On the Thickness Ratio in the Quasigeostrophic Two-Layer Model of Baroclinic Instability

NOBORU NAKAMURA AND LEI WANG

Department of the Geophysical Sciences, University of Chicago, Chicago, Illinois

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ABSTRACT

It is shown that the classical quasigeostrophic two-layer model of baroclinic instability possesses an optimal ratio of layer thicknesses that maximizes the growth rate, given the basic-state shear (thermal wind), beta, and the mean Rossby radius. This ratio is interpreted as the vertical structure of the most unstable mode. For positive shear and beta, the optimal thickness of the lower layer approaches the midheight of the model in the limit of strong criticality (shear/beta) but it is proportional to criticality in the opposite limit. For a set of parameters typical of the earth’s midlatitudes, the growth rate maximizes at a lower-layer thickness substantially less than the midheight and at a correspondingly larger zonal wavenumber.

It is demonstrated that a turbulent baroclinic jet whose statistical steady state is marginally critical when run with equal layer thicknesses can remain highly supercritical when run with a nearly optimal thickness ratio.

1. Introduction

The quasigeostrophic two-layer model, developed more than 60 yr ago by Phillips (1951), contains all essential ingredients of large-scale dynamics in the midlatitude yet its solutions are accessible through elementary mathematics and affordable numerical integrations. Held (2005) refers to the model as the “E. coli of climate models” (p. 1612), drawing analogy to the guiding roles played by the microorganism in molecular biology.

In applications to atmospheric baroclinic instability, the rest thicknesses of the two layers are often assumed equal. Unlike the oceans in which an observed pycnocline serves as a basis for the two-layer representation, the troposphere lacks a clear distinction between the upper and lower layers. Given this, assigning equal depths to the two layers may be considered an unbiased choice. However, since the thickness ratio also defines the vertical structure of modes, the consequence of its choice merits careful examination. Despite the thorough documentation of the unstable normal modes (Phillips 1954; Pedlosky 1987, 551–574; Vallis 2006, 271–280), we feel that this aspect of Phillips’s model has received little attention and hence the writing of this paper.

2. Two-layer model with unequal layer thicknesses

The thickness ratio in Phillips’s model affects baroclinic instability in two ways. First, it sets the steering level of an unstable mode, where the phase speed of the mode equals the basic-state zonal wind. In the two-layer model the phase speed of an unstable mode is bounded by the maximum basic-state zonal wind and by the minimum Doppler-shifted Rossby wave speed (Pedlosky 1987, p. 515). Except at small wavenumbers, this falls between the upper- and lower-layer winds, making the layer interface the steering level. If we define the penetration depth of a mode using the steering level, the thickness of the lower layer provides exactly that depth.

Second, the thickness ratio regulates the gradients of the basic-state potential vorticity (PV) that the modes recognize. Suppose that beta and shear (the basic-state thermal wind) are both positive so their contributions to the lower-layer PV gradient are opposite in sign. Since modes recognize only the vertically averaged PV gradient in each layer, shear’s contribution is “diluted” as the thickness of the layer increases. As a result, the critical shear necessary to keep the lower-layer PV gradient negative (and hence the flow unstable; Charney and Stern 1962) increases with the thickness of the lower layer.

Our main question is at what thickness ratio (vertical structure of the mode) the growth rate of instability is maximized given shear, beta, and other parameters of the flow. To address this question, we start from the
standard, linear normal-mode PV equations and nondimensionalize them in a form convenient for our purpose:

\[
(U_1^* - c^*) \left[ -(k^{*2} + l^{*2})\psi_1^* + \frac{f_0^{*2}}{g^*H_1^*} (\psi_2^* - \psi_1^*) \right] \\
+ \left( \beta^* + \frac{2f_0^{*2}\Delta^*}{g^*H_1^*} \right) \psi_1^* = 0, \quad (1a)
\]

\[
(U_2^* - c^*) \left[ -(k^{*2} + l^{*2})\psi_2^* - \frac{f_0^{*2}}{g^*H_2^*} (\psi_2^* - \psi_1^*) \right] \\
+ \left( \beta^* - \frac{2f_0^{*2}\Delta^*}{g^*H_2^*} \right) \psi_2^* = 0, \quad (1b)
\]

where the subscripts 1 and 2 denote the upper and lower layers, respectively, and the asterisk emphasizes that a quantity is in dimensional form. Here, \(U^*\) and \(H^*\) are the zonal wind and layer thickness of the basic state, and \(f_0^*, \beta^*, \) and \(g^*\) are the Coriolis parameter, beta, and reduced gravity, respectively. Note that shear \(\Delta^*\) is defined as \(U_1^* = U_2^* + 2\Delta^*\), and a normal-mode solution of the form \(\psi^* \propto \exp[ik^*(x^* - c^*t^*) + il^*y^*]\) is assumed for streamfunction. We define external length scales and the thickness ratio as

\[
H_0^* = \frac{H_1^* + H_2^*}{2}; \quad L_D^* = \frac{g^*H_0^*}{2f_0^*}; \quad \delta = \frac{H_2^*}{H_0^*}; \quad \epsilon = 1 - \delta.
\]

(2)

Note 0 < \delta < 2 and -1 < \epsilon < 1, (\delta, \epsilon) = (1, 0) corresponding to equal layer depths. Now nondimensionalize \(U_1^*, U_2^*, \) and \(c^*\) by \(\Delta^*; \) \(k^{*2}\) and \(l^{*2}\) by \(L_D^*; \) and \(\beta^*\) by \(\Delta^*L_D^{*2}; \) Then (1a) and (1b) become

\[
(U_1 + \frac{1}{2}) \left[ -(1 + \epsilon)k_1^2 + l_1^2\right] \psi_1 + \frac{1}{2}(\psi_2 - \psi_1) \\
+ [(1 + \epsilon)\beta_1 + 1] \psi_1 = 0, \quad (3a)
\]

\[
(U_1 - \frac{1}{2}) \left[ -(1 - \epsilon)k_1^2 + l_1^2\right] \psi_2 - \frac{1}{2}(\psi_2 - \psi_1) \\
+ [(1 - \epsilon)\beta_1 - 1] \psi_2 = 0, \quad (3b)
\]

where \(\hat{u} = [(U_1 + U_2)/2] - c\) and the asterisk has been dropped. The dispersion relation is

\[
\mu(\mu + 1)\hat{u}^2 - [2\mu(p - \epsilon) + p] \hat{u} - \mu^2 + \mu + p(p - \epsilon) = 0, \quad (4)
\]

where \(\mu = (1 - \epsilon^2)(k_1^2 + l_1^2)\) and \(p = \beta(1 - \epsilon^2).\) The solution is supercritical (unstable for certain wavenumbers) when \(\beta^{-1} > 1 - \epsilon = \delta, \) which demonstrates the proportional relationship between the critical shear and the lower-layer thickness (\(\beta^{-1}\) is routinely referred to as the criticality parameter). Marginal stability is found at

\[
k^2 + \hat{l}^2 = \left( \frac{1 + \epsilon \beta}{2(1 - \epsilon^2)} \right) \left[ 1 + (1 + \epsilon)\beta(1 - \epsilon) \right]^{1/2}.
\]

(5)

corresponding to short- and long-wave cutoffs. For \(\beta = 0\) the long-wave cutoff coincides with \(k = l = 0, \) whereas the short-wave cutoff wavenumber is

\[
k^2 + \hat{l}^2 = (1 - \epsilon^2)^{-1/2} = [\delta(2 - \delta)]^{-1/2}.
\]

(6)

In (6) the cutoff wavenumber minimizes at \(k^2 + \hat{l}^2 = 1\) when \(\delta = 1\) (equal layer thickness), whereas it becomes infinite as \(\delta \to 0\) or 2 (i.e., when the thickness of either layer vanishes). The growth rate of the unstable mode is given by

\[
|k_c| = \left| k\hat{u}_1 \right| = \left| \frac{\mu(1 - \mu)}{1 + \mu} \right| \left| \frac{\epsilon \mu^2 - \beta(1 - \epsilon^2)}{(1 - \epsilon^2)\mu(\mu + 1)^2} \right|^{1/2}.
\]

(7)

These results have been widely known under different notations (Pedlosky 1987; Vallis 2006). To see how \(\epsilon\) affects the maximum growth rate, it is instructive to consider the transverse mode \([l = 0, k = \mu/(1 - \epsilon^2)].\) Upon substitution, (7) may be arranged into

\[
(k_c)^2 = \frac{\mu(1 - \mu)}{1 + \mu} \left| \frac{\epsilon \mu^2 - \beta(1 - \epsilon^2)}{(1 - \epsilon^2)\mu(\mu + 1)^2} \right|^{1/2}.
\]

(8)

Since the last term on the right-hand side is nonpositive and the first term depends only on \(\mu, \) the maximum growth rate occurs for a value of \(\mu\) that maximizes the first term and a combination of \(\epsilon\) and \(\beta\) that makes the last term vanish for this \(\mu:\)

\[
|k_c|_{\text{max}} = \sqrt{2} - 1,
\]

(9)

\[
\mu_{\text{max}} = \sqrt{2} - 1, \quad \epsilon_{\text{max}}(\sqrt{2} - 1)^2 - \beta(1 - \epsilon_{\text{max}})^2/2 = 0. \quad (9)
\]

From (9) the optimal thickness ratio is given by

\[
\epsilon_{\text{max}} = -\frac{(\sqrt{2} - 1)^2}{\beta} + \left[ 1 + \frac{(\sqrt{2} - 1)^4}{\beta^2} \right]^{1/2}
\]

(10
The other root of \( \varepsilon \) is negative and unsuitable for \( \beta > 0 \) [see (9)]. Alternatively, by substituting \( \varepsilon_{\text{max}} = 1 - \delta_{\text{max}} \) in (9) and assuming \( \delta_{\text{max}} \ll 2 \), an approximate expression

\[
\delta_{\text{max}} \approx \frac{(\sqrt{2} - 1)^2}{(\sqrt{2} - 1)^2 + \beta} = \frac{1}{1 + 5.83\beta} \quad (11)
\]

may be obtained. Note \( \delta_{\text{max}} \to 1 \) as \( \beta \to 0 \); the optimal lower-layer thickness for baroclinic instability approaches the midheight of the model (equal layer thickness) in the limit of strong criticality. In the opposite limit (\( \beta \gg 1 \)) \( \delta_{\text{max}} \propto \beta^{-1} \); namely, the optimal thickness becomes proportional to criticality. Once \( \varepsilon_{\text{max}} \) (and \( \delta_{\text{max}} \)) is obtained, the wavenumber \( k \) that maximizes the growth rate may be computed from (9) and (11) as

\[
k_{\text{max}} = \left( \frac{\mu_{\text{max}} [\delta_{\text{max}}/(2 - \delta_{\text{max}})]}{1/2} \right)^{1/2} \approx \frac{0.644 + 3.75\beta}{\sqrt{1 + 11.7\beta}}. \quad (12)
\]

Without the meridional convergence of the zonal-mean eddy momentum flux, the vertically integrated zonal-mean eddy PV flux vanishes (Pedlosky 1987, p. 551). With the PV flux written as the product of PV gradient and eddy diffusivity \( K \), this may be written as

\[
K_1[(2 - \delta)\beta + 1] + K_2(\delta\beta - 1) = 0. \quad (13)
\]

From (13) and (11), the diffusivity ratio \( K_2/K_1 \) that corresponds to \( \delta_{\text{max}} \) is

\[
(K_2/K_1)_{\text{max}} = 1 + \beta[2.41 + 0.416/(1 + 4.83\beta)]. \quad (14)
\]

Thus \( K_2 > K_1 \) for \( \beta > 0 \) and the ratio increases with increasing \( \beta \).

Figure 1 shows the growth rate of the transverse mode as a function of \( k \) and \( \delta \) for \( \beta = 0, 0.5, \) and 1.0. With \( \beta = 0 \), the growth rate maximizes at \( k_{\text{max}} = 0.644, \delta_{\text{max}} = 1 \), and it is symmetric about \( \delta = 1 \). There is no long-wave cutoff, whereas the short-wave cutoff wavenumber increases toward \( \delta = 0 \) and 2 as (6) predicts (Fig. 1a). A positive \( \beta \) breaks the symmetry and the growth rate maximum shifts to smaller \( \delta \) and larger \( k \), although the value of the maximum growth rate does not change. A long-wave cutoff is also introduced (Fig. 1b). With \( \beta = 1 \), the flow is unstable only for \( \delta < 1 \). This corresponds to the supercriticality condition \( (\beta^{-1} > \delta, \text{Fig. 1c}) \). The area of unstable domain becomes substantially smaller, with further shifts in \( \delta_{\text{max}} \) and \( k_{\text{max}} \).

Figure 2 plots \( \theta_{\text{max}} \), \( k_{\text{max}} \), and \( (K_2/K_1)_{\text{max}} \) as functions of \( \beta \), using both exact and approximate formulæ. As \( \beta \) increases \( \delta_{\text{max}} \) decreases quickly, whereas the corresponding increase in \( k_{\text{max}} \) is more gradual. The diffusivity ratio \( (K_2/K_1)_{\text{max}} \) increases nearly linearly with \( \beta \), indicating that eddying motion is increasingly confined to the lower layer.

3. Discussion

Interpreting \( \delta \) as the mode’s vertical scale allows one to draw analogy between Phillips’s (1951) and Charney’s (1947) models. Unstable normal modes in Charney’s model do not have short-wave cutoff because they can adjust their vertical scale (critical level) as the wavelength decreases (Bretherton 1966). In Phillips’s model short waves are stabilized when their vertical scale falls below the prescribed \( \delta \). However, if \( \delta \) is allowed to vary with the horizontal wavenumber, Phillips’s model, too, can trace the most unstable mode to an arbitrarily large wavenumber. Figure 3a shows the maximum growth rate of the transverse mode for varying \( \delta \) (here \( \beta = 1 \) is
assumed) as a function of the corresponding wavenumber $k_m(\delta)$. It resembles the growth rate of the Charney mode in Fig. 3b ($k > 0.88$) for a comparable choice of parameters: there is no short-wave cutoff.

One might also compare (11) with the vertical scale of the most unstable mode in Charney’s model, $h^*$:

$$
\frac{h^*}{H^*} \approx \frac{1}{1 + (\beta^* N^* f^*)/(\Lambda^* f^*)} = \frac{1}{1 + (\beta^* L_D^2)/(\Lambda^* H^*)}.
$$

(15)

where $H^*$ is scale height, $N^*$ is Brunt–Väisälä frequency, and $\Lambda^*$ is vertical shear (Held 1978). In both models the mode depth is proportional to criticality in the limit of weak criticality and approaches a constant in the opposite limit (midheight and scale height, respectively). However, while $h^*$ is proportional to the horizontal scale of the Charney mode, it is the square root of $\delta_{\text{max}}$ that is proportional to the horizontal scale of the unstable modes in Phillips’s model (see (12)). As a result, for modes with similar horizontal scales $\delta_{\text{max}}$ is much smaller than $h^*/H^*$. For example, let $2\Delta^* = \Lambda^* H^* = 40 \text{ m s}^{-1}$, $\beta^* = 1.6 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$, $L_D^* = 800 \text{ km};$ typical for the earth’s midlatitude. This gives $\beta^* 

\approx 0.51$ in (11) so $\delta_{\text{max}} \approx 0.25$, whereas (15) predicts $h^*/H^* = 0.8$, although the zonal wavenumbers of the modes are comparable. Such a large discrepancy might appear surprising considering that a stratified model can be cast into a layered model using isentropic coordinate. It suggests that the lower boundary in Charney’s model, where the

![Fig. 2](http://journals.ametsoc.org/jas/article-pdf/70/5/1505/3871178/jas-d-12-0344_1.pdf)

**Fig. 2.** Properties of the most unstable mode in Phillips’s model as functions of $\beta$ for (a) $\delta_{\text{max}}$, (b) $k_{\text{max}}$, and (c) $(K_2/K_1)_{\text{max}}$. Solid curves are exact solutions and dashed curves are approximate solutions based on (11). See text for details.

![Fig. 3](http://journals.ametsoc.org/jas/article-pdf/70/5/1505/3871178/jas-d-12-0344_2.pdf)

**Fig. 3.** (a) Maximum growth rate of the unstable mode in Phillips’s model as a function of $k_m(\delta)$, where $k_m(\delta)$ is the zonal wavenumber that maximizes the growth rate for the given $\delta$ when $\beta = 1$, $l = 0$. (b) Growth rates of the unstable modes in Charney’s model as a function of nondimensional zonal wavenumber for $(\beta^* N^* f^*)/(\Lambda^* f^*) = 1$, $l^* = 0$, where $L_D^* = N^* H^*/f^*$ is the Rossby radius, $N^*$ is the Brunt–Väisälä frequency, $H^*$ is scale height, and $\Lambda^*$ is vertical shear. Wavenumber and growth rate are nondimensionalized by $L_D^* f^*$ and $f^* N^*$, respectively.

![Fig. 4](http://journals.ametsoc.org/jas/article-pdf/70/5/1505/3871178/jas-d-12-0344_3.pdf)

**Fig. 4.** Vertically averaged eddy kinetic energy at the center latitude of the channel in the forced dissipative simulations for $\delta = 1.0$ (gray) and $\delta = 0.25$ (black).
isentropes intersect the ground, invalidates this analogy and makes surface quasigeostrophic dynamics (Held et al. 1995) dictate the solution in which the scaling of PV leads to $h^* \propto (k^* + l^*)^{-1/2}$.

When one specifies $\delta$ in the two-layer model, one effectively singles out a vertical structure of eddies. How does this choice affect the result of, and on what basis should one choose $\delta$ in, an initial-value problem? The last question is perhaps more straightforward for the oceans, since the observed pycnocline depth may be compared directly to the interface height of the model. Indeed, the two-layer model with unequal layer depths and its variants have been used in a number of stability analysis studies for ocean gyres (e.g., Gill et al. 1974; Robinson and McWilliams 1974; Flierl 1978; Killworth 1980). A shallow mixed layer overlying a deep bottom...
layer is like a vertically inverted model of ours with small \( \delta \), except that the unstable modes are trapped in the top layer when shear is negative (e.g., surface current is westward).

Apart from the observed stratification, two choices for \( \delta \) seem equally plausible based on the following theoretical considerations: (i) \( \delta = \beta^{-1} \) and (ii) \( \delta = \delta_{\text{max}} \). In Phillips’s model tall modes are generally less unstable when \( \beta \neq 0 \) because shear’s negative contribution to the lower-layer PV gradient is diluted. All modes taller than \( \delta_{\text{c}} = \beta^{-1} \) are stable as shear becomes subcritical (Fig. 1c). The critical value \( \delta_{\text{c}} \) may be interpreted as the depth of the heat-transporting eddy required to neutralize the flow (Lindzen and Farrell 1980; Stone 1978). Evaluating \( \beta^{-1} \) from the observed shear suggests that \( \delta_{\text{c}} \geq 1 \); that is, to maintain the atmosphere marginally critical one needs deep eddies (Green 1970; Held 1982). Then, \( \delta = \beta^{-1} \approx 1 \) will be a reasonable choice if one aims to simulate a marginally critical state of an Earth-like atmosphere. However, such state is preconditioned to deep, slowly growing eddies and a priori eliminates shallower, fast-growing eddies that could emerge if \( \delta \) were smaller. For the set of parameters introduced earlier, \( \delta = 1 \) gives a maximum growth rate 0.332 at \( k \approx 0.74 \) and \( K_2/K_1 \approx 3.1 \), whereas a much greater growth rate 0.414 is achieved if we choose \( \delta = \delta_{\text{max}} = 0.25 \) at \( k_{\text{max}} \approx 0.97 \) and \( K_2/K_1 \approx 2.2 \). (Note that the equal layer thicknesses produce a greater asymmetry in diffusivity). Thus, by choosing \( \delta = 0.25 \) one would render the flow significantly more unstable and decrease the horizontal scale of eddy.

In Figs. 4 and 5 we compare two forced dissipative simulations of a turbulent baroclinic jet on a wide plateau channel using a two-layer model similar to Esler (2008). The mean shear is relaxed toward a jet (gray curves in Figs. 5c,d), while the lower-layer vorticity is damped by Ekman friction. The two runs are identical except \( \delta = 1 \) in the first and \( \delta = 0.25 \) (close to the optimal thickness at the axis of the jet) in the second, and the Ekman damping in the lower layer is 4 times stronger in the latter. The model is run for 900 days (the time series of eddy kinetic energy is shown in Fig. 4) and statistics are constructed from the last 400 days of the simulations. Although the equilibrium wind profile is identical, with \( \delta = 1 \) the corresponding lower-layer PV gradient is negative only near the center of the channel (gray dashed curve in Fig. 5a), whereas with \( \delta = 0.25 \) it is everywhere negative and strongly so near the axis of the jet (Fig. 5b). In statistical steady state, the negative PV gradient in the former is nearly eliminated, whereas the lower-layer gradient in the latter, though substantially reduced, still remains strongly negative (Figs. 5a,b). The mean shear in the statistical steady state with \( \delta = 0.25 \) is much weaker than that with \( \delta = 1 \) (Figs. 5c,d), whereas eddy kinetic energy is higher (Fig. 4), thus the conversion of available potential energy to kinetic energy is more efficient.

Since the two-layer model prescribes the vertical structure of baroclinic eddies with a preset thickness ratio, any single realization of statistical steady state should be interpreted with care; our results demonstrate that changing this parameter with all other parameters fixed can deliver a range of climate states, from marginally critical to strongly supercritical. Typically a supercritical state emerges when the width of unstable baroclinic zone is much wider than the horizontal scale of eddies (Pavan and Held 1996; Nakamura 1999; Jansen and Ferrari 2012). In the two-layer model, this condition is readily met under earth-like parameters when the lower-layer thickness is significantly less than the mid-height. By considering only equal layer thicknesses one could miss a large class of climate states with high eddy activity that may be equally realizable.

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