Estimation of Atmospheric Duct Structure Using Radar Sea Clutter

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ABSTRACT

Retrieving atmospheric refractivity profiles from the sea surface backscattered radar clutter is known as the refractivity-from-clutter (RFC) technique. Because the relationship between refractivity and radar sea clutter is clearly nonlinear and ill posed, it is difficult to get analytical solutions according to current theories. Previous works treat this problem as a model parameter estimation issue and some optimization algorithms are selected to get approximate solutions. Two main factors that limit the accuracy of the estimation are that 1) the refractive environments are described by using some idealized refractivity parameter models that cannot describe the exact information of the refractivity profile, and 2) accurate modeling of the sea surface radar cross section (RCS) is very difficult. Rather than estimating a few model parameters, this paper puts forward possibilities of using the variational adjoint approach to jointly retrieve the every-height refractivity values and sea surface RCS using radar clutter data. The derivation of the adjoint model is accomplished by an analytical transformation of the parabolic equation (PE) in the continuous domain. Numerical simulations including range-independent and range-dependent RCS cases are presented to demonstrate the ability of this method for RFC estimations. Making use of the refractivity retrievals, propagation loss predictions are also presented.

1. Introduction

An accurate knowledge of atmospheric refractive conditions is important for the performance of shipboard radar and communication systems. However, direct measurement of temperature, pressure, and humidity distributions, which determine refractivity, is difficult and expensive. Recently, remote sensing of refractivity using radar clutter returns that are backscattered from the ocean surface has been developed. An attractive feature of inferring refractivity from clutter (RFC) is that it is unnecessary to use additional hardware or extra meteorological/electromagnetic measurements (Yardim 2007). In RFC estimations, pulses transmitted from shipboard radar are processed to yield clutter power as a function of range. This clutter return is a function of the one-way propagation loss from the transmitter to the range cell and the radar cross section (RCS). The propagation loss depends on the index-of-refraction profile in the troposphere, through which the radiated beam propagates. For example, the capping inversion of the marine boundary layer often produces a trapping layer, or downward-refracting region, which increases the energy trapped near the surface and, therefore, the clutter strength well beyond the horizon (Kraut et al. 2004).

Based on the experiments launched at Wallops Island (called Wallops98) and other measurement data, some optimization procedures for RFC have been proposed (Krolik et al. 1999; Rogers et al. 2000, 2005; Gerstoft et al. 2003a,b, 2004; Barrios 2004; Yardim et al. 2006, 2007, 2008, 2009; Vasudevan et al. 2007; Douvenot et al. 2008, 2010; Wang et al. 2009; Park and Fabry 2011). Most of these RFC techniques use the split-step Fourier parabolic equation (PE) method to model electromagnetic wave propagation. Brief introductions of the above different RFC algorithms can be found in Karimian et al. (2011) and Zhao and Huang (2011). Although all these algorithms show promise for RFC estimations, two weaknesses still exist:

1) Some idealized refractivity profile models, such as log-linear evaporation duct models based on similarity theory (Jeske 1973; Paulus 1990a; Fairall et al. 1996) and bilinear, trilinear, and five-parameter models, are used to describe a probable characterization of the refractive index structure. These idealized models,
However, cannot describe the information of the real refractivity profile. Small perturbations of the gradients of the profile are ignored, resulting in modeling errors. In general, the statistics of these modeling errors are unknown.

2) In received signal power simulations, accurate computation of RCS is very difficult, which hampers the ability to estimate the refractivity profile.

Besides global optimization algorithms, the variational adjoint approach has also been testified as an effective method for remote sensing of refractivity structure (Zhao et al. 2011; Zhao and Huang 2011). Other than estimating a few refractivity parameters, it can retrieve the refractivity values at each sampling point over height, which is helpful to describe the detailed vertical information of the refractivity profile. Zhao et al. (2011) first introduced the variational adjoint approach into refractivity estimations for tomography geometry, where the source-receiver configuration is assumed to be bistatic. Then, with the aid of the Dirac delta function, they extended this method for RFC geometry inversions (Zhao and Huang 2011). In their works, the observations were assumed to be the electric field $u$, which was a complex field. But in practical RFC estimations, the received clutter power $P_r$ is a real number field. Through the PE model, $P_r$ is determined by $|u|$—that is, the absolute value of $u$, but not $u$ (Barrios 1991; Gerstoft et al. 2003b). How to establish the relationship between the complex equation and the real number relationship is an intractable problem.

This paper is an extension of the work of Zhao and Huang (2011) that $P_r$ is directly used for the cost function construction. Moreover, in $P_r$ simulations, RCS is considered as an unknown parameter, which will be estimated jointly with the refractivity profile structure. Both range-dependent and range-independent RCS models are considered and discussed. The reminder of this paper is organized as follows. Forward simulation of $P_r$ is introduced in section 2. Section 3 gives a brief theoretical description of the implementation of the variational adjoint approach for RFC geometry estimations. Finally, numerical experiments and analysis are given in section 4.

2. Forward model

Before performing RFC estimations, a forward simulation of $P_r$ has to be computed. In the absence of receiver noise, the received signal power from the clutter can be modeled as a function of the one-way propagation loss $L_{\text{loss}}$ in decibels (Gerstoft et al. 2003b):

$$P_r(x, \mathbf{m}) = -2L_{\text{loss}}(x, \mathbf{m}) + 10 \log_{10}(\sigma^0(x)) + C,$$

(1)

where $\mathbf{m}$ is the parameter vector describing refractive environment, $\sigma^0(x)$ is the normalized sea RCS at range $x$, and $C$ is a constant that includes transmitter power, antenna gain, etc. Usually, $\sigma^0$ is assumed to be range-independent. Krolik et al. (1999) found improved results by allowing some compliance for range-dependency in $\sigma^0$; however, it is not possible yet to draw a general conclusion as to if, or how much, compliance is useful (Gerstoft et al. 2001). In our theoretical derivation, $\sigma^0$ is assumed to be a constant.

Since $C$ has no impact on the inversion results, for brevity we denote $\sigma^0 + C$ by $\sigma$. Then Eq. (1) can be modified as

$$P_r(x, \mathbf{m}) = -2L_{\text{loss}}(x, \mathbf{m}) + 10 \log_{10}(\sigma) + C.$$  

(2)

From Eq. (2), $P_r$ is mainly determined by $\sigma$ and $L_{\text{loss}}$. Despite the considerable progress in low-grazing-angle backscatter modeling (Brown 1998; Voronovich and Zavorotny 2000; Song et al. 2000), it is still an open problem to compute RCS accurately. Thus, we treat $\sigma$ as a parameter to be estimated in this paper. Propagation loss can be computed as (Barrios 1991)

$$L_{\text{loss}}(x,z) = -10 \log_{10} \left( \frac{\lambda^2|u(x,z)|^2}{(4\pi)^2} \right),$$

(3)

where $\lambda$ is the wavelength, and $u$ is the electric field that can be computed numerically using the split-step Fourier PE method. For a smooth, perfectly conducting surface and horizontal polarization conditions, the PE model can be described as an initial-value problem:

$$\partial_z^2 u + 2ik_0 \partial_x u + k_0^2 (m^2 - 1) u = 0,$$

$$u(x,0) = 0,$$

$$u(0,z) = \phi(z),$$

(4a)

(4b)

(4c)

>where $\partial_x = \partial/\partial x$ and $\partial_z = \partial/\partial z$. Also, $k_0$ is the free-space wavenumber, and $m = m(x,z)$ is the modified index of refraction with the earth’s curvature included. It is defined by $m = m + z/a_c$, $n$ being the index of refraction and $a_c$ being the radius of the earth. Finally, $\phi(z)$ gives the initial field at the source range. A detailed description of the split-step Fourier PE solution has been given in Kuttler and Dockery (1991) and Barrios (1992, 1994).

It should be noted that, using the PE method, the field values at all sampling range and height points can be obtained. However, the measured radar sea clutter power is only one value at each range. From the PE model, the lower boundary condition is fixed. Hence, it typically
approximates the signal that is scattered at a given range from the PE field at a designated height \( z_0 \) near the surface (Gerstoft et al. 2003a,b; Vasudevan et al. 2007; Yardim 2007). Substituting Eq. (3) into Eq. (2) yields
\[
P_r(x) = \frac{20}{\ln(10)} \ln |u(x, z_0)|^2 + \sigma \\
+ 40 \log_{10} \left( \frac{A}{4\pi} \right) - 10 \log_{10}(x). \tag{5}
\]

3. Adjoint model

The spatial change of atmospheric refractivity is larger with height than with range and generally the range variations can be neglected. In open-ocean conditions, it was found that calculations of propagation enhancements based on a single profile were correct in 86% of the cases (Hitney et al. 1985). Therefore, the refractive conditions are considered laterally homogeneous [i.e., \( m = m(z) \)].

Although the primary concerns in this paper are different from our previous published papers, the derivation process of the adjoint model is similar to that of Zhao and Huang (2011). Here, the main steps and results are presented. Detailed derivations by a different method (through introducing a strong constraint) are given in the appendix.

a. Cost function

The cost function is defined as
\[
J(m, \sigma) = \frac{1}{2} \int_0^L \left[ P_r(x) - P_r^{obs}(x) \right]^2 dx, \tag{6}
\]
where \( P_r^{obs}(x) \) is the field measurement along the propagation distance \( L \), and \( P_r(x) \) is the field predicted by the forward model.

b. Adjoint model

To obtain the gradient of \( J \) with respect to \( m \), the adjoint field should be computed. According to the derivation in the appendix, the adjoint field satisfies
\[
\partial_z^2 w(x, z) + 2ik_0b_1w(x, z) + k_0^2[m^2(z) - 1]w(x, z) = g(x, z), \tag{7a}
\]

\[
w(x, 0) = 0, \tag{7b}
\]

\[
w(L, z) = 0, \tag{7c}
\]

\[
g(x, z) = \frac{40}{\ln(10)} \frac{P_r(x) - P_r^{obs}(x)}{\pi(x, z)} \Delta(z - z_0), \tag{8}
\]
where \( w \) is the adjoint field, \( \Delta \) is the Dirac delta function, and \( \overline{u} \) is the conjugate function of \( u \).

c. Solution of the adjoint model

Making use of the solution to PE and the property of the Dirac delta function integral, the adjoint model can also be solved by the split-step Fourier algorithm:
\[
W(x_{k+1}, p) = \exp \left[ \frac{k_0^2(m^2 - 1) - p^2}{2ik_0} \right] W(x_k, p) + \frac{1}{2ik_0} \\
\times \int_{x_{k+1}}^{x_k} G(x, p) \exp \left[ \frac{k_0^2(m^2 - 1) - p^2}{2ik_0} (x - x_k) \right] dx, \tag{9}
\]
where \( \delta x \) is the range increment, given by \( \delta x = x_{k+1} - x_k \).

Also, \( W(x, p) \) and \( G(x, p) \) are the Fourier transforms of \( w(x, z) \) and \( g(x, z) \), respectively:
\[
W(x, p) = F[w(x, z)] = \int_{-Z}^{Z} w(x, z) \exp(-ipz) \, dz, \tag{10}
\]
\[
G(x, p) = F[g(x, z)] = \int_{-Z}^{Z} \frac{40}{\ln(10)} \frac{P_r(x) - P_r^{obs}(x)}{\pi(x, z)} \exp(-ipz) \, dz.
\]
\[
\times \frac{40}{\ln(10)} \frac{P_r(x) - P_r^{obs}(x)}{\pi(x, z)} \exp(-ipz_0), \tag{11}
\]
where \( p \) is the transform variable. The solution in \( z \) space can be computed by performing the inverse Fourier transform of Eq. (9).

d. Gradient of the cost function

The gradient of \( J \) with respect to \( m \) and \( \sigma \) can be expressed as
\[
\nabla_m J = -2k_0^2 \int_0^L \text{Re}(\overline{u} \cdot w) \, dx, \tag{12}
\]
\[
\nabla_\sigma J = \int_0^L [P_r(x) - P_r^{obs}(x)] dx, \tag{13}
\]
where \( \text{Re}(u) \) is the real component of a complex variable \( u \).

e. Iteration process

With the gradient of \( J \), minimization could be generally accomplished through using the standard iterative gradient-based methods:
4. Numerical experiments and analysis

In the propagation assessment, a quantity frequently used to describe atmospheric refractive conditions is the modified refractivity $M$, which is related to the modified index of refraction as follows:

$$M = 10^6 \times (m - 1). \quad (16)$$

In our simulations, an evaporation-duct profile measured by a low-altitude captive balloon at the East China Sea is used to model as the observed profile, and the corresponding radar sea clutter power at height $z_0$ computed by the split-step Fourier PE method is used as the observed radar data. According to the suggestions from Gerstoft et al. (2003a,b) and Yardim (2007), $z_0$ is selected as 1 m. The initial field $\phi(z)$ is determined by the antenna pattern (Barrios 1994). Here, the transmitter simulating an omnidirectional source operating at 2 GHz is set at a height of 15 m.

a. RCS is range independent

In our derivations, RCS is assumed to be range-independent. Now that $C$ is a constant with no impact on the inversion results, we set $C = 0$ and $\sigma_0 = -90 \text{ dB}$. Here, $-90 \text{ dB}$ is a typical value for the normalized RCS in the evaporation-duct environments (Paulus 1990b). In our computations, the boundaries for $M$ are constrained from 250 to 500 $M$ units. The initial refractivity is given as a linear profile with the typical gradient of 0.118 $M$ units $m^{-1}$ and $M$ value of 330 $M$ units at sea level (see the dashed line in Fig. 2). The boundaries of $\sigma$ are constrained from 0 to $-200 \text{ dB}$. The initial value of $\sigma$ is generated randomly between the upper and lower boundaries. The propagation distance and range increment are set to be 50 km and 50 m, respectively.

Figure 2a gives the observed modified refractivity profile (solid line), the initial profile (dashed line), and the values of the retrieved profile (dotted line). The corresponding values of the base-10 logarithm of the cost function at iterations 1–1000 are given in Fig. 2b. For 1000 iteration numbers, the retrieved value of $\sigma$ is $-89.96 \text{ dB}$.

When RCS is a constant (this is an idealized case), the retrieved results of refractivity are similar to that in Zhao and Huang (2011). Here, the retrievals fluctuate surrounding the initial profile but follow the synoptic gradients of the observed profile, while the retrievals in Zhao and Huang (2011) converge to the observed profile. The difference between the results in this paper and the results in Zhao and Huang (2011) originates in the fact that the absolute value of $P_r$ is greater than $|\mu|$ by approximately $10^5$, which makes the inversions rely much more on the duct structures rather than on the exact values.

Vasudevan et al. (2007) found that the most important information is in the slopes of the refractivity rather than their exact values. Douvenot et al. (2008) also pointed out that the aim of RFC estimation is not to give the exact refractivity profile, but rather to propose a potential structure that could render an approximation of the real atmospheric conditions to predict propagation for assessing the performance of both communications and radar systems. Based on the original retrievals in Fig. 2, a moving-average filter is adopted to perform curve fitting (Makridakis and Wheelwright 1977). This filter can be realized by the “smooth” function in Matlab. In the process of curve fitting, the value at sea level is not included and the values above the upper $\frac{1}{4}$ field are replaced by extending the lower values with the standard gradient of 0.118 $M$ units per meter. The filtered results are given in Fig. 3.

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**Fig. 1.** Scheme of iteration process.

In this paper, an efficient quasi-Newton gradient technique for large-scale bound-constrained optimization [limited-memory Broyden–Fletcher–Goldfarb–Shanno (L-BFGS-B)] is applied (Zhu et al. 1995). The flowchart of the whole iteration process is shown in Fig. 1, where “TC satisfied?” (TC is terminal condition) means that as long as one of the following conditions is satisfied, the program will stop: 1) the cost function is less than a predetermined small positive number; 2) the gradient of the cost function approaches zero; 3) the value of the cost function no longer descends; and 4) the maximum iteration number achieves.
Figure 4 shows the coverage diagrams (dB) of the modeled one-way propagation loss computed by the split-step Fourier PE method, which gives an indication of how well the inverted profile is able to predict the propagation characteristics. The entire area of the diagrams is 0–50 km in range and 0–100 m in height. The propagation losses in Figs. 4a and 4b are computed from the observed profile and the postprocessed retrievals in Fig. 3, respectively. The absolute difference between the above two figures is shown in Fig. 4c. Most of the differences are below 4 dB, which indicates that the inversion is of favorable quality.

For most of the RFC inversion algorithms, RCS is assumed to be a constant with range (Rogers et al. 2000; Kraut et al. 2004; Yardim et al. 2006; Vasudevan et al. 2007; Wang et al. 2009). In practical implementation, RCS at any range is a mean RCS value plus a stochastic random variable due to changing sea surface and waves. Thus, some little horizontal variability of RCS should be considered. In the next simulations, two kinds of perturbations are added onto RCS. But RCS is still assumed to be a constant in the inversion process.

Figure 5 shows the postprocessed refractivity retrievals for RCS range-independent assumptions. In Fig. 5a, 3-dB sine-wave perturbations are added onto RCS and the retrieved value of \( \sigma \) is \(-88.53\) dB. In Fig. 5b, 3-dB Gaussian perturbations are added onto RCS and the retrieved value of \( \sigma \) is \(-90.31\) dB. From Fig. 5, it can be seen that when some perturbations are added onto RCS, the whole refractivity estimations are degraded. However, 3-dB Gaussian noise works much better than 3-dB sine noise. This is because the mean value of \( \sigma \) at any range is still \(-90\) dB when adding 3-dB Gaussian noise whereas adding a nonstochastic 3-dB sine noise over the range creates a range-varying mean RCS.

Compared with the RCS constant case, the corruption of estimations is possibly ascribed to the approximation of RCS. More accurate refractivity inversions might be obtained by more accurate description of RCS range variations.

b. RCS is range dependent

In this section, the case of range-dependent RCS is considered. To retrieve \( \sigma \) at each sampling point over...
propagation range will increase the dimensionality of the estimated parameter vector significantly, which in turn increases the computational complexity. In practical modeling, however, this handling way is unnecessary. Within local intervals, RCS could be considered to be a constant. Recurring to the Heaviside function,

\[ H(x) = \begin{cases} 1, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases} \]

(17)

The range-dependent \( \sigma \) can be modeled as

\[ \sigma(x) = \sum_{i=0}^{K-1} (\sigma_{i+1} - \sigma_i)H(x - X_i), \]

(18)

where \( \sigma_0 = 0 \). Thus, the dimensionality of \( \sigma \) is reduced to \((\sigma_1, \sigma_2, \ldots, \sigma_K)^T\). Using Eq. (18), Eq. (13) can be modified as

![Fig. 4.](image1.png)

**Fig. 4.** (a) Coverage diagram corresponding to the observed refractivity profile, (b) coverage diagram corresponding to the inverted postprocessed profile, and (c) absolute difference between (a) and (b).

![Fig. 5.](image2.png)

**Fig. 5.** The postprocessed retrievals for RCS range-independent assumptions for (a) 3-dB sine-wave perturbations and (b) 3-dB Gaussian perturbations.
In the present paper, RCS is assumed to be constant within the interval of 1 km. For a 50-km propagation range, the dimensionality of $s$ is equal to $K = 50$. Figure 6 shows the postprocessed refractivity retrievals for range-dependent RCS cases, in which the initial values of $\sigma$ are set to be the retrieved results from range-independent assumptions. Figure 6a and 6b show the results for 3-dB RCS sine wave perturbations and 3-dB RCS Gaussian perturbations, respectively. Comparing Fig. 6 with Fig. 5, improved results can be obtained by allowing some compliance for range dependency in RCS. The corresponding estimations of $\sigma$ for sine-wave perturbations are given in Fig. 7a. Figures 7b and 7c respectively

\[
\mathbf{V}_{\sigma, J} = \int_{X_{-1}}^{X} \left[ P_r(x) - P^{\text{obs}}_r(x) \right] dx.
\]

Fig. 6. The postprocessed retrievals for RCS range-dependent cases for (a) 3-dB sine-wave perturbations and (b) 3-dB Gaussian perturbations.

Fig. 7. (a) Retrievals of $\sigma$ from 3-dB RCS sine-wave perturbations, (b) absolute difference of the propagation loss computed by the observed profile and the retrieved profile in Fig. 6a, and (c) absolute difference of the propagation loss computed by the observed profile and the retrieved profile in Fig. 6b.
show the absolute difference between the propagation loss computed by the observed profile and the propagation loss computed by the postprocessed retrievals.

Here, it should be noted that RCS “range-independent” and “range-dependent” mentioned in this paper do not focus on the properties of RCS itself. The term “range-independent” means that in the inversion process, RCS is treated as a constant, whereas “range-dependent” means that in the inversion process, RCS is treated as a function of range.

5. Conclusions

This paper has shown how the adjoint of PE can be used to invert the refractivity profile and RCS from radar sea clutter. The detailed derivation of the adjoint model and the data assimilation process are presented. Numerical simulations are adopted to validate the feasibility of the theoretical algorithm. When RCS is constant, both retrievals of the refractivity profile and σ are favorable. When RCS is contaminated with noise, constant assumptions of σ bring about some corruptions of the refractivity inversions. However, improved results can be obtained by allowing some compliance for range dependency in RCS.

Although the simulation results are promising, there remain three aspects worth noting. 1) The adjoint process is a local optimization method and the initial guess has an impact on the inversion accuracy (Hursky et al. 2004; Huang and Wu 2005; Zhao and Huang 2011). Only a standard linear initial profile is discussed in this paper; more appropriate initialization may be obtained from historical observations and/or from the output of numerical weather prediction models. 2) Barrios (1992) pointed out that using split-step Fourier algorithm to invert the refractivity profile and RCS from radar sea clutter. The detailed derivation of the adjoint of PE can be obtained by allowing some compliance for range dependency in RCS.

3) Because of an absence of real observed radar data, simulations are performed. In practical operations, the available measurements are often contaminated by much more noise. Previous studies of the inverse problem have indicated that these problems are nonlinear and ill-posed (Tarantola 2005). With the aid of regularization techniques, introducing an appropriate regularization term may be capable of dealing with the ill-posedness and making computations stable.

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APPENDIX

Derivation of the Adjoint Field

To make the algorithm in the present paper easier to follow, the derivation of the adjoint field is provided. Introducing PE as a strong constraint into the cost function, Eq. (6) can be modified as

\[
J(m, \sigma) = \frac{1}{2} \int_0^L [P_r(x) - P_r^{\text{obs}}(x)]^2 dx - \Re \left\{ \int_0^L w \frac{\partial^2}{\partial z^2} u + 2i k_0 \partial_z u + k_0^2 (m^2 - 1) u \right\} dx dz, \tag{A1}
\]

where \( w \) is the Lagrange multiplier [i.e., the adjoint field in Eq. (7)], \( \overline{\sigma} \) is the conjugate function of \( u \), and \( \Re(u) \) is the real component of \( u \).

Note that

\[
J_0(m, \sigma) = \frac{1}{2} \int_0^L [P_r(x) - P_r^{\text{obs}}(x)]^2 dx, \tag{A2}
\]

\[
J_1(m, \sigma) = \Re \left\{ \int_0^L w \frac{\partial^2}{\partial z^2} u + 2i k_0 \partial_z u + k_0^2 (m^2 - 1) u \right\} dx dz. \tag{A3}
\]

The variational formalism of \( J \) can be computed as

\[
\delta J(m, \sigma) = \delta J_0(m, \sigma) - \delta J_1(m, \sigma), \tag{A4}
\]
\[ \delta I_0 = \text{Re} \left\{ \int_0^L \left[ \frac{40}{\ln(10)} \frac{P_r(x) - P_{\text{obs}}(x)}{m(x, z_0)} \Delta u(x, z_0) \right] dx \right\} + \int_0^L \left[ P_r(x) - P_{\text{obs}}(x) \right] \delta \sigma \, dx \]
\[ = \text{Re} \left\{ \int_0^L \int_0^Z \left[ \frac{40}{\ln(10)} \frac{P_r(x) - P_{\text{obs}}(x)}{\pi} \Delta(z - z_0) \Delta u \right] \, dx \, dz \right\} + \int_0^L [P_r(x) - P_{\text{obs}}(x)] \delta \sigma \, dx, \quad (A5) \]
\[ \delta I_1 = \text{Re} \left\{ \int_0^L \int_0^Z w \left[ \frac{\partial^2 \Delta u}{\partial x^2} + 2ik_0 \partial z (\Delta u) + k_0^2 (m^2 - 1) (\Delta u) + 2k_0 um (\Delta m) \right] \, dx \, dz \right\}, \quad (A6) \]

where \( \delta \) is the variational operator, \( \delta u \) is an arbitrary increment of \( u \), \( \delta m \) is an arbitrary increment of \( m \), \( \delta \sigma \) is an arbitrary increment of \( \sigma \), and \( \Delta \) is the Dirac delta function.

In Eq. (A6),
\[ \int_0^L \int_0^Z w \cdot \partial_z^2 (\Delta u) \, dx \, dz = \int_0^L \int_0^Z w \cdot \partial_z^2 (\Delta u) \, dx \, dz + \int_0^L \left[ w \cdot \partial_z (\Delta u) - \partial_z w \cdot (\Delta u) \right]_{z=0} \, dx, \quad (A7) \]
\[ \int_0^L \int_0^Z w \cdot 2ik_0 \partial_x (\Delta u) \, dx \, dz = \int_0^L \int_0^Z 2ik_0 w \cdot (\Delta u) \, dx \, dz - \int_0^L \left[ 2ik_0 w \cdot (\Delta u) \right]_{x=0} \, dz, \quad (A8) \]
\[ \int_0^L \int_0^Z w \cdot k_0^2 (m^2 - 1) (\Delta u) \, dx \, dz = \int_0^L \int_0^Z k_0^2 (m^2 - 1) w \cdot (\Delta u) \, dx \, dz, \quad (A9) \]
\[ \int_0^L \int_0^Z w \cdot 2k_0 um (\Delta m) \, dx \, dz = \int_0^L \int_0^Z 2k_0 um \partial_z (\Delta m) \, dx \, dz. \quad (A10) \]

Substituting Eqs. (A5)–(A10) into Eq. (A4) leads to
\[ \delta J(m, \sigma) = \text{Re} \left\{ \int_0^L \left[ g(x, z) - [\partial_z^2 w + 2ik_0 \partial_x w + k_0^2 (m^2 - 1)w] \right] \Delta u \, dx \, dz \right\} \]
\[ + \int_0^L \left[ P_r(x) - P_{\text{obs}}(x) \right] \delta \sigma \, dx - \text{Re} \left\{ \int_0^L \int_0^Z 2k_0^2 m \partial_z m \cdot \Delta m \, dx \, dz \right\} \]
\[ - \text{Re} \left\{ \int_0^L \left[ w \cdot \partial_z (\Delta u) - \partial_z w \cdot (\Delta u) \right]_{z=0} \, dx \right\} + \int_0^Z \left[ 2ik_0 w \cdot (\Delta u) \right]_{x=0} \, dz \right\}. \quad (A11) \]

where
\[ g(x, z) = \frac{40}{\ln(10)} \frac{P_r(x) - P_{\text{obs}}(x)}{\pi} \Delta(z - z_0). \quad (A12) \]

On the other hand, from the definition of the variational formalism, \( J \) can be expressed as
\[ \delta J(m, \sigma) = \int_0^Z (V_{m} J \cdot \Delta m) \, dz + V_{\sigma} J \cdot \Delta \sigma, \quad (A13) \]
where \( V_{m} J \) is the gradient of \( J \) with respect to \( m \), and \( V_{\sigma} J \) is the gradient of \( J \) with respect to \( \sigma \).

Equation (A11) minus Eq. (A13) produces
Recurring to the initial boundary condition of PE, the conjugate initial boundary condition of the adjoint field can be set as \( w(x, 0) = 0, w(x, Z) = 0 \), and \( w(L, z) = 0 \) to satisfy
\[
0 = \operatorname{Re} \left( \int_0^L \int_0^Z \left\{ g(x, z) - \left[ \partial_z^2 w + 2ik_0 \partial_z w + \frac{k_0^2}{m^2} - 1 \right] w \right\} \overline{\partial u} \, dx \, dz \right) \\
+ \left\{ \int_0^L [P_r(x) - P_r^{obs}(x)] \, dx - \mathbf{V}_\sigma^F \right\} \delta \sigma - \left\{ \int_0^Z \left[ 2k_0^2 m \right] \int_0^L \operatorname{Re}(\overline{\partial w}) \, dx + \mathbf{V}_m^F \right\} \cdot \delta m \, dz \\
- \operatorname{Re} \left\{ \int_0^L \left[ w \cdot \partial_z \left( \overline{\partial u} \right) - \partial_z w \cdot \left( \overline{\partial u} \right) \right] \theta = Z \, dx + \int_0^Z \left[ 2ik_0 w \cdot \left( \overline{\partial u} \right) \right] \theta = 0 \, dz \right\}. \tag{A14}
\]

Since \( \delta u, \delta m, \) and \( \delta \sigma \) are arbitrary, and Eq. (A14) equals to zero, we can get the following adjoint model and the gradients of \( J \):
\[
\partial_z^2 w(x, z) + 2ik_0 \partial_z w(x, z) + \frac{k_0^2}{m^2} \left[ m^2 - 1 \right] w(x, z) \\
= g(x, z), \tag{A17a}
\]
\[
w(x, 0) = 0, \tag{A17b}
\]
\[
w(L, z) = 0, \tag{A17c}
\]
\[
\mathbf{V}_m^F = -2k_0^2 m \int_0^L \operatorname{Re}(\mathbf{\pi} \cdot w) \, dx, \tag{A18}
\]
\[
\mathbf{V}_\sigma^F = \int_0^L \left[ P_r(x) - P_r^{obs}(x) \right] \, dx. \tag{A19}
\]

REFERENCES


