
<tr>
<th></th>
<th>Power rain</th>
<th>Graupel</th>
<th>Snow</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>842.0</td>
<td>124.14</td>
<td>2.35</td>
<td>m(^{-b}) s(^{-1})</td>
</tr>
<tr>
<td>( b )</td>
<td>0.8</td>
<td>0.66</td>
<td>0.11</td>
<td>—</td>
</tr>
<tr>
<td>( N_0 )</td>
<td>( 8.0 \times 10^6 )</td>
<td>( 4.0 \times 10^6 )</td>
<td>( 2.0 \times 10^7 )</td>
<td>m(^{-4})</td>
</tr>
<tr>
<td>( \rho )</td>
<td>1000</td>
<td>400</td>
<td>100</td>
<td>kg m(^{-3})</td>
</tr>
</tbody>
</table>

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**Fig. 4.** Fractional collection rate of snow (s\(^{-1}\)) vs power rain mixing ratio, for 0.5 g kg\(^{-1}\) of snow. Collection efficiency of unity is assumed. Values of other parameters are given in Table 1. Dark squares indicate the conventional Wisner approximation, open squares indicate the general Wisner approximation, and stars indicate the series solution taken to 40 terms. Horizontal line indicates fractional collection of unity in 5-s time step.
that the collection of faster graupel hydrometeors by slower rain hydrometeors is significant.

Mathematically, for $a > b$, both greater than or equal to one, then $\Gamma(a + c)\Gamma(b) > \Gamma(a)\Gamma(b + c)$ for $c$ positive. The difference becomes more extreme as the difference between $a$ and $b$ increases [For example, $\Gamma(7)\Gamma(1) = 720$, but $\Gamma(6)\Gamma(2) = 120$]. Therefore, there is a tendency for the $v_{Sc}$ gamma products in (14) to exceed those of $v_{Sc}$ when $b_S$ is too close to $b_F$. Even for power rain collecting snow ($b_F = 0.8; b_S = 0.11$), we have $\Gamma(1)\Gamma(6 + b_S) > \Gamma(1 + b_S)\Gamma(6)$, although for the other gamma products in (14) the $v_{Fc}$ versions are greater. Therefore, when $(\lambda_F/\lambda_S)$ becomes large enough, the GWA for the collection rate of snow by power rain will become negative as well. For power rain collecting graupel, all the $v_{Sc}$ gamma products exceed the corresponding $v_{Fc}$ versions; thus $v_{Fc} > v_{Sc}$ is required for a positive GWA.

\[
\begin{align*}
\frac{dr_S}{dt} &= -\frac{r_S F_p P_r p_0}{24 F_F} \sum \frac{a_k}{\lambda_F^2} \left( \Gamma(3 + k)\Gamma(4) + 2\Gamma(2 + k)\Gamma(5) \frac{\lambda_F}{\lambda_S} + \Gamma(1 + k)\Gamma(6) \left( \frac{\lambda_F}{\lambda_S} \right)^2 \right) \\
&\quad - v_{Sc} \left[ \Gamma(3)\Gamma(4 + b_S) + 2\Gamma(2)\Gamma(5 + b_S) \left( \frac{\lambda_F}{\lambda_S} \right) + \Gamma(1)\Gamma(6 + b_S) \left( \frac{\lambda_F}{\lambda_S} \right)^2 \right].
\end{align*}
\]

Figure 6 shows an example of the collection of snow by the polynomial rain fit of RH83 and RH84, where $a_0 = -0.267, a_1 = 5150, a_2 = -1.0225 \times 10^6,$ and $a_3 = 7.55 \times 10^7$ in MKS units. A numerical integration was used to determine the exact solution. When the numerical integration was applied to the power rain case, the rates were virtually identical to those of the series solution, confirming the validity of the series approach. We see that the GWA remains nearly as accurate as in the polynomial rain case over virtually the same range.

5. Slow-falling collector, fast-falling collected

We now examine the case in which the faster species is being collected, and masses and fall velocities obey power laws. In this case $m_i(D_S)$ in (9) now corresponds to $m_i(D_F)$, thereby changing the expression for $G(D_S, D_F)$. However, it is possible to deduce what the GWA should be in this instance. We first transform the fast-falling collector equation (14) from $F, S$ to $x, y$, using $F = y$ and $S = x$. We then note that, if the fast species is the one being collected, the only change in the GWA...
derivation in terms of \(x, y\) labels is that, instead of using \((v_x - v_y)\) in the integrand, we now use \((v_x - v_y)\). Thus, the integrand in the slow-falling collector case is simply the negative of what it would be in the fast-falling collector case, so the transformed GWA equation for a slow-falling collector is the negative of the transformed fast-falling collector GWA equation. We then transform back using \(y = S\) and \(x = F\), and obtain

\[
\frac{dr_F}{dt} = - \frac{2r_F r_S \lambda_S p_0}{24 \rho_S} \left\{ \nu_S \left[ \Gamma(3) \Gamma(4 + b_F) + 2 \Gamma(2) \Gamma(5 + b_F) \left( \frac{\lambda_S}{\lambda_F} \right) + \Gamma(1) \Gamma(6 + b_F) \left( \frac{\lambda_S}{\lambda_F} \right)^2 \right] \right. \\
- \nu_S \left[ \Gamma(3 + b_S) \Gamma(4) + 2 \Gamma(2 + b_S) \Gamma(5) \left( \frac{\lambda_S}{\lambda_F} \right) + \Gamma(1 + b_S) \Gamma(6) \left( \frac{\lambda_S}{\lambda_F} \right)^2 \right] \right\}.
\]

(20)

The net effect is interchanging the labels \(F\) and \(S\) in (14), and then taking the negative of the entire equation.

The correction to obtain the exact solution can be shown to be

\[
\frac{dr_F}{dt} \bigg|_{\text{corr}} = - \frac{2r_F r_S \lambda_S p_0}{24 \rho_S} \nu_S \times \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1 + j)} \left( \frac{\nu_S}{\nu_F} \right)^{(j+1)/b_F} \Gamma \left[ 3 + b_S + \frac{b_S}{b_F} (4 + j) \right] \frac{b_F}{(4 + j)(b_F + 4 + j)} \right. \\
+ 2 \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1 + j)} \nu_S \nu_F \left( \frac{\nu_S}{\nu_F} \right)^{(6 + j)/b_F} \Gamma \left[ 1 + b_S + \frac{b_S}{b_F} (6 + j) \right] \frac{b_F}{(6 + j)(b_F + 6 + j)} \left( \frac{\lambda_S}{\lambda_F} \right)^2 \right\}.
\]

(21)

As before, the necessary and sufficient conditions for convergence are \(b_S < b_F\), for \(b_S \neq b_F\), and \(v_{Sc} < v_{Fc}\), for \(b_S = b_F\). Also, as before, the series show that the GWA equation for the slow-falling collector should be valid whenever \(v_{Sc}/v_{Fc}\) is small. Note, however, that the lowest order term in the series is \((v_{Sc}/v_{Fc})^{(b_F-1)}\), as opposed to \((v_{Sc}/v_{Fc})^{(b_F-1)}\) in the fast-falling collector case. Hence, the slow-falling collector series should converge more rapidly, and extend the range of GWA validity.

We again compare the CWA to the GWA in the latter’s range of validity. This is simplest to do in \(x,y\) space. We note from the fast-falling collector equations (16) and (17) that the \(v_x\) terms are greater in the CWA than in the GWA; the \(v_y\) terms are either of the same or lesser magnitude in the CWA than in the GWA. Now, for the slow-falling collector case, if the GWA is valid, then \(v_{Sx} \geq v_{Sy}\), so the absolute value function takes the negative, and we obtain the negative of the \(x,y\) form of (16). However, as previously mentioned, the slow-falling collector GWA equation in \(x,y\) space is also the negative of the fast-falling collector expression. Therefore, we may make the same correspondence of CWA terms to GWA terms in \(x,y\) space as before. The only difference is that for a slow-falling collector the \(v_y\) terms are the dominant ones, and for these the CWA velocity moments are the same or less than those in the GWA equation. As a consequence, the CWA must underestimate the true collection rate for a slow-falling collector when the GWA is valid. Mathematically, the combination of the \(m_F(D_F)\) term with the collection kernel means that the collection rate depends on average velocities of higher order than the mass average for the fast species, so the use of the third-order mass average is an underestimate.

Figure 7 shows the fractional collection rate of power rain by snow versus mixing ratio of snow for 0.5 g kg\(^{-1}\) of rain. Snow collecting rain is a sink for rain in RH84 and Hodur (1997), among other models. The GWA should be appropriate for snow mixing ratios below certain thresholds. Even for 0.5 g kg\(^{-1}\) of rain, however, these thresholds are so large that in Fig. 7 there is no perceptible difference between the GWA and the series solution. This is even true for the case of graupel-collecting rain.
lecting power rain, also a sink for rain in the above
models, with barely noticeable differences near 4 g kg⁻¹
of graupel (Fig. 8). In both figures, as we would expect,
the CWA underestimates the collection rate. Figure 9
shows the rate of snow collecting rain using the poly-
nomial fall velocity in the slow-falling collector equa-
tion. Even though we have not rigorously established
the accuracy of the polynomial GWA, we can see em-
pirically that it is also highly accurate.

Note that in the case of 0.5 g kg⁻¹ of both power rain
and graupel, the GWA is nearly perfect in predicting
the rate of graupel collecting power rain (Fig. 8), but
is quite poor in predicting the rate of power rain collect-
ing graupel (Fig. 5). This demonstrates the greater ac-
curacy of the slow-falling collector GWA, as predicted.

It is also apparent that the GWA fractional collection
rates are greater for the slow-falling collector cases than
the fast-falling collector cases. In contrast to the fast-
falling collector case, the \( v_F \) gamma function products
in (20) are guaranteed to be larger than the correspond-
ing ones of \( v_S \), for \( b_F > b_S \), so the GWA collection rate
remains positive even when \( v_F = v_S \). The same char-
acteristic fall velocities can produce a negative collection
rate in the fast-falling collector equation.

It is of interest to examine the \( v_F/v_S \approx 1 \) asymptotic
case, when the faster hydrometeor, despite its larger
velocity exponent, is present in such small quantities
that \( v_F \) is negligible compared to \( v_S \). This case requires
a large number of terms from the appropriate infinite
series evaluated extremely accurately, and the GWA is
clearly not appropriate in this regime. However, we can
argue that in this limit the species with the smaller ve-
locity exponent can be treated as if it were the fast
species because in this limit the largest velocities are
those of the larger members of \( S \). So we replace \( v_F(D_F) - v_S(D_S) \)
in the integrands of (11) with \( v_S(D_S) - v_F(D_F) \), again neglect the first integral in (11), and, thus

find that the \( v_F/v_S \approx 1 \) limit is simply the negative of
(14), for a fast-falling collector, or (20), for a slow-
falling collector. This is confirmed by Fig. 10, showing
the fractional collection rate of 2.0 g kg⁻¹ of graupel by
power rain. The negative of the GWA equation[(14)]
becomes increasingly accurate as the amount of rain
decreases.

6. Relative bulk conversion rates

We now summarize our results. In an interaction be-
tween two hydrometeors with power-law fall velocities,
we denote one as the fast-falling species and the other
the slow-falling species depending on the relative mag-
nitudes of the fall velocity exponents, which determines

*FIG. 8. Fractional collection rate of power rain \((s^{-1})\) vs graupel
mixing ratio, for 0.5 g kg⁻¹ of power rain. Otherwise, same as in
Fig. 5.*

*FIG. 9. Fractional collection rate of polynomial rain \((s^{-1})\) vs
snow mixing ratio for 0.5 g kg⁻¹ of polynomial rain. Otherwise as
in Fig. 4.*

*FIG. 10. Fractional collection rate of graupel \((s^{-1})\) vs power rain
mixing ratio, for 2.0 g kg⁻¹ of graupel. Otherwise, same as in Fig.
5 except that open squares denote negative of GWA equation
(14).*
threshold. How small \( /H9271 \) small (see Figs. 11 and 12), which tends to occur when-

the relative fall velocities at larger sizes. We will define a regime where the GWA is appropriate as a GWA regime. This regime occurs when \( v_S/v_F \) is sufficiently small (see Figs. 11 and 12), which tends to occur whenever the amount of the fast species \( F \) exceeds a certain threshold. How small \( v_S/v_F \) must be in the GWA regime depends on the specific hydrometeors and whether the fast species is collecting or is being collected, as shown in appendix B. In the GWA regime one of two situations is appropriate, depending on whether the collector is faster falling or slower falling. The CWA equation overestimates the collection rate in the former case and underestimates it in the latter. These characteristics are summarized in Tables 2 and 3.

Lin et al. (1983) classify microphysical interactions in a bulk model as two-component and three-component processes. In the former, the result of the interaction between two different species is the transfer of mass from one species to the other; the amount of mass is determined by applying the collection equation to the collected species. There is no mass transferred in the opposite direction, so in a two-component process the mass is transferred to the fast species as accretion and one where mass is transferred to the slow species as interception. From our results we see that in the GWA regime, the CWA equation will overestimate an accretion rate, but underestimate an interception rate.

In a three-component process, which we will denote as conversion, the result of an interaction between two species is the transfer of mass from both to a third species. Thus, for any conversion interaction between a fast species \( F \) and a slow species \( S \), we need to apply the collection equation twice, once for \( F \) collecting \( S \) and the other for \( S \) collecting \( F \). The third species gains mass at the expense of both \( F \) and \( S \). We are interested, however, in relating the rate of mass depletion of \( F \) to the mass depletion of \( S \) in their mutual conversion process. We again assume a GWA regime. The tendency of the ratio \( r_S r_F \) is given by

\[
\frac{d}{dt} \left( \frac{r_S}{r_F} \right) = \frac{r_S}{r_F} \frac{1}{24} \frac{d}{dr_S} \left( \frac{1}{r_S} \frac{dr_S}{dt} - \frac{1}{r_F} \frac{dr_F}{dt} \right),
\]

and so depends on the respective fractional depletion rates. Our examples have shown that the GWA fast species fractional depletion tends to exceed slow species fractional depletion; if this is the case, then a conversion process will preferentially deplete the fast species.

To be more precise, we substitute in (14) and (20) to give

\[
\frac{d}{dt} \left( \frac{r_S}{r_F} \right) = \frac{r_S}{r_F} \frac{1}{24} \frac{d}{dr_S} \left( \frac{n_{0S}}{n_{0F}} \right) \left[ \Gamma(1) \Gamma(6 + b_S) \left( \frac{\lambda_F}{\lambda_S} \right)^2 + \cdots - \Gamma(1 + b_S) \Gamma(6) \left( \frac{\lambda_F}{\lambda_S} \right)^2 - \cdots - \Gamma(3 + b_S) \Gamma(6) \left( \frac{\lambda_F}{\lambda_S} \right)^2 \cdots \right],
\]

where we have only displayed the terms with the largest gamma function products and the only term that is zeroth order in \( \lambda_F/\lambda_S \). The terms with \( n_{0S}/n_{0F} \) derive from the \( dr_F/dt \) equation, whereas the others are from \( dr_S/dt \).

As noted in section 4, we can expect the largest of the gamma products to be \( \Gamma(6 + b_S) \Gamma(1) \); the other three products shown should be of comparable magnitude. This term is further increased relative to the other
terms, except $\Gamma(1 + b_F) \Gamma(6)$, by being multiplied by $v_{yk}$ instead of the smaller $v_{yr}$. Hence, it should be this term that determines the sign of the relative depletion; being positive, we indeed find that the slow species relative fraction should increase over time. Even within the GWA regime, this conclusion can be incorrect for two reasons: 1) if $n_{0k}/n_{0F}$ is very small, then $dr_k/dt$ will be larger in magnitude and the fast species relative fraction will increase in time; 2) if $\lambda_F/\lambda_S$ is very small, eventually the $\Gamma(3 + b_F) \Gamma(4)$ term will prevail, again causing the fast species relative fraction to increase.

Physically, it may seem at first counterintuitive that a fast species will be depleted at the expense of a slow species in mutual interaction. But this is a reflection of the fact that the fall velocity of the slow species is relatively independent of mass, but is strongly weighted to the higher mass hydrometeors of the fast species. Thus, a coalescence is more effective in terms of fractional mass loss at defeating the fast species than the slow species. Since $N = n_S/\lambda_S$, where $N$ is the total number concentration of a hydrometer, we see that as a general rule the slow species will be preferentially depleted only if $N_{0k}/N_{0F}$ is small enough. The more hydrometeors over which a given species mass is distributed, the more difficult it is to remove that mass in collisions with a given number of another hydrometeor species (see, for instance, Lin et al. 1983; Curic and Janc 1997).

For the snow–power rain interaction that we have analyzed, the ratio $n_{0k}/n_{0F}$ is 2.5, and achieving even $\lambda_F/\lambda_S = 0.5$ requires $r_F = 64r_S$ (mixing ratio is proportional to $\lambda^3$ here). So the fractional rain conversion rate should exceed that of snow if the mixing ratio of snow is any significant fraction of the mixing ratio of rain. We can compare the two conversion rates for the case of 0.5 g kg$^{-1}$ of both power rain and snow by utilizing Figs. 4 and 7. It is confirmed that the fractional rain depletion is greater, both using the GWA solution and the exact solution. For the graupel–power rain interaction, $n_{0k}/n_{0F}$ is 0.5, and $\lambda_F/\lambda_S = 0.5$ requires $r_F = 80r_S$, so, while the characteristic diameters of the two species should be comparable, graupel is relatively sparse in concentration, and its ability to deplete rain is reduced. A comparison of the exact solutions from Figs. 5 and 8 shows that for 0.5 g kg$^{-1}$ of graupel and rain the fractional depletion rates are very close. However, for these mixing ratios the collection of graupel by rain is clearly outside the GWA regime because the GWA solution is negative. A plot of the fractional depletion rate of power rain by graupel for 4.0 g kg$^{-1}$ of power rain (Fig. 13) allows, along with Fig. 5, a comparison of the collection rates for 4.0 g kg$^{-1}$ of rain and 0.5 g kg$^{-1}$ of graupel. Although the GWA solution is still not very accurate for the collection of graupel, we see that the rain depletion rate is clearly greater than that of graupel, both in the GWA and exact solutions.

## 7. Discussion

From a numerical modeling standpoint, both accretion and conversion are commonly represented processes. Interception is less commonly represented; such a process suggests that water mass (either solid or liquid) is transferred from faster-falling to slower-falling hydrometeors during descent. We have seen how the CWA tends to overestimate an accretion rate in a GWA regime, sometimes by large amounts. For a 5-s time step, complete depletion of a species by the CWA can become quite common; for mesoscale model time steps on the order of tens of seconds, complete depletion in a time step becomes even more likely, resulting in a potential distortion of the true microphysical processes.

We have shown evidence that in a GWA regime the conversion rate of a fast species $F$ exceeds that of the corresponding slow species $S$. The only exception is when the concentration of $S$ is much greater than the concentration of $F$, and this is not the case for the two kinds of interactions described above. In the CWA within the GWA regime, the collection rate of slower hydrometeors is enhanced while that of faster hydrometeors is suppressed. As a consequence, the preference of $S$ in a conversion interaction is suppressed by the CWA. We see in Figs. 4 and 7 that the CWA decreases the differential collection rate for 0.5 g kg$^{-1}$ of power

## Table 2. Behavior of approximate equations vs velocity ratio for the fast-falling collector case.

<table>
<thead>
<tr>
<th>$v_{ys}/v_{yr}$</th>
<th>$v_{ys}/v_{yr} \ll 1$</th>
<th>$v_{ys}/v_{yr} \cong 1$</th>
<th>$v_{ys}/v_{yr} \gg 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approximation: equations</td>
<td>GWA: (14)</td>
<td>None—use exact: (14) + (15)</td>
<td>Negative GWA: $-1 \times (14)$</td>
</tr>
<tr>
<td>CWA bias</td>
<td>CWA $&gt;$ exact</td>
<td>Indeterminate</td>
<td>Indeterminate</td>
</tr>
<tr>
<td>GWA behavior</td>
<td>Nearly exact</td>
<td>GWA $&lt;$ exact</td>
<td>Inapplicable</td>
</tr>
</tbody>
</table>

## Table 3. Behavior of approximate equations vs velocity ratio for the slow-falling collector case.

<table>
<thead>
<tr>
<th>$v_{ys}/v_{yr}$</th>
<th>$v_{ys}/v_{yr} \ll 1$</th>
<th>$v_{ys}/v_{yr} \cong 1$</th>
<th>$v_{ys}/v_{yr} \gg 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approximation: equations</td>
<td>GWA: (20)</td>
<td>None—use exact: (20) + (21)</td>
<td>Negative GWA: $-1 \times (20)$</td>
</tr>
<tr>
<td>CWA bias</td>
<td>CWA $&lt;$ exact</td>
<td>Indeterminate</td>
<td>Indeterminate</td>
</tr>
<tr>
<td>GWA behavior</td>
<td>Nearly exact</td>
<td>GWA $&lt;$ exact</td>
<td>Inapplicable</td>
</tr>
</tbody>
</table>
Fig. 13. As in Fig. 8 but for 4.0 g kg\(^{-1}\) of power rain.

8. Conclusions

For exponentially distributed hydrometeors with power-law fall velocities, the exact collection equation has been integrated to yield one of two power series solutions in terms of the characteristic fall velocities of the hydrometeors involved. Because these fall velocities are quite often widely separated, a useful approximation to the exact collection equation in these situations can be found involving only the lowest order terms. With the help of these general Wisner approximations it has been demonstrated that, unless the quantity of the faster hydrometeor category is small, the conventional Wisner approximation overestimates the collection of a slower hydrometeor but underestimates the collection of a faster hydrometeor. Some of the implications for the numerical modeling of cloud microphysics have been discussed.

A sequel to this study will focus on more applied aspects of using the general Wisner approximation in a numerical model, as well as examine quantitatively the magnitudes of the fractional collection rates by various schemes.

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APPENDIX A

Series Solutions to Exact Collection Equation

We wish to evaluate the double integral:

\[
\frac{dr_s}{dt}_{\text{corr}} = -\frac{\pi \rho_0 \eta_0 \rho_F}{4 \rho_0} \int_{D_F=0}^{D_F=\infty} \int_{D_S=0}^{D_S=\infty} 2 \left[ e^{-\lambda_D D_F} e^{-\lambda_S D_S} dD_F dD_S \right],
\]

where \(\lambda_D\) and \(\lambda_S\) are the fall velocity exponents for the faster and slower hydrometeors, respectively. The solution can be expressed in terms of the characteristic fall velocities of the hydrometeors involved. For two hydrometeors with similar terminal velocities, the GWA method will not work well, and the particular functional form of the number distributions will become more important, as well as the presence of turbulent fluctuations in the mean quantities. Clearly, neither graupel nor, especially, snow obey spherical mass relationships and may not be exponentially distributed. On the other hand, we note that the exponential distributions are most numerous at the smallest sizes, precisely those sizes at which the GWA is least valid. So we expect that the GWA should be even more accurate, when gamma number distributions are involved, because the small-size behavior is relatively suppressed. Generalizations to different hydrometeor shapes or number distributions are not performed here, but should be relatively straightforward.
for the case of a fast-falling collector species, and power-law velocity relations for both species. This is the first part of the expression in (10) and is the correction that must be added to the GWA solution to yield the exact solution to the collection equation. In the power-law fast-falling collector case, \( G(D_S, D_F) \) is a six-term sum, given by

\[
G(D_S, D_F) = \frac{\pi D_S^3}{6} \left( \alpha F D_S^2 D_F + 2 \alpha F D_S^4 D_F^{1+b_F} + a_F D_S^3 D_F^{1+b_F} \right) - a_S D_S^{5+b_S} - 2 a_S D_S^{4+b_S} D_F - a_S D_S^{3+b_S} D_F^2.
\]

(A2)

For the sake of brevity, we will rewrite \( G(D_S, D_F) \) as \( \alpha F \sum_{i=6}^{i=6} H_i(D_S) D_F^{(i)} \). We can then evaluate the \( D_F \) integral as

\[
\frac{dr_x}{dt} = -\frac{\pi^2 n_{0,0} n_{0,F} \rho S}{24 \rho_0} \int_{D_S=0}^{D_S=\infty} \sum_{i=6}^{i=6} \int_{D_F=0}^{D_F=\infty} H_i(D_S) \gamma[i(i + 1, \lambda_i D)] dD_S.
\]

(A3)

where \( \gamma(a, z) \) is one of the two incomplete gamma functions.

According to Abramowitz and Stegun (1970), the incomplete gamma function \( \gamma(a, z) \) can be written as

\[
\gamma(a, z) = \int_0^z e^{-t} t^{a-1} dt = \sum_{j=0}^{\infty} \frac{(-1)^j z^{a+j}}{\Gamma(j + 1)(a + j)}.
\]

(A4)

using a Taylor series of \( e^{-t} \). For the current situation, \( \nu_S(D_S) = a_S D_S^5 \) and \( \nu_F(D_F) = a_F D_F^2 \). So \( D(D_S) \), defined by \( a_D D_D^{b_D} = a_S D_S^{b_S} \), is given by \( (a_S/a_F)^{b_D} D_S^{b_D} \). Thus, the incomplete gamma function in (A3) can be written as

\[
\gamma[m(i) + 1, \lambda_i D] = \int_0^{x_F} e^{-t} t^{m(i)} dt = \sum_{j=0}^{\infty} \frac{(-1)^j (\lambda_i a_F a_S)^{b_D} D_S^{b_D} (m(i) + 1 + j)}{\Gamma(j + 1)(m(i) + 1 + j)}.
\]

(A5)

One can then substitute (A5) into (A3) and integrate over \( D_S \), using \( m(i) = b_F, 1 + b_F, 2 + b_F, 0, 1, 2, \) and \( H_i(D_S) = a_S D_S^3, 2 a_S D_S^5, a_F D_F^2, -a_S D_S^{5+b_S} - 2 a_S D_S^{4+b_S} - a_S D_S^{3+b_S} \). After integration the correction to be added to (14) in the exact solution is found to be
After some algebra, the correction can be written as

\[
dr_S \bigg|_{\text{corr}} = -\frac{2r_F r_S \lambda_p p_0}{24p_S} v_{sc} \times \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1+j)} \left( \frac{v_{sc}}{v_{fc}} \right)^{(3+j)b_F} \Gamma \left[ 4 + b_S + \frac{b_S}{b_F} (3+j) \right] b_F \frac{b_F}{(3+j)(b_F + 3+j)} \right\}
\]

\[+ 2 \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1+j)} \left( \frac{v_{sc}}{v_{fc}} \right)^{(2+j)b_F} \Gamma \left[ 5 + b_S + \frac{b_S}{b_F} (2+j) \right] b_F \frac{b_F}{(2+j)(b_F + 2+j)} \frac{\lambda_F}{\lambda_S} \]

\[+ \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1+j)} \left( \frac{v_{sc}}{v_{fc}} \right)^{(1+j)b_F} \Gamma \left[ 6 + b_S + \frac{b_S}{b_F} (1+j) \right] b_F \frac{b_F}{(1+j)(b_F + 1+j)} \left( \frac{\lambda_F}{\lambda_S} \right)^2. \quad (A7)\]

The ratio of the \(j + 1\) to \(j\) terms in the last summation is given by

\[
\left[ \frac{v_{sc}}{v_{fc}} \right]^{b_F (6 + b_S + \frac{b_S}{b_F} (1+j))} \left( b_F + 2 + j \right) \frac{\Gamma \left[ 6 + b_S + \frac{b_S}{b_F} (2+j) \right]}{(2+j)\Gamma \left[ 6 + b_S + \frac{b_S}{b_F} (1+j) \right]}.
\]

The first bracketed factor is a constant independent of \(j\). The second factor approaches unity as \(j\) becomes large. To take the limit of the third factor, we can apply the relation (Abramowitz and Stegun 1970)

\[
\lim_{\varepsilon \to 0} \frac{\Gamma(z + c)}{\Gamma(z)} \sim \varepsilon^c.
\]

(A8)

With \(c = b_S/b_F\) and \(z = 6 + b_S + (b_S/b_F)(1+j)\), we have

\[
\lim_{j \to \infty} \frac{b_F}{b_S} \frac{\Gamma(z + c)}{\Gamma(z)} \sim (z - 6 - b_S) + 1. \quad (A9)
\]

As \(z\) and hence \(j\) become large, the magnitude of the terms in the summation approach zero if \(c = b_S/b_F < 1\), but become unbounded if \(b_S/b_F > 1\), by the ratio test for series convergence. The same result applies to the other summations as well. If \(b_S = 0\), the third bracketed term is just \(1/(2+j)\), which also clearly approaches zero for large \(j\). So \(b_S/b_F < 1\) is the necessary condition for the exact solution of (14) plus (A7) to apply. Since the series are alternating, when the absolute values of the individual terms approach zero, convergence of the summation is assured. Hence, \(b_S/b_F < 1\), is also a sufficient condition for the applicability of this solution. The only exceptional case is \(b_S = b_F\), which will converge to a solution if \(v_{sc} < v_{fc}\) but will diverge otherwise.

For the case of power-law velocities but a slow-falling collector, we can apply the same procedure, but this time the form of \(G(D_S, D_F)\) is different because \(m_F(D_F)\) appears rather than \(m_S(D_S)\). The correction in this case can be shown to be

\[
\frac{dr_F}{dt} \bigg|_{\text{corr}} = -\frac{2r_F r_S \lambda_p p_0}{24p_S} v_{sc} \times \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1+j)} \left( \frac{v_{sc}}{v_{fc}} \right)^{(4+j)b_F} \Gamma \left[ 3 + b_S + \frac{b_S}{b_F} (4+j) \right] b_F \frac{b_F}{(4+j)(b_F + 4+j)} \right\}
\]

\[+ 2 \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1+j)} \left( \frac{v_{sc}}{v_{fc}} \right)^{(5+j)b_F} \Gamma \left[ 2 + b_S + \frac{b_S}{b_F} (5+j) \right] b_F \frac{b_F}{(5+j)(b_F + 5+j)} \frac{\lambda_S}{\lambda_F} \]

\[+ \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1+j)} \left( \frac{v_{sc}}{v_{fc}} \right)^{(6+j)b_F} \Gamma \left[ 1 + b_S + \frac{b_S}{b_F} (6+j) \right] b_F \frac{b_F}{(6+j)(b_F + 6+j)} \left( \frac{\lambda_S}{\lambda_F} \right)^2. \quad (A10)\]

These summations similarly can be shown to converge for \(b_S < b_F\) and diverge for \(b_F > b_S\).
APPENDIX B

Scale-Based Ordering of Terms in Exact Solution

The exact solution for the fast-falling collector case consists of the GWA equation (14), which can be written:

$$\frac{dr_s}{dt} = - \frac{rs_p \lambda_p \rho_0 v_{Sc}}{24 \rho_r} \left[ \Gamma(3 + b_p) \Gamma(4) + 2 \Gamma(2 + b_p) \Gamma(5) \beta_\lambda + \Gamma(1 + b_p) \Gamma(6) \beta_\lambda^2 - \beta_\lambda \Gamma(3) \Gamma(4 + b_s) \right]$$

$$+ 2 \Gamma(2) \Gamma(5 + b_s) \beta_\lambda + \Gamma(1) \Gamma(6 + b_s) \beta_\lambda^2 \right] \right\},$$

(B1)

plus the correction (15), which can be written:

$$\frac{dr_s}{dt} \bigg|_{\text{corr}} = - \frac{2rs_p \lambda_p \rho_0 v_{Sc}}{24 \rho_r} \beta_s \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1 + j)} \left( \frac{v_{Sc}}{v_{Fc}} \right)^{(3+j)/b_F} \right\} \Gamma \left[ 4 + b_s + \frac{b_s}{b_F} (3 + j) \right] \right\} \frac{b_F}{(3 + j)(b_F + 3 + j)}$$

$$+ 2 \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1 + j)} \left( \frac{v_{Sc}}{v_{Fc}} \right)^{(2+j)/b_F} \Gamma \left[ 5 + b_s + \frac{b_s}{b_F} (2 + j) \right] \frac{b_F}{(2 + j)(b_F + 2 + j)} \beta_\lambda$$

$$+ \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(1 + j)} \left( \frac{v_{Sc}}{v_{Fc}} \right)^{(1+j)/b_F} \Gamma \left[ 6 + b_s + \frac{b_s}{b_F} (1 + j) \right] \frac{b_F}{(1 + j)(b_F + 1 + j)} \beta_\lambda^2 \right\}$$

(B2)

where $\beta_s = v_{Sc}/v_{Fc}$, and $\beta_\lambda = \lambda_F/\lambda_S$. All terms in these equations belong to one of three groups, depending on whether the exponent of $\beta_s$ is 0, 1, or 2. Consider the terms for one of the groups. The only mixing-ratio-dependent parameter that determines the relative magnitudes of terms within a group is $\beta_s$, so this factor may be used to scale the terms. The terms in (B1) are either of zeroth order or of order $\beta_s$. The lowest-order terms in (B2) contain an additional factor of at least $\beta_s^{\lambda/\lambda_F}$.

Therefore, (B1) is the first-order approximation for the exact solution within the group, provided $b_s \approx b_F$, and will become an increasingly accurate approximation as $\beta_s$ becomes small. As a consequence, in the set of all the approximations to the exact collection equation of form $a + b_\lambda \beta_s + c \beta_s^2$ that are first order in $\beta_s$, the GWA equation (B1) will be the most accurate, for $b_s \approx b_F$, and $\beta_s$ sufficiently small. In particular, it will be more accurate than the conventional Wisner equation, which under these conditions is a member of that set. Furthermore, the relative accuracy of (B1) increases as $\beta_s$ decreases. From these consequences the comparative discussion in the main section of this paper is justified.

However, when computing the actual magnitude of all the terms in (B1) and (B2), two caveats must be kept in mind, which can alter the ordering of the terms even when $\beta_s$ is less than unity. The first is the fact that $\beta_\lambda$ introduces another mixing-ratio-dependent scaling that can counteract the scaling of $\beta_s$. The second is that the constants in (B1) and (B2) can differ considerably in magnitude and, in fact, can be much greater in the $\beta_s$ group than in the others [e.g., for $b_s = 0.66$, $\Gamma(1)\Gamma(6 + b_s) = 384.3$, while $\Gamma(3)\Gamma(4 + b_s) = 29.1$].

Because of these considerations, it is possible that the magnitude of terms from a group in the series solution (B2) can exceed those of some terms from another group retained in the GWA equation (B1), even for $\beta_s < 1$. We now investigate under what conditions this may happen. We will use the two main cases in the text, one involving “power rain” ($b_F = 0.88$) collecting snow ($b_s = 0.11$) and the other involving power rain collecting graupel ($b_s = 0.66$). The first possibility is that $\beta_\lambda$ becomes so small that the first-order $\beta_\lambda$ term in the GWA equation becomes smaller than the lowest-order $\beta_s^{\lambda/\lambda_F}$ term in (B2). (We will always define order as the exponent of the $\beta_s$ factor.) This criterion becomes a function of both $\beta_\lambda$ and $\beta_s$. However, the larger constants in the $\beta_\lambda$ term and the additional factor of $\beta_s^{\lambda/\lambda_F}$ in the $\beta_s^{\lambda/\lambda_F}$ term conspire to make it unlikely that the latter will exceed the former. Furthermore, $\beta_\lambda$ tends to stay near unity because it is proportional to the fourth root of $r_s/r_F$ for spherical hydrometeors, and in the cases where it is smallest $\beta_s^{\lambda/\lambda_F}$ in the series term will also be the smallest. A calculation (not shown) determined that the above criterion is never satisfied for plausible power rain–graupeal interactions and is only satisfied for power rain–snow interactions for snow characteristic sizes less than approximately ten microns, which is a dubious application of these formulas anyway. In brief, we need not be concerned about the first possibility.

The second possibility is that $\beta_\lambda$ becomes so large that the lowest-order $\beta_s^{\lambda/\lambda_F}$ term in the series exceeds that of the first-order $\beta_s^{\lambda/\lambda_F}$ term in (B1). This is more likely than the previous possibility because the $\beta_\lambda$ has larger constants, and is only multiplied by a factor of $\beta_s^{\lambda/\lambda_F}$. Furthermore, increasing $\beta_\lambda$ tends to decrease $\beta_s^{\lambda/\lambda_F}$, has...
tening the fulfillment of this criterion. It is easily shown that the lead $\beta_e^1$ term in (B2) exceeds the first-order $\beta_{s}^0$ term in (B1) when

$$\frac{\lambda_F}{\lambda_s} > \left[ \frac{\Gamma(3)\Gamma(4 + b_S)(b_F + 1)}{2b_F \left( \frac{\nu_{sc}}{\nu_{Fc}} \right)^{1/b_F} \Gamma \left( 6 + b_S + \frac{b_S}{b_F} \right)} \right]^{0.5}. \quad (B3)$$

For the case of power rain collecting graupel (parameters given in Table 1), this criterion is indeed satisfied for $\nu_{sc}/\nu_{Fc} \approx 0.2$ and $\lambda_F/\lambda_s \approx 0.35$. (The exact value is slightly dependent on the actual mixing ratios.) Therefore, this is the region where the velocity-scaled ordering and the GWA begins to break down for power rain as the fast-falling collecting species and graupel as the slow-falling collected species. The GWA would still be quite accurate because it would contain five of the six largest terms in the exact solution expansion. However, it can also be shown that, when

$$\beta_e = \frac{\nu_{sc}}{\nu_{Fc}} > \left[ \frac{\Gamma(1)\Gamma(6 + b_S)(b_F + 1)}{2b_F \Gamma \left( 6 + b_S + \frac{b_S}{b_F} \right)} \right]^{b_F}, \quad (B4)$$

which for the current case gives $\nu_{sc}/\nu_{Fc} > 0.32$, the lowest order $\beta_e^1$ term in (B2) even exceeds the first-order $\beta_{s}^0$ term in (B1). Not only does this invalidate the velocity-scaled ordering of terms within the $\beta_e^1$ group, but it means that a term from the series in (B2) exceeds what is often the second largest of the six terms in the GWA equation. At this point (occurring when $\beta_e = 0.6$) the GWA would no longer be even approximately valid.

Repeating the calculation for the power rain–snow case shows that the largest term from (B2) exceeds the smallest term in (B1) for $\beta_{s} \approx 0.5$, with $\beta_{s}$ ranging from 0.4 for small mixing ratios to 0.6 for larger ones. However, for the criterion in (B4) to be satisfied, a velocity ratio $\beta_{s} > 0.91$ is required. This is contrary to the assumption of small $\beta_{s}$, and furthermore is unlikely for power rain–snow interactions unless the characteristic size for power rain is quite small (e.g., a $D_c$ of at least $400 \mu m$ of power rain would require a $D_c$ of at least $13.2 \mm$ for snow). The difference from the graupel case is directly related to a much greater difference between the snow and power rainfall velocity exponents.

To summarize the fast-falling collector case: the GWA equation will be the low-order representation of the exact solution for $b_F > b_S$ and $\beta_{s} = \nu_{sc}/\nu_{Fc}$ sufficiently small. For values of $\beta_{s}$ slightly larger but still less than unity, the magnitude of some neglected terms for the GWA equation will begin to exceed some of those retained. However, the GWA equation should still be more accurate than the CWA under these circumstances because it has the correct representation of the largest terms (i.e., those of order $\beta_e^0$), whereas the CWA equation does not. For even larger values of $\beta_{s}$, neglected terms in the GWA equation become larger than all first-order terms retained, and the GWA will no longer be a good approximation. The smaller the difference in fall velocity exponents, the more limited the range of GWA applicability. For a case of $b_F = 0.8$ and $b_S = 0.11$, the GWA is very good for virtually the whole parameter space; for a case of $b_F = 0.8$ and $b_S = 0.66$, the GWA is only accurate for $\beta_e < 0.32$. A similar scaling can be performed on the slow-falling collector solution (20) plus (21). The lowest-order term in (21) is of order $1 + 4/b_F$; the terms in (20) are of order 0 and 1, as before. As in the fast-falling collector case, the GWA equation (20) is the most accurate first-order form for small $\beta_s$ and $b_S \leq b_F$. But, compared with the fast-falling collector case, there is a considerably greater separation between the orders of the (20) terms and the (21) terms. Therefore, for a given value of $\beta_e$, the slow-falling collector GWA equation should be more accurate than the fast-falling collector GWA equation. In addition, the constants in the series are considerably smaller than the largest ones in the GWA equation. In brief, provided $\beta_e < 1$, the appropriateness of the slow-falling collector GWA equation is virtually assured.

REFERENCES


