A Semihydrostatic Theory of Gravity-Dominated Compressible Flow

THOMAS DUBOS
Laboratoire de Météorologie Dynamique, IPSL, Ecole Polytechnique, Palaiseau, France

FABRICE VOITUS
Groupe d’étude de l’Atmosphere Météorologique, CNRM, Météo-France, Toulouse, France

(Manuscript received 25 March 2014, in final form 24 July 2014)

ABSTRACT

From Hamilton’s least-action principle, compressible equations of motion with density diagnosed from potential temperature through hydrostatic balance are derived. Slaving density to potential temperature suppresses the degrees of freedom supporting the propagation of acoustic waves and results in a soundproof system. The linear normal modes and dispersion relationship for an isothermal state of rest on $f$ and $\beta$ planes are accurate from hydrostatic to nonhydrostatic scales, except for deep internal gravity waves. Specifically, the Lamb wave and long Rossby waves are not distorted, unlike with anelastic or pseudoincompressible systems.

Compared to similar equations derived by A. Arakawa and C. S. Konor, the semihydrostatic system derived here possesses an additional term in the horizontal momentum budget. This term is an apparent force resulting from the vertical coordinate not being the actual height of an air parcel but its hydrostatic height (the hypothetical height it would have after the atmospheric column it belongs to has reached hydrostatic balance through adiabatic vertical displacements of air parcels). The Lagrange multiplier $\lambda$ introduced in Hamilton’s principle to slave density to potential temperature is identified as the nonhydrostatic vertical displacement (i.e., the difference between the actual and hydrostatic heights of an air parcel).

The expression of nonhydrostatic pressure and apparent force from $\lambda$ allow the derivation of a well-defined linear symmetric positive definite problem for $\lambda$. As with hydrostatic equations, vertical velocity is diagnosed through Richardson’s equation. The semihydrostatic system has therefore precisely the same degrees of freedom as the hydrostatic primitive equations, while retaining much of the accuracy of the fully compressible Euler equations.

1. Introduction

So far most atmospheric numerical modeling systems have specialized as limited-area mesoscale models, global climate models, and global weather prediction models, each typically solving different sets of equations of motion—either the Euler equations or approximate sets derived from them using various assumptions [see, e.g., Tort and Dubos (2014b)]. Climate models resolve horizontal scales at which the hydrostatic approximation is accurate and cost effective. Mesoscale models need equations that remain accurate at small horizontal scales and typically solve either the compressible Euler equations or anelastic/pseudoincompressible equations [in the following, we refer to “anelastic/pseudoincompressible” as both anelastic equations (Ogura and Phillips 1962; Lipps and Hemler 1982) and pseudoincompressible equations (Durran 1989, 2008; Klein and Pauluis 2012)]. Global weather forecasting models need accuracy at both small and large scales, which may rule out the anelastic equations, which distort long Rossby waves (Davies et al. 2003, hereinafter DSWT03; Arakawa and Konor 2009, hereinafter AK09; Dukowicz 2013).

The Euler equations support the propagation of acoustic waves. Resolving these meteorologically unimportant motions not only involves taking care of useless degrees of freedom but also creates numerical difficulties. Although much progress has been achieved toward efficient and accurate solutions to these difficulties (Skamarock and Klemp 2008; Weller et al. 2013; Smolarkiewicz et al. 2014), it can still be desirable to identify “unified” equations of motion that would not support acoustic waves while retaining accuracy at large and small scales. This is especially desirable if the
specialization of modeling systems is to be abandoned in favor of all-scale, unified modeling systems—a strong current trend. Even if such equations are eventually not chosen as the basis of a numerical model, they may help identify the independent degrees of freedom of the atmospheric flow to be modeled and how the dependent fields are related to the independent fields.

A major step toward the identification of such a unified system of equations has been achieved by AK09 [see also Konor (2014)]. To discuss their work, it is useful to recall how acoustic waves are suppressed from the hydrostatic and anelastic/pseudoincompressible equations, what the remaining degrees of freedom are and how they relate to large-scale waves. The hydrostatic equations of motion neglect vertical acceleration in the vertical momentum balance. A consequence is that vertical velocity is not prognostic any more. Furthermore, for a given profile of potential temperature, adequate top and bottom boundary conditions (either imposed pressure or imposed altitude at the top of the domain), and a given column-integrated mass, there exists a unique profile of density that satisfies hydrostatic balance. This fact is particularly apparent if the flow is described using a Lagrangian or mass-based vertical coordinate (e.g., Dubos and Tort 2014). From this point of view hydrostatic balance is the result of a vertical hydrostatic adjustment (i.e., adiabatic vertical displacement of air parcels, which adjust their specific value until the resulting pressure profile reaches hydrostatic balance). Therefore, the remaining degrees of freedom are horizontal velocity, potential temperature, and column-integrated mass. Vertical velocity can be diagnosed by time-differentiating hydrostatic balance, yielding Richardson’s equation—a one-dimensional vertical elliptic equation (Richardson 1922). The 2 degrees of freedom lost, vertical velocity and density, were precisely those that allowed the acoustic waves to propagate.

On the other hand, with the anelastic/pseudoincompressible equations, density is locally prescribed or diagnosed from potential temperature (Ogura and Phillips 1962; Lipps and Hemler 1982; Durran 1989, 2008; Klein and Pauluis 2012) through a horizontally uniform hydrostatic background atmospheric profile assumed time independent (Ogura and Phillips 1962; Lipps and Hemler 1982; Durran 1989, 2008; Klein and Pauluis 2012) or allowed to vary in time (Almgren 2000; O’Neill and Klein 2014). This approach relies on the hypothesis that the actual atmosphere does not depart too much from this background profile. Because of the anelastic/pseudoincompressible constraint, only two of the three velocity components are prognostic. Although other choices are possible (Jung and Arakawa 2008), one may consider horizontal velocity components as prognostic, vertical velocity being then diagnosed from the anelastic/pseudoincompressible constraint. Although it involves vertical acceleration, the vertical momentum budget is then not a prognostic equation. Instead, combined with the horizontal momentum budget, it leads to a Poisson-like, three-dimensional elliptic problem for the deviation of pressure from its reference value. Compared to the hydrostatic equations, the anelastic/pseudoincompressible equations have lost 2 two-dimensional degrees of freedom: the column-integrated mass and the divergence of the column-integrated mass flux (or potential temperature flux). As a consequence, the anelastic/pseudoincompressible equations do not support the Lamb wave (DSWT03). The Lamb wave is by itself not important meteorologically, but the ability of the barotropic flow to be divergent is crucial to the correct propagation of long barotropic Rossby waves (Cressman 1958), which are meteorologically important (Madden 2007). This damages the validity of the anelastic/pseudoincompressible equations at large scales, although the seriousness of this issue remains a subject of debate (Smolarkiewicz et al. 2014).

Generally speaking, acoustic waves are suppressed if the feedback between velocity divergence and pressure is suppressed, which the hydrostatic and anelastic/pseudoincompressible equations achieve by slaving density to potential temperature, although in different ways. Based on the above discussion, AK09 concluded that accuracy at large scales requires that density be slaved the hydrostatic way (i.e., deduced from potential temperature and column-integrated mass, and not the anelastic/pseudoincompressible way). They recognized that this assumption does not by itself imply the neglect of vertical acceleration in the vertical momentum budget, only the converse being true. This observation enabled them to derive nonhydrostatic equations of motion where density satisfies the hydrostatic constraint. A delicate part of their derivation is the formulation of an elliptic problem yielding the deviation of pressure from its hydrostatic value. They obtained an elliptic problem similar to that of pseudoincompressible equations. However, one source term proportional to the second time derivative of density remained unspecified, and they suggested a numerical time extrapolation procedure to evaluate it in a time-marching numerical scheme. Furthermore, additional ad hoc arguments related to the conservation of energy had to be invoked in order to ensure a unique solution to the elliptic problem. Nevertheless, normal-mode analysis of the unified system for small deviations from an isothermal state of rest revealed its excellent accuracy from large to small horizontal scales.
Historically, the vast majority of approximate sets of equations used in geophysical fluid dynamics have been derived heuristically, using a combination of scale analysis, intuition, and ingenuity to achieve both accuracy and dynamical consistency, defined as the existence of conservation principles for suitably defined energy, potential vorticity, and momentum. However, it is now well established that Hamilton’s principle of least action provides a systematic procedure to derive dynamically consistent equations of motion, provided all approximations are made directly in the expression of the action integral prior to invoking its stationarity (Salmon 1983; Morrison 1998; Holm et al. 2002). Nevertheless, this approach remains the exception (Tort and Dubos 2014a) rather than the rule, although most useful equations of atmospheric motion can be shown to derive from a variational principle (Tort and Dubos 2014b). Especially soundproof equations are obtained through the introduction of a Lagrange multiplier enforcing the constraint satisfied by density (Cotter and Holm 2013; Tort and Dubos 2014b).

Assuming that the AK09 equations conserve energy, potential vorticity, and momentum, the question emerges naturally as to whether these can be obtained from a variational principle. Such a formulation would provide additional insight into their structure and may help designing numerical schemes with desirable properties such as discrete conservation of energy or self-adjointness of the discrete elliptic problem. In Tort and Dubos (2014b) a Lagrangian description of the flow is adopted (Salmon 1983; Morrison 1998). This makes the variational calculus simple, but would complicate the expression of the hydrostatic constraint because it involves a vertical derivative that is not straightforwardly expressed with Lagrangian variables. The most suitable framework is therefore an Eulerian description of the flow and the relevant variational calculus (Holm et al. 2002; Cotter and Holm 2013). The work presented hereafter started as an attempt to derive the AK09 equations from an Eulerian variational principle. However, it turned out that a slightly different set of equations was obtained, which called for a physical interpretation of the additional terms and a normal-mode analysis to check its accuracy.

The paper is organized as follows. Section 2 recalls the various forms of the inviscid, compressible Euler equations for an arbitrary equation of state, as well as the AK09 unified equations. Section 3 applies the physical idea underpinning AK09 within an Eulerian variational framework, introducing a Lagrange multiplier \( \lambda \) to impose hydrostatic balance. The equations of motion are directly obtained in vector-invariant form. Comparison with AK09 shows that an additional term has appeared, which suggests the interpretation of \( \lambda \) as a non-hydrostatic vertical displacement from which all other nonhydrostatic perturbations derive. In section 4 the conservation laws, which necessarily hold owing to the variational approach, are explicitly derived and confirm the physical interpretation of \( \lambda \). The elliptic problem satisfied by \( \lambda \) is obtained as well, invoking only the equations of motion and rigid-boundary boundary conditions. The normal-mode analysis performed in section 5 shows that the equations of motion filter acoustic waves and support accurate inertia–gravity waves and undistorted long Rossby waves, with some caveats with respect to deep internal gravity waves. Section 6 summarizes and concludes.

2. The Euler and AK09 unified equations

a. Inviscid fully compressible Euler equations

For adiabatic motion of a compressible fluid, the mass and entropy budgets are

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div}\rho \mathbf{u} &= 0, \\
\frac{Ds}{Dt} &= 0,
\end{align*}
\]

where \( \rho, v = 1/\rho, \) and \( s \) are the density, specific volume, and specific entropy, respectively, and \( \mathbf{u} = D\mathbf{x}/Dt \) is the velocity, where \( \mathbf{x} \) is the position in the three-dimensional Cartesian coordinates \( x, y, z \) and \( D/Dt \) is the material (Lagrangian) time derivative. The momentum budget can then be obtained from Hamilton’s principle of least action, using either the Lagrangian or the Eulerian description of the flow. If the Lagrangian description is used (Salmon 1983; Morrison 1998) then the equations of motion are obtained in Euler–Lagrange form:

\[
\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \text{div} p + ge_z = 0,
\]

where \( e(v, s) \) is the specific internal energy and \( p(\rho, s) = -\partial e/\partial v \) is the pressure. The local conservation law for total energy (kinetic, internal, and potential)

\[
E = \rho \left[ \frac{\mathbf{u} \cdot \mathbf{u}}{2} + e(1/\rho, s) + g z \right] \quad \text{and} \quad (3a)
\]

\[
\frac{\partial E}{\partial t} + \text{div} [(E + p)\mathbf{u}] = 0 \quad (3b)
\]

follows quite straightforwardly from (2).

On the other hand an Eulerian variational principle can be used (Newcomb 1962; Holm et al. 2002):

\[
\delta \int L \, dt = 0, \quad L[\rho, \mathbf{u}, s] = \int l(\rho, \mathbf{u}, s, \mathbf{x}) \, d\mathbf{x}. \quad (4)
\]

In Hamilton’s principle (4), \( L[\rho, \mathbf{u}, s] \) is a functional of the 3D, instantaneous Eulerian fields \( \rho, \mathbf{u}, s \) while...
l = lE(ρ, u, s, x) is an ordinary function (of eight scalar variables):

\[ l_E(\rho, u, s, x) = \rho \left[ \frac{u \cdot u}{2} - e(1/\rho, s) - gz \right]. \]  (5)

From (4) the equations of motion are naturally obtained in the curl form or the vector-invariant form [see Holm et al. (2002) and section 3]:

\[ \partial_t u + (curl u) \times u + V \left( \frac{u \cdot u}{2} + e + \frac{P}{\rho} - Ts + gz \right) \]
\[ + sVT = 0. \]  (6)

The Lagrangian conservation of Ertel’s potential vorticity

\[ q = \frac{1}{\rho} V s \cdot curl u, \quad Dq/Dt = 0 \]  (7)

follows relatively easily from (6).

Finally, the conservation law for momentum is apparent in the flux form of the equations:

\[ \partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p + \rho g e_z = 0. \]  (8)

This form is easy to obtain from the Euler–Lagrange form \([2]\). It does not directly follow, but can be obtained, from an Eulerian variational principle if the Lagrangian is invariant with respect to translations. As in \([2]\), (8) permits us to identify unambiguously the forces acting on the fluid.

Assuming an ideal perfect gas, \(udp = \theta d\pi\), where \(\theta = \theta(s) = T(p/p_0)^{-R/c_p}\) is potential temperature and \(\pi = \partial e/\partial \theta = c_p(p/p_0)^{R/c_p}\) is the Exner function \((R, c_p, p_0\) being the constant \(pu/T,\) the specific heat capacity at constant pressure, and an arbitrary reference pressure). A form often encountered in the literature is then

\[ \frac{Du}{Dt} + \theta V \pi + \rho g e_z = 0. \]  (9)

It must be stressed that this form is neither an Euler–Lagrange form, a curl form, or a flux form. It is a mix between the Euler–Lagrange form \([5]\) and the curl form \([6]\). Furthermore, for a general equation of state, although one can still define a potential temperature \(\theta(s) = T(s, p_0)\) and an Exner function \(\pi = \partial e/\partial \theta = T ds/d\theta\), \(udp - \theta d\pi = d(e + pu - \theta \pi) \neq 0\) so that the contribution from thermodynamics \(V(e + pu - \theta \pi)\) does not generally vanish, as in (6).

b. \(AK09\) unified equations

\(AK09\) have derived unified equations based on the idea that the density field satisfies hydrostatic balance while velocity does not. Their approximation strategy uses (9) as a starting point, and they expand the Exner function \(\pi\) into a quasi-static contribution \(\pi_{qs}\) determined by hydrostatic balance and the equation of state:

\[ \partial_z p_{qs} + \rho g = 0, \]  (10a)

\[ p_{qs} = p(\rho, \theta), \quad \pi_{qs} = \pi(\rho, \theta) = c_p \left( \frac{p_{qs}}{p_0} \right)^{\theta/\rho g \pi}, \]  (10b)

and a nonhydrostatic contribution \(\pi'\) to be determined by an elliptic equation deduced from the mass budget:

\[ \partial_t \rho + V \cdot (\rho u) = 0 \quad \text{and} \quad Dq/Dt = 0. \]  (11a)

\[ D\mathbf{u}/Dt + \theta V \pi_{qs} + \rho g e_z = -\theta V \pi'. \]  (11b)

\(AK09\) do not explicitly derive a flux form for (11b). Such a form should be obtained by multiplying (11b) by the density \(\rho\) obeying the mass budget \([11a]\). Then \(\rho \theta V \pi_{qs} = V p_{qs}\) because of (10b) but it is not obvious how \(\rho \theta V \pi'\) can also be transformed into a pure divergence.

3. Constrained variational principle

In this section the physical idea underpinning \(AK09\) is implemented using a different strategy: instead of manipulating the equations of motion, the Lagrangian is augmented using a Lagrange multiplier \(\lambda\) in order to enforce \((10a)\). The equations of motion follow, which turn out to differ from \((11b)\). A physical interpretation of additional terms is given and confirmed in the next section by analyzing the momentum budget in flux form.

a. Lagrange multiplier

Equation \((10a)\) is an additional relationship that we wish to impose. It will restrict the motions accessible to the flow and is therefore a constraint. A constraint can be introduced into a variational principle though a Lagrange multiplier \(\lambda\) provided the constraint acts only on positions, not velocities. Such a constraint is termed holonomic [see Bernardet (1995) for a discussion in the context of the anelastic approximation]. From a fluid parcel point of view, \(\rho\) and \(s\) can be deduced from positions \(\text{Salmon 1983; Morrison 1998}\) so the hydrostatic constraint is indeed holonomic. Since the constraint holds at every time and position, a field \(\lambda(x, t)\) is necessary to enforce it. We therefore add to the Lagrangian the contribution

\[ L_\lambda[\rho, s, \lambda] = - \lambda \left( \frac{\partial p_{qs}}{\partial z} + \rho g \right) d^3x = \left( p_{qs} \frac{\partial \lambda}{\partial z} - \rho g \lambda \right) d^3x, \]  (12)
where $\lambda$ has the dimension of a length and boundary conditions will be discussed later. As in AK09, we note $p_{qs} = p(\rho, s)$.

The constrained variational principle is therefore
\begin{equation}
\delta \int \mathcal{L} \, dt = 0, \quad \mathcal{L}[\rho, u, s, \lambda] = \int \left( l(\rho, u, s, \lambda) \right) \, dx, \tag{13} \end{equation}
where $l$ is now a function of 10 variables including $\lambda$ and $\lambda_z = \partial \lambda / \partial z$ and $\mathcal{L}$ is a functional of the Eulerian fields $\rho$, $u$, $s$, $\lambda$:
\begin{equation}
l(\rho, u, s, \lambda, \lambda_z) = l_E + l_n \quad \text{and} \quad l_n(\rho, s, \lambda_z) = p_{qs} \lambda_z - \rho p. \tag{14a} \end{equation}
Vanishing variations with respect to $\lambda$ impose
\begin{equation} \begin{align*}
0 = \delta \mathcal{L} &= \left( \frac{\partial l}{\partial \lambda} + \frac{\partial l}{\partial \lambda_z} \right) \, dx = \left( \frac{\partial l}{\partial \lambda} - \frac{\partial}{\partial z} \frac{\partial l}{\partial \lambda_z} \right) \, dx;
\end{align*} \tag{15a} \end{equation}
hence,
\begin{equation} \frac{\partial l_n}{\partial \lambda} = \frac{\partial}{\partial z} \frac{\partial l_n}{\partial \lambda_z}. \tag{15b} \end{equation}

\begin{itemize}
\item[(i.e.,)] hydrostatic balance [\cite{10a}].
\end{itemize}

\begin{itemize}
\item[b.] Eulerian variational principle
\end{itemize}

We now obtain the equations of motion from Hamilton’s principle of least action. What follows is essentially independent because $\rho, u, s$ are not mutually independent because $\rho, u, s$ must satisfy the purely kinematic transport equations:
\begin{equation}
\begin{align*}
\frac{\partial \rho}{\partial t} + \text{div} \rho u &= 0, \quad \frac{\partial s}{\partial t} + u \cdot \nabla s = 0. \tag{16} \end{align*} \end{equation}
Therefore, $\partial \rho / \partial t$, $\partial u / \partial t$, $\partial s / \partial t$ must satisfy
\begin{equation}
\begin{align*}
\frac{\partial}{\partial t} \lambda &= \frac{\partial}{\partial \lambda} \frac{\partial l_n}{\partial \lambda_z}, \\
\frac{\partial}{\partial t} \lambda_z &= \frac{\partial}{\partial \lambda_z} \frac{\partial l_n}{\partial \lambda_z}. \tag{17} \end{align*} \end{equation}
The general solution obtained by Newcomb (1962) is
\begin{equation}
\begin{align*}
\delta \rho &= -\text{div}(\rho \xi), \quad \delta u = \frac{\partial \xi}{\partial t} + \nabla \xi \cdot u - u \cdot \nabla \xi, \\
\delta s &= -\xi \cdot \nabla s, \tag{18} \end{align*} \end{equation}
where the virtual displacement $\xi$ is arbitrary in the interior of the space–time domain \cite{Newcomb1962, Holm1998, Holm2002}. We use standard tensor notation; that is, $u \cdot \nabla u$ and $v \cdot \nabla u$ are the vectors with components $\sum (\partial_i u_j) v_j$ and $\sum v_i (\partial_i u_j)$, respectively. For a fixed spatial domain, boundary conditions are that $\xi = 0$ at $t = t_1, t_2$ and $\xi$ must be tangential at the boundaries of the domain. Hamilton’s principle [(4)] holds for variations of the form of (18) only, not for arbitrary, independent variations of $\rho, s, u$. Letting
\begin{equation}
\begin{align*}
v &= \frac{1}{\rho} \frac{\partial l}{\partial \rho} \\
\text{and taking into account (18), the Lagrangian density varies as}
\end{align*} \tag{19} \end{equation}

\begin{equation}
\begin{align*}
\delta l &= -\frac{\partial l}{\partial \rho} \text{div} \rho \xi + \rho v \cdot \left( \frac{\partial \xi}{\partial t} + \nabla \xi \cdot u - u \cdot \nabla \xi \right) - \frac{\partial l}{\partial \rho} \frac{\partial l}{\partial \xi} \cdot \nabla s \\
&= -\text{div} \left( \frac{\partial l}{\partial \rho} \frac{\partial l}{\partial \xi} \right) \frac{\partial l}{\partial \rho} \frac{\partial l}{\partial \xi} \cdot \nabla s \\
&= -\rho v \cdot u \cdot \xi - \frac{\partial l}{\partial \rho} \frac{\partial l}{\partial \xi} \cdot \nabla s. \tag{20a} \end{align*} \end{equation}

\begin{itemize}
\item[The identities]
\end{itemize}
\begin{equation}
\begin{align*}
\text{div}(v \otimes \rho u) &= \text{div}(\rho u) v + \rho v \cdot \rho u, \\
V(u \cdot v) &= v \cdot V u + u \cdot V v, \\
V v \cdot u - u \cdot V v &= (V \times v) \times u. \tag{20b} \end{align*} \end{equation}

\begin{itemize}
\item[yield]
\end{itemize}
\begin{equation}
\begin{align*}
\delta l &= \frac{\partial}{\partial t} \left( \rho v \cdot \xi \right) + \text{div} \left( \rho v \cdot \xi \right) u - [\partial l / \partial \xi] u \\
&= \frac{\partial}{\partial t} [\frac{\partial l}{\partial \xi} u] + [\frac{\partial l}{\partial \psi} + (V \times v) \times u + VB + \frac{\partial l}{\partial s} V s] \cdot \rho \xi, \tag{20c} \end{align*} \end{equation}

where $B = u \cdot v - \partial l / \partial \rho$ is the Bernoulli function. Hamilton’s principle of least action then yields the evolution equation for $v$ in the curl form:
\begin{equation}
\delta \int \mathcal{L} \, dt = -\int \left[ \frac{\partial l}{\partial \xi} u + \left( V \times v \right) \times u + VB + \frac{\partial l}{\partial s} V s \right] \cdot \rho \xi \, dx \, dt \tag{21a} \end{equation}
and
\begin{equation}
\delta \int \mathcal{L} \, dt = 0 \iff \frac{\partial l}{\partial \xi} u + \left( V \times v \right) \times u + VB + \frac{\partial l}{\partial s} V s = 0. \tag{21b} \end{equation}
Notice that with the Lagrangians [(5) and (14a)] considered in this work one finds simply \( \mathbf{v} = \mathbf{u} \). In a more general setting the Lagrangian would include a contribution generating the Coriolis force, resulting in \( \mathbf{v} \) being absolute instead of relative velocity. The Lagrangian [(5)] yields the Euler equations (6) and we focus here on the additional terms resulting from the constraint enforced by \( l \). Using

\[
\frac{\partial l}{\partial \rho} = c^2 \frac{\partial \lambda}{\partial z} - g \lambda, \quad \frac{\partial l}{\partial s} = \frac{\partial p}{\partial s} \frac{\partial \lambda}{\partial z},
\]

\[
\frac{\partial l}{\partial \lambda} = -\rho g, \quad \frac{\partial l}{\partial \lambda_z} = p_{qs},
\]

(22)

and introducing the speed of sound \( c = \sqrt{\frac{\partial p}{\partial \rho}} \), we find

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{V} \left( \frac{\partial \mathbf{u} \cdot \mathbf{u}}{2} + e + \rho \nu \nabla \right) - TVs = \frac{\partial l}{\partial \rho} - \frac{1}{\rho} \frac{\partial l}{\partial s} \mathbf{V} = \mathbf{V} \left( c^2 \frac{\partial \lambda}{\partial z} - g \lambda \right) - \frac{1}{\rho} \frac{\partial \rho}{\partial s} \frac{\partial \lambda}{\partial z} \mathbf{V}. \]

(23)

c. Comparison with AK09

To facilitate the comparison of (23) with AK09 we consider, in this subsection only, an ideal perfect gas and replace \( s \) by \( \theta \). As a consequence, \( T = \rho \theta \) becomes \( \pi_{qs} = \rho \theta / \theta \) in (23). Noticing that \( \partial \rho / \partial \theta = \partial^2 \rho / \partial \theta \partial \theta = -\partial \pi / \partial \theta = \rho \partial \pi / \partial \rho \), the rhs of (23) can be rewritten as

\[
\mathbf{V} \left( c^2 \frac{\partial \lambda}{\partial z} - g \lambda \right) - \frac{1}{\rho} \frac{\partial \rho}{\partial \theta} \frac{\partial \lambda}{\partial z} \mathbf{V} = -g \nabla \lambda - \frac{\partial \rho'}{\rho} \mathbf{V} + \pi' \mathbf{V} \theta,
\]

(24)

where we have defined

\[
\rho' = -\rho \frac{\partial \lambda}{\partial \rho}, \quad \rho'' = \rho \frac{\partial \rho}{\partial \rho} = -\rho c^2 \frac{\partial \lambda}{\partial \rho},
\]

\[
\pi' = \rho \frac{\partial \pi}{\partial \rho} = -\rho \frac{\partial \lambda}{\partial \rho} \theta.
\]

(25)

Finally, \( ud\bar{\pi} = \theta d\pi \) implies \( p'/\rho = \theta \pi' \); hence,

\[
\frac{Du}{Dt} + \theta \nabla \pi_{qs} + g \mathbf{e}_z = -\theta \nabla \pi' - g \nabla \lambda. \tag{26}
\]

Clearly, the variational approach yields equations of motion [(26)] that differ from (11b) of AK09.

Assuming a characteristic vertical scale \( L_z \), (25) yields \( \rho c^2 \lambda \sim L_z p' \); hence, \( g \lambda (L_z/H) \theta \pi' \) where \( H = c^2 / g \approx 10 \text{ km} \) is the scale height. This suggests that the extra term \( g \nabla \lambda \) is generally small compared to the nonhydrostatic contributions already taken into account by AK09, except for motions whose vertical scale is comparable to or larger than the scale height.

d. Physical interpretation and boundary conditions for \( \lambda \)

The additional term \( -g \mathbf{V} \lambda \) found in (26) conveys at this stage no obvious physical interpretation. All that can be said is that it cannot be reexpressed in terms of \( \rho ' \) or \( \pi ' \) because they depend on \( \partial l / \partial \lambda \) and do not carry information about the horizontal variations of \( \lambda \). However, if we rewrite (26) as

\[
\frac{Du}{Dt} + \mathbf{V} \left( g(z + \lambda) \right) + \theta \mathbf{V} (\pi_{qs} + \pi') = 0, \tag{27}
\]

then a possible physical interpretation emerges: the true height of the air parcel may in fact not be \( z \) but \( Z = z + \lambda \). This is consistent with \( \pi = \pi_{qs} + \pi' \) and the full density being not \( \rho \) but \( \mu = \rho + \rho' = \rho (1 + \partial \lambda / \partial \lambda) \). Indeed, assuming \( \partial \lambda / \partial \lambda \ll 1 \) or, equivalently, \( \rho' / \rho, \mu \approx \rho (1 + \partial \lambda / \partial \lambda) \). If we consider a single atmospheric column with density and specific entropy profiles \( \mu(Z), s(Z) \) close to hydrostatic balance, label each air parcel by its initial position \( Z \), and fictitiously and adiabatically displace each parcel to another altitude \( Z = Z - \lambda \), the density profile becomes \( \mu (dz / dZ) = \rho \). Hence, \( \rho \) is the density of the atmospheric column after all parcels have been displaced from their true height \( Z \) to the height \( z \).

Now the internal and potential energy of the atmospheric column depends on the final position \( z(Z) \) of air parcels as

\[
\mathcal{P}[z] = \int \left[ e \left( \frac{\partial Z}{\mu} s(Z) \right) + gZ \right] \mu dZ \quad \text{and} \tag{28a}
\]

\[
\delta \mathcal{P} = \int \left[ p \left( \frac{\partial Z}{\mu} s(Z) \right) \delta Z + \mu g \delta Z \right] dZ = \int \left[ p \left( \frac{\partial Z}{\mu} s(Z) \right) + \mu g \right] \delta Z dZ, \tag{28b}
\]

where a rigid lid is assumed so that \( \delta Z = 0 \) at the top and bottom of the atmospheric column. Hence, hydrostatic balance \( \delta Z_{qs} + \rho g = 0 \) is satisfied precisely if the altitudes \( z \) are such that \( \mathcal{P} \) is minimized. The hypothetical process moving the air parcels from their true altitude \( Z \) to the altitude \( z \) is therefore an (adiabatic) hydrostatic adjustment. Height \( z \) is the altitude an air parcel would have after its atmospheric column has undergone a hydrostatic adjustment, and \( \lambda \) is the difference between the true height \( Z \) and the adjusted height \( z \), assumed small. This interpretation provides the boundary conditions for \( \lambda \).
For rigid-lid boundary conditions one should take \( \lambda = 0 \) at the top and bottom.

The Lagrangian \([14a]\) then appears as the result of an asymptotic expansion:

\[
\mathcal{L} = \rho \left[ \frac{\mathbf{u}_H \cdot \mathbf{u}_H}{2} + \frac{(Dz/ Dt + D\lambda/ Dt)^2}{2} - \frac{e}{\rho} \left( 1 + \frac{\partial \lambda}{\rho}, s \right) - g(z + \lambda) \right] dz \, dx
\]

\[
\simeq \rho \left[ \frac{\mathbf{u}_H \cdot \mathbf{u}_H}{2} + \frac{(Dz/ Dt)^2}{2} - \frac{e}{\rho} \left( \frac{1}{\rho}, s \right) + \rho \frac{\partial \lambda/ \partial z - g(z + \lambda) \rho}{\rho} \right] dz \, dx. \tag{29}
\]

where we have used the constraint \((15b)\). Replacing \(l_\lambda\) and its derivatives by their expressions \((22)\) as a function of \(\rho, s, \lambda\) yields

\[
\partial_t \rho \mathbf{u} + \text{div} (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p_{qs} + \rho e_z = 0
\]

\[
= \mathbf{V} \left( \rho \frac{\partial \lambda}{\partial z} \right) - \mathbf{V} \left( \frac{\partial \lambda}{\partial z} \right) + \frac{\partial}{\partial z} \left( p_{qs} \nabla \lambda \right). \tag{32a}
\]

This shows that \((26)\), contrary to \((11b)\), clearly admits a flux-form momentum budget. This budget is to be interpreted as follows:

- **The vertical budget is**

\[
\rho \frac{D\mathbf{u}_H}{Dt} = -\frac{\partial p'}{\partial z}. \tag{32b}
\]

The weight of an air parcel is \(gdm\), where \(dm = \rho dx dy dz = (\rho + \rho') dx dy dz(z + \lambda)\) is the true elementary mass. The vertical gradient of hydrostatic pressure \(dp_{qs}\) exactly cancels, by design, this weight on the left-hand side. Hence the vertical acceleration is solely due to nonhydrostatic pressure \(p'\) defined as the extra pressure that arises if air parcels undergo an adiabatic vertical displacement from their hydrostatically adjusted height \(z\) to their true height \(z + \lambda\). Indeed, as discussed previously, if the parcels initially at altitudes \(z\) and \(z + dz\) move to \(z + \lambda\) and \(z + dz + \delta \lambda/ \delta z dz\), the volume they occupy is multiplied by \(1 + (\delta \lambda/\delta z)\)—hence, the expressions \((25)\) for \(p'\) and \(p''\).

- **The horizontal budget is**

\[
\rho \frac{D\mathbf{u}_H}{Dt} + \nabla H p_{qs} = -\nabla H p' - \nabla H \left( \frac{\partial \lambda}{\partial z} \right) + \frac{\partial}{\partial z} \left( p_{qs} \nabla \lambda \right), \tag{32c}
\]

where \(\mathbf{u}_H\) and \(\nabla H\) are the horizontal velocity and gradient. Apart from \(\nabla H p'\) already discussed, two additional terms appear. They make sense if this momentum budget is interpreted as a momentum budget over a vertically displaced infinitesimal domain, located between \(x\) and \(x + dx, y\) and \(y + dy\), and
z + λ and z + dz + λ + ∂λ/∂zdz. The upper and lower boundaries of this domain are actually sloping, with a slope V_Hλ. The pressure force exerted by this upper surface has a horizontal component p_qw V_Hρ. The upper and lower contributions sum up to ∂/∂z(p_qw V_Hλ). Furthermore, the lateral boundaries have height (1 + ∂λ/∂z)dz. The extra pressure forces exerted on these boundaries sum up to −V_H[p_qw(∂λ/∂z)].

Therefore, the physical interpretation of λ as the vertical displacement between the true parcel position and its position after the atmospheric column has undergone hydrostatic adjustment is confirmed by the analysis of the momentum budget. In this budget, two categories of new terms arise. The first term involves the additional internal and potential energy fluxes found in the local budget of the atmosphere.

Now invariance of the Lagrangian with respect to time yields

\[
\frac{\partial l}{\partial t} = \frac{\partial l}{\partial \rho} + \frac{\partial l}{\partial s} \partial_s \rho + \frac{\partial l}{\partial \lambda} \partial_\lambda \rho + \frac{\partial l}{\partial z} \partial_z \rho.
\]

(33a)

Let us note

\[
E' = -l_{\lambda} = \rho g \lambda - p_{qs} \partial_\lambda \rho.
\]

(34a)

which represents the additional internal and potential energy induced by the displacement λ. Notice that hydrostatic balance [(15b)] implies that \(\int E' \, dx = 0\). Now, using \(\partial l/\partial \rho = -\rho g \lambda - p' = -E' - p' - p_{qs} \partial_\lambda \rho\) and collecting the standard terms and the new terms, the full energy budget reads

\[
\partial_t (E + E') + \text{div}[(E + E' + p_{qs} + p')u] + \text{div}[p_{qs}(u \partial_\lambda \rho + \partial_\lambda \rho)] = 0.
\]

(34b)

Boundary conditions of no-normal velocity and \(\lambda = 0\) imply that no energy crosses the boundaries of the domain, so that the domain-integrated energy, equal to the usual energy \(\int E \, dx\), is constant.

The last term of this energy budget makes sense if we remember that the air parcels are actually at altitude z + λ. Their actual velocity is \(u + u'\) with \(u' = (Da/Dt)e_z\). Furthermore, the slope of the surfaces at altitude z + λ and the variation in “thickness” between surfaces at altitude z + λ and z + dz + dλ should be taken into account. These effects are already taken into account in \(\rho\)—hence, in E but not in \(p_{qs}\). They sum up to \(p_{qs} u' - p_{qs} u \cdot V_\lambda + p_{qs} u \partial_\lambda \lambda = p_{qs}(u \partial_\lambda \rho + \partial_\lambda \rho e_z)\). Hence, the energy fluxes found in the local budget of \(E + E'\) can all be interpreted either as the work of the pressure \(p_{qs} + p'\) or as due to the displacement \(\lambda\).

d. Diagnostic equation for the nonhydrostatic displacement

The nonhydrostatic displacement \(\lambda\) must be diagnosed in order to close the semi-hydrostatic system of equations. The hydrostatic constraint [(15b)] does not involve \(\lambda\) and cannot be used to diagnose it. Instead it constrains \(\rho\), which is slaved to \(s\). Additional hidden constraints, obtained by time-differentiating the hydrostatic balance, must exist. Because \(\partial p/\partial t\) and \(\partial s/\partial t\) do not depend on \(\lambda\), the first hidden constraint will not be solved for \(\lambda\) but for \(w\). Only after a second time differentiation does an equation for \(\lambda\) appear. These calculations are done below assuming a flat bottom, but similar calculations can be done in the presence of orography.

One time differentiation of hydrostatic balance yields

\[
\partial_\lambda \left[ c^2 \partial_\lambda (\rho w) + \frac{\partial p}{\partial s} \partial_s u \partial_\lambda s + gw \right] + \partial_\lambda \left[ c^2 \partial_\lambda (\rho u) + \frac{\partial p}{\partial s} \partial_s u \partial_\lambda u + g \partial_\lambda (\rho u_H) \right] = 0.
\]

(35)

Using −\(\rho g = \partial_\lambda p_{qs} = c^2 \partial_\lambda \rho + (\partial p/\partial s) \partial_\lambda s\), \(\partial_\lambda p_{qs} = c^2 \partial_\lambda \rho + (\partial p/\partial s) \partial_\lambda s\) one finds the elliptic equation for \(w\):

\[
A \cdot w + B \cdot u_H = 0,
\]

(36a)

where \(A; w \rightarrow \partial_\lambda (\rho c^2 \partial_\lambda w),\)

\[
B; u_H \rightarrow \partial_\lambda (\rho c^2 \partial_\lambda u_H + u_H \partial_\lambda u_{qs}) + g \partial_\lambda (\rho u_H),\]

(36c)
Assuming a flat bottom, \( w = 0 \) at the top and bottom; \( A \) is self-adjoint, positive definite; while \( B \) has adjoint \( B^* \) (with horizontally periodic BCs and \( \lambda = 0 \) at the top and bottom). Equation (36a) is Ooyama’s “neater form” of Richardson’s equation that arises when solving the primitive equations in Eulerian coordinates (Richardson 1922; Ooyama 1990). It is also a particular case of the form derived by Dubos and Tort (2014) for general quasi-hydrostatic systems. Notice that this form differs from Richardson’s original formulation, which suffers from the hydrostatic systems. Notice that this form differs from the near cancellation of two large terms (AK09), unlike the neater form (Ooyama 1990).

Now the momentum balance [(32a)] is precisely of the form

\[
\partial_t w - \frac{1}{\rho} A \cdot \lambda = \text{rhs} \quad \text{and} \quad (37a)
\]

\[
\partial_t u_H - \frac{1}{\rho} B^* \cdot \lambda = \text{rhs}, \quad (37b)
\]

where rhs collects terms that do not involve \( \lambda \). Time differentiating (36a) and eliminating \( \partial_t w \) therefore yields the self-adjoint problem:

\[
\begin{pmatrix}
B^* \rho & A \\
A & -\rho
\end{pmatrix}
\begin{pmatrix}
\lambda \\
\partial_t w
\end{pmatrix} = \text{rhs}, \quad (37c)
\]

where \( \lambda \) and \( \partial_t w \) satisfy homogeneous Dirichlet boundary conditions. Eliminating \( \partial_t w \) yields a single, positive definite problem for \( \lambda \):

\[
\begin{pmatrix}
A & B^* \rho
\end{pmatrix}
\lambda = \text{rhs} \quad (37d)
\]

with inhomogeneous Dirichlet boundary conditions for \( A \lambda \). One sees that the “strange” terms in (32a) are crucial to obtain an elliptic problem for \( \lambda \).

e. Computational procedure

At this point, it is possible to outline a computational procedure to solve the semi-hydrostatic equations. Using the form [(26)] and hydrostatic height \( z \) as a vertical coordinate, possible prognostic variables of the system are \( u_H, \theta, \rho \) (the computational procedure will be similar if the flux- or vector-invariant form of the system is considered instead). The major diagnostic variables are \( p_{qs}, \lambda, w \). Prognostic variables are assumed to be known at time level \( n \). An explicit time-stepping organization for advancing from the current time level \( n \) to the next one \( n + 1 \) can be sketched as follows:

(i) Hydrostatic quantities \( p_{qs}^{(n)}, \pi_{qs}^{(n)} \) are diagnosed from \( \theta^{(n)} \) and \( \rho^{(n)} \) using the equation of state [(10b)].

(ii) Vertical wind \( w^{(n)} \) is diagnosed from Richardson’s equation [(36a)] with adequate boundary conditions.

(iii) The full wind field \( \mathbf{u}_{H}^{(n)} \), \( w^{(n)} \) being now available, thermodynamic variables \( \theta^{(n+1)} \) and \( \rho^{(n+1)} \) are prognostically obtained using (16).

(iv) The diagnostic equation for the nonhydrostatic displacement \( \lambda^{(n)} \) [(37d)] is solved at the current time level with upper and lower rigid-wall boundary conditions for \( \lambda \). Collaterally, the nonhydrostatic departure \( \pi' \) at time level \( n \) is obtained via (25).

(v) Horizontal wind \( \mathbf{u}^{(n+1)}_H \) is prognostically obtained from (26), adding the extra tendency associated to \( \lambda^{(n)} \) to the right-hand-side term of the horizontal momentum equation.

Notice that density can be predicted by the mass continuity equation without altering the sound-filtering ability of the system provided that the hydrostatic constraint is initially satisfied and later maintained via the resolution of Richardson’s equation [(36a)]. AK09 have devised a computational procedure that avoids solving Richardson’s equation. This procedure may be applicable to the present system, but we leave the investigation of this point for future work.

5. Normal-mode analysis

a. Resting isothermal base flow

Having explicitly verified the dynamical consistency of (23), we now proceed to assess their accuracy. For this we reintroduce the Coriolis force, left aside for brevity in the previous section, and examine the dispersion relationship of infinitesimal perturbations to an isothermal, resting atmosphere. We assume an ideal perfect gas and use potential temperature \( \theta \) instead of entropy \( s \). Including the Coriolis force, the full set of equations of motion is

\[
\frac{D\mathbf{u}}{Dt} = -f \mathbf{e}_z \times \mathbf{u} - \frac{1}{\rho} \left( 1 + \frac{\partial \lambda}{\partial z} \right) \nabla p_{qs} + \frac{1}{\rho} \left( \rho c^2 \frac{\partial \lambda}{\partial z} \right) - g \mathbf{V} \lambda - g \mathbf{e}_z, \quad (38a)
\]

\[
\frac{D\theta}{Dt} = 0, \quad (38b)
\]

\[
\frac{D\rho}{Dt} = -\rho \mathbf{V} \cdot \mathbf{u}, \quad \text{and} \quad (38c)
\]
\[ \frac{\partial}{\partial z} p_{qs} = -\rho g, \]  
\( \text{where } p_{qs}(\rho, \theta) \text{ satisfies } \rho R \theta = p_{qs}/p_{qs}' \) with \( \kappa = R/C_p \) and \( p_{00} \) a reference pressure. For the purpose of the linearization of this system, the atmospheric basic-state \( \lambda^* \) is chosen to be stationary, resting, horizontally homogeneous, and hydrostatically balanced. Hence, \( \theta^* = \theta^*(z) \), \( p_{qs}^* = p_{qs}^*(z) \), and \( \rho^* = \rho^*(z) \), linked by the hydrostatic balance \( dp_{qs}^*/dz = -\rho^*g \). In addition, the temperature of the basic state \( T^* = p_{qs}^*/R \rho^* \) is assumed to be constant in order to be able to find analytic solutions. The corresponding profiles are
\[ p_{qs}^*(z) = p_{qs}^*(0) e^{-z/H}, \]  
\( \rho^*(z) = [p_{qs}^*(0)/RT^*] e^{-z/H^*} - \rho_s^* e^{-z/H_s}, \) and
\[ \theta^*(z) = T^*[p_{qs}^*(0)/p_{00}^*]^{-\kappa} e^{-z/H^*} = \theta_s^* e^{-z/H_s}, \]
where \( H_s = RT^*/\kappa g \) denotes the characteristic height scale of the standard atmosphere and the basic surface pressure \( p_{qs}^*(0) \) is an arbitrary constant. All other variables of the basic state are set to zero, \( u^* = v^* = w^* = \lambda^* = 0 \), where \( u, v, w \) are the velocity components. The analysis is performed for small perturbations around the basic-state \( \lambda^\prime = \lambda - \lambda^* \). The perturbations \( \lambda^\prime \) are assumed to remain small; thus, only the first-order perturbed quantities is considered (i.e., any product of primed perturbation terms is neglected). Moreover, the domain is restricted to a vertical plane along \((x, z)\) directions for clarity. Finally, the linearized system is written
\[ \frac{\partial u'}{\partial t} = fu' - \frac{1}{\rho^*} \frac{\partial}{\partial z} (p_{qs}' + (c_s^2 \beta' - g) \partial_z \lambda'), \]
\[ \frac{\partial v'}{\partial t} = -fu', \]
\[ \frac{\partial w'}{\partial t} = \frac{1}{\rho^*} \partial_z (\rho^* c_s^2 \beta' \partial_z \lambda'), \]
\[ \frac{\partial b'}{\partial t} = -\frac{N_s^2}{g} w', \]
\[ \frac{\partial p_{qs}'}{\partial t} = -\rho g, \] and
\[ (1 - \kappa) \frac{p_{qs}'}{p_{qs}^*} = \frac{b'}{\rho^*} + b', \]
with \( b' = \theta'/\theta^* \), \( c_s^2 = (C_p/C_v)RT^* = g H_s/(1 - \kappa) \) the square of the Brunt–Väisälä frequency. From (40a) one sees again that the extra term \( \kappa g \) is significant only for vertically large motion \( \partial_z \leq 1/H_s \). For an isothermal profile \( c_s, N_s, \) and \( H_s \) are constants and further progress can be made analytically. In what follows, the normal-mode analysis will be along the lines of DSWT03. Specifically rigid lower and upper boundaries at respectively \( z = 0 \) and \( z = z_T \) are assumed. However, assuming \( z_T \gg H_s \), it is unlikely that a different choice of upper boundary condition would materially change the conclusions; see Dukowicz (2013). We first investigate slow modes on \( f \) and \( \beta \) then external and internal modes on the \( f \) plane.

b. Slow modes

For constant \( f \), (40a)–(40e) admit stationary solutions (degenerate Rossby modes). The constraint \( \partial/\partial t = 0 \) leads to \( \omega = \hat{u} = 0 \), corresponding to a nondivergent flow. Furthermore, \( \partial (\rho^* \beta \partial z) / \partial z = 0 \) together with the rigid-wall boundary conditions \( \lambda(0) = \lambda(z_T) = 0 \) implies \( \lambda = 0 \). Therefore, only the geostrophic and the hydrostatic balances hold true for the perturbed hydrostatic pressure \( p_{qs}^* \), the normal wind component \( \nu' \), and the density \( \rho' \), as expected.

We now sketch the solution of (40a)–(40e) on a mid-latitude \( \beta \) plane (i.e., \( f = f_s + \beta \gamma \)). The issue is to assess the accuracy of the dispersion relationship for Rossby modes, especially long external Rossby modes since these modes, which are meteorologically meaningful (Madden 2007), are strongly distorted in the pseudoincompressible and anelastic/pseudoincompressible systems (DSWT03). Therefore, the regime of interest is the quasigeostrophic regime, characterized by \( \partial/\partial x = O(f_s/c_s) \) and \( \partial/\partial t = O(\beta_c f_s) = O(\kappa f_s) \), where \( \kappa = \beta_c f_s \approx 1 \). Expanding (40a)–(40g) in powers of \( \kappa \) yields at zero order the hydrostatic and geostrophic balances, with zero nonhydrostatic displacement. Nonzero \( \lambda \) appears only at order \( \kappa \) but \( \lambda \) is not involved in the expression of potential vorticity, whose transport by the zero-order geostrophic velocity determines the dispersion relationship. Therefore, one ends up with, as in AK09, the same dispersion relationship as with hydrostatic primitive equations (i.e., Rossby modes are undistorted).

c. External modes on an \( f \) plane

We are now interested in nonstationary modes for constant \( f = f_s \). All coefficients in (40a)–(40e) being horizontally homogeneous and time independent, solutions are sought in the normal-mode form \( X' = X(z) \exp(i k x - i \omega t) \) for \( X = u, v, w, b, c_s, p_{qs}, \) \( \lambda \) where \( k \) is a prescribed horizontal
wavenumber and $\sigma \neq 0$ is an unknown pulsation. Inserting this form into (40a)–(40e) yields $d\hat{p}/dz = -\hat{\rho} g$ and

$$ (f^2 - \sigma^2) \hat{u} = -\sigma k \left[ \frac{\hat{p}}{\rho^*} - c_0^2 \left( \frac{d}{dz} - \frac{g}{c_0^2} \right) \hat{\lambda} \right], \tag{41a} $$

$$ \hat{v} = -\frac{f}{\sigma} \hat{u}, \tag{41b} $$

$$ \hat{w} = \frac{c_0^2}{\sigma} \left( \frac{d}{dz} - \frac{1}{H_s^*} \right) \hat{\lambda}, \tag{41c} $$

$$ \hat{\rho} = \frac{N_s^2}{\sigma g \hat{w}}, \tag{41d} $$

$$ \frac{\hat{p}}{\rho^*} = \frac{1}{\sigma} \left[ k \hat{u} + \left( \frac{d}{dz} - \frac{1}{H_s^*} \right) \hat{\lambda} \right], \tag{41e} $$

$$ \frac{\hat{\rho}}{\rho^*} = c_0^2 \left( \frac{\hat{p}}{\rho^*} + \hat{\rho} \right). \tag{41f} $$

Before focusing on modes that satisfy $\hat{w} \neq 0$, which correspond to the internal modes, we first consider modes without vertical motion, $\hat{w} = 0$, corresponding to the external modes of the system. Together with rigid upper and lower boundary condition on $\lambda$, this leads to $\hat{b} = \hat{\rho} = \hat{\lambda} = 0$. Eliminating $\hat{u}$, $\hat{v}$, $\hat{\rho}$ leaves $\left( \sigma^2 - f^2 - c_0^2 k^2 \right) \hat{\rho} = 0$. Assuming $\hat{\rho} \neq 0$, the pulsation $\sigma$ is determined by

$$ \sigma^2 - f^2 - c_0^2 k^2 = 0, \tag{42} $$

while hydrostatic balance $(d\hat{p}/dz - g/c_0^2) \hat{\rho} = 0$ determines the vertical structure of the mode:

$$ p_{\rho} = \hat{p}_0 \exp\left[ -\left( 1 - \frac{\kappa}{\kappa' z/H_s^*} \right) e^{i(kx - \omega t)} \right], \tag{43a} $$

$$ \rho' = \frac{(1 - \kappa)}{RT^*} \hat{p}_0 \exp\left[ -\left( 1 - \frac{\kappa}{\kappa' z/H_s^*} \right) e^{i(kx - \omega t)} \right], \tag{43b} $$

$$ u' = -\frac{\sigma k}{f^2 - \sigma^2} \hat{p}_0 \exp(\kappa z/H_s^*) e^{i(kx - \omega t)}, \tag{43c} $$

$$ u' = \frac{f k}{f^2 - \sigma^2} \hat{p}_0 \exp(\kappa z/H_s^*) e^{i(kx - \omega t)}, \tag{43d} $$

where $\hat{p}_0$ denotes the value of the perturbed hydrostatic pressure at surface ($z = 0$). Note that the vertical modal structure [(43a)–(43d)] is also valid for the external degenerate external Rossby mode (i.e., with $\sigma = 0$). Consequently, the semihydrostatic system correctly captures the frequency and vertical structure of the external modes, for Lamb modes as well as degenerate Rossby modes. The external normal mode are exactly identical to those found for fully compressible equations, as with the AK09 unified equations, and unlike anelastic and pseudoincompressible systems, see DSWT03 and AK09.

d. Internal modes on an $f$ plane

We now consider the internal modes ($\hat{w} \neq 0$). By combining (41d)–(41f) with the hydrostatic balance equation, we obtain a first constraint (44a), equivalent to Richardson’s equation [(36a)]. By eliminating $\hat{\lambda}$ and $\hat{\rho}$ from (41a) using respectively (41c) and (41f), one obtains a second constraint [(44b)] that links $\hat{u}$ to $\hat{w}$:

$$ \left( \frac{d}{dz} - \frac{N_s^2}{g} \right) \hat{u} = -\left( \frac{d}{dz} - \frac{1}{H_s^*} \right) \frac{d\hat{w}}{dz} \tag{44a} $$

$$ (f^2 - \sigma^2 + c_0^2 k^2) \left( \frac{d}{dz} - \frac{1}{H_s^*} \right) \frac{d\hat{u}}{dz} = \frac{c_0^2 k}{\sigma^2} \left( \frac{d}{dz} - \frac{g}{c_0^2} \right) \left[ \frac{\sigma^2}{c_0^2} \left( \frac{d}{dz} - \frac{1}{H_s^*} \right) \frac{d\hat{w}}{dz} \right]. \tag{44b} $$

Hence, eliminating $\hat{u}$ in favor of $\hat{w}$ provides the vertical structure equation:

$$ \left( \sigma^2 - f^2 \right) \Delta^2_{z} + k^2 \left[ \left( N_s^2 - \sigma^2 \right) \Delta^*_{\rho} - \frac{N_s^2}{c_0^2} \right] \hat{w} = 0, \tag{44c} $$

where $\Delta^*_{\rho} = \left( \frac{d}{dz} - 1/H_s^* \right) \frac{d}{dz}$. \tag{44d}

The eigenmodes of the left-hand-side differential operator of (44c) are the same as those of $\Delta^2_{z} = \left( d\hat{p}/dz - 1/2H_s^* \right)^2 - 1/4H_s^2$. The latter are of the form $\hat{w}(z) = \hat{w}_0 \sin(mz) \exp(z/2H_s^*)$, where the vertical wavenumber $m$ is quantized as $m = n\pi/\pi_T$ ($n \in \mathbb{N}$) because of the rigid-wall boundary conditions. Replacing $\Delta^*_{\rho}$ by its eigenvalue $-m^2 - 1/4H_s^2$ in (44c) yields

$$ \left( m^2 + \frac{1}{4H_s^2} + \Gamma_+^2 \right) \left( m^2 + \frac{1}{4H_s^2} - \Gamma_-^2 \right) = 0, \tag{45a} $$

with

$$ \Gamma_\pm^2 = \frac{k^2}{2} \left[ \frac{\left( N_s^2 - \sigma^2 \right)^2}{\sigma^2 - f_s^2} + \frac{4N_s^2 \sigma^2}{c_0^2 k^2 (\sigma^2 - f_s^2)} \right]^{1/2} $$

$$ + \frac{k^2}{2} \left( \frac{N_s^2 - \sigma^2}{\sigma^2 - f_s^2} \right), \tag{45b} $$
where \( \Gamma_{-/-+} \) are real positive values. Here, we assume that \( N_0^2 > f^2 \) and \( \sigma^2 > f^2 \). Since \( m \) is real owing to boundary conditions and \( \Gamma_+ > \Gamma_- \), only the second factor in (45a) can vanish. Hence boundary conditions restrict the permissible frequencies \( \sigma \) to those that satisfy the condition \( \Gamma_- = \frac{1}{2} m^2 + 1/4 H_0^2 > 1/2 H_0 \). Some more algebra yields the dispersion relation

\[
\sigma^2 = f^2 + \frac{k^2(N_0^2 - \delta_m f^2)}{\delta_m^2 k^2 + m^2 + \Gamma_0^2} \tag{45c}
\]

with

\[
\delta_m = 1 - \frac{\kappa(1 - \kappa)}{(m H_0)^2} = \frac{m + 1}{m^2 + \Gamma_0^2}, \quad \Gamma_0 = \frac{1}{2H_0}, \quad \Gamma_1 = (1 - 2\kappa)\Gamma_0. \tag{45d}
\]

Equation (45c) and other dispersion relationships are obtained by appropriately setting the switches \((\delta_V, \delta_A, \delta_U)\) in the more general dispersion relation:

\[
\sigma^2 = f^2 + \frac{k^2(N_0^2 - \delta_V f^2)}{\delta_V k^2 + m^2 + \Gamma_1^2 + \delta_A C_\theta^2 + \delta_V \delta_A \delta_U \sigma^2}, \tag{46}
\]

where \((\delta_V, \delta_A, \delta_U) = (1, 1, 1)\) corresponds to the fully compressible system, \((\delta_V, \delta_A) = (0, 1)\) to the hydrostatic primitive equations, and \((\delta_V, \delta_A) = (1, 0)\) to the pseudo-incompressible system (Durran 1989; DSWT03). Setting \((\delta_V, \delta_A, \delta_U) = (1, 1, 0)\) yields the AK09 unified system while, using \(N_0^2/C_\theta^2 = \Gamma_0^2 - \Gamma_1^2\), (45c) corresponds to \((\delta_V, \delta_A, \delta_U) = (\delta_m, 1, 0)\). Since (45c) has no quartic terms in \( \sigma \), it provides only two solutions \( \sigma(k) \) where (46) with \((\delta_V, \delta_A, \delta_U) = (1, 1, 1)\) would provide four. Whenever \( \delta_m \approx 1 \), (45c) yields a frequency close to the one arising in the AK09 soundproof system. Therefore, the modes filtered by the semi-hydrostatic equations are indeed acoustic.

Notice first that for \( m H_0 \gg 1, 1 - \delta_m \ll 1 \). In this regime, assuming in addition \( f \ll N_0 \), the values of \( \sigma \) given by (46) with \((\delta_V, \delta_A, \delta_U) = (1, 1, 1), (\delta_V, \delta_A, \delta_U) = (1, 1, 0), \) and \((\delta_V, \delta_A, \delta_U) = (1, \delta_m, 0)\) all simplify to the common limit:

\[
\sigma^2 = f^2 + \frac{k^2(N_0^2 - f^2)}{k^2 + m^2}. \tag{47a}
\]

Furthermore, \( \kappa(1 - \kappa) < 0.25 \) so that \( 1 - \delta_m \) is about 2\% when \( m H_0 \) is on the order of 3. With \( H_0 = 10 \text{ km}, m H_0 = 3 \) corresponds to a vertical wavelength of 20 km. Therefore, \( \delta_m \ll 1 \) actually covers a large number of real situations.

The situation is more complex when \( m H_0 = O(1) \). Interesting limits to the dispersion relation [(45c)] are then as follows:

- The steep gravity wave regime \( k \gg m \). Since \( \delta_m \) multiplies the horizontal wavenumber \( k \) in the denominator of the right-hand side of (45c), the effect of \( \delta_m \) is most significant in this regime:

\[
\sigma^2 \approx \frac{N_0^2}{\delta_m}. \tag{47b}
\]

- The deep gravity wave regime where \( k \) is \( O(\Gamma) \). In this regime \( \sigma = O(N_0) \) and

\[
\sigma^2 = \frac{k^2}{\delta_m k^2 + m^2 + \Gamma_0^2} N_0^2. \tag{47c}
\]

- The shallow gravity wave regime where \( f \ll \Gamma \) (note that \( f \ll N_0^2 \) implies \( \Gamma \gg f \)). In this regime \( f \ll \sigma \ll N_0^2 \):

\[
\sigma^2 \approx \frac{k^2 N_0^2}{m^2 + \Gamma_0^2}. \tag{47d}
\]

- The inertia–gravity wave regime where \( k \ll O(f) \). In this regime \( \sigma = O(f) \):

\[
\sigma^2 \approx f^2 + \frac{k^2 N_0^2}{m^2 + \Gamma_0^2}. \tag{47e}
\]

Equation (46) with \((\delta_V, \delta_A, \delta_U) = (1, 1, 1)\) coincides with the above limits only if \( k \ll \Gamma \) in the inertia gravity wave and shallow gravity wave regimes. Conversely (46) with \((\delta_V, \delta_A, \delta_U) = (1, 1, 1)\) and (45c) disagree in the steep and deep gravity wave regimes. In fact in these regimes vertical momentum balance implies \( H_0^2 \lambda \sim (a/C_\theta^2)w \), hence \( \delta_m \lambda / w \sim \sigma^2 H_0^2 / C_\theta^2 = O(1) \), which violates the assumption \( D\lambda / Dt \ll Dz \) (29), required for the accuracy of approximation (29).

Notice that when the wavenumber region \( k \sim m \sim \Gamma \), acoustic modes and internal modes both have \( \sigma = O(N_0) \). Therefore, accurately separating acoustic from gravity modes for \( k \sim m \sim \Gamma \) may present a fundamental difficulty. Furthermore, within the anelastic/pseudo-incompressible approximations \( \Gamma_0 \) is replaced by \( \Gamma_1 \), which distorts the dispersion relationship at scales larger than \( H_0 \) (DSWT03). These points are discussed more in depth in the conclusion (section 6).

Overall, except when \( k \sim m \sim \Gamma_0 \), one can expect that (45c) differs very little from the exact dispersion relationship at all scales in the geophysically relevant
case \( f \ll N_s \). As an illustration, Fig. 1 presents \( \sigma \) computed via (45c) and compared to the exact frequency with a basic-state temperature \( T^* = 250 \) K and an upper rigid lid at \( z_T = 80 \) km as in DSWT03. As expected, it is seen that the major discrepancy is most noticeable where \( kH_* \approx 1 \) and \( mH_* < 3 \). Note that the frequencies of the shallow internal gravity modes such as \( mH_* \approx 5 \) are almost exactly captured by the system for any horizontal scale.

We finally examine the full mode structure, obtained by back substituting the modal form first into (44a) and (44b) and then into (41a)–(41f). To compare with DSWT03, it is important to realize that the perturbations obtained so far are at altitude \( z + \lambda' \). Furthermore, the total pressure perturbation is the sum of the hydrostatic and nonhydrostatic perturbations \( \hat{p}_{qs} \) and \( -\rho \alpha c_s^2 \partial \lambda/\partial z \). Hence, the perturbation of Exner pressure at fixed \( z \) is deduced from \( \hat{p}_{qs} \) and \( \lambda \) as

\[
\hat{\pi}_{tot} = \hat{\pi}_{qs} - \rho \alpha c_s^2 \frac{\partial \lambda}{\partial z} - \lambda \frac{\partial \hat{p}_{qs}^*}{\partial z} = \hat{p}_{qs} + \rho \alpha \left( g - c_s^2 \frac{d}{dz} \right) \hat{\lambda},
\]

which yields

\[
\lambda' = \hat{\lambda}_0 \sin(mz) \exp\left( \frac{z}{2H_*} \right) e^{ik(x - \alpha t)}, \tag{48a}
\]

\[
\omega' = \hat{\omega}_0 \sin(mz) \exp\left( \frac{z}{2H_*} \right) e^{ik(x - \alpha t)}, \tag{48b}
\]

\[
\theta' = \hat{\theta}_0 \sin(mz) \exp\left[ \frac{1}{2} \kappa \right] \frac{z}{H_*} e^{ik(x - \alpha t)} , \tag{48c}
\]

\[
u' = \hat{\nu}_0 \Gamma_1 \sin(mz) - m \cos(mz) \exp\left( \frac{z}{2H_*} \right) e^{ik(x - \alpha t)}, \tag{48d}
\]

\[
p_{qs}' = \hat{p}_{qs} + \pi_{qs}^* \sin(mz) - m \cos(mz) \exp\left( \frac{z}{2H_*} \right) e^{ik(x - \alpha t)}, \tag{48e}
\]

\[
\pi_{tot}' = \hat{\pi}_{tot} \Gamma_1 \sin(mz) - m \cos(mz) \exp\left[ \frac{1}{2} \kappa \right] \frac{z}{H_*} e^{ik(x - \alpha t)}, \tag{48f}
\]

with

\[
\hat{\lambda}_0 = -\frac{\delta_m \sigma^2}{N_s^2 - \delta_m \sigma^2} \frac{\theta^* \hat{\pi}_0}{c_s^2}, \tag{49a}
\]

\[
\hat{\omega}_0 = \frac{\sigma (m^2 + \Gamma_1^2)}{N_s^2 - \delta_m \sigma^2} \frac{\theta^* \hat{\pi}_0}{\pi_s^*}, \tag{49b}
\]

\[
\hat{\theta}_0 = \frac{g(m^2 + \Gamma_1^2)}{N_s^2 - \delta_m \sigma^2} \frac{\theta^* \hat{\pi}_0}{\pi_s^*}, \tag{49c}
\]

\[
\hat{\nu}_0 = \frac{\sigma k}{\alpha^2 - f^2} \frac{\theta^* \hat{\pi}_0}{\pi_s^*}, \tag{49d}
\]

\[
\hat{p}_0 = \frac{N_s^2}{N_s^2 - \delta_m \sigma^2} \frac{\theta^* \hat{\pi}_0}{\pi_s^*}, \tag{49e}
\]

where \( \pi_s^* = c_p T^*/\theta_s^* \), the amplitude \( \hat{\pi}_0 \) is considered given, and \( \sigma \) is given either by the dispersion relation \([45c]\) for the internal gravity modes or \( \sigma = 0 \) for the degenerate internal Rossby modes.

The above modal structure is to be compared to that of the fully compressible system \([i.e., (5.15)–(5.24) of DSWT03]\), setting all the switches to one (notice that the DSWT03 definition of Exner function \( \sigma \) and ours differ by a factor \( c_p \)). Examination of (48a)–(48f) shows that...
the height scale of the internal modes and the location of modal zeroes, determined by the vertical phase angle \( \varphi_m = \arctan(m/T_1) \), are both correctly captured. The amplitudes \((49a)–(49c)\) also coincide with (5.21)–(5.24) of DSWT03 when \( \sigma = 0 \) (i.e., for internal degenerate Rossby modes). However, this is not the case when \( \sigma \neq 0 \) because of \( \delta_m \neq 1 \). Consequently, there is some energy redistribution for internal gravity modes. Nevertheless, \((49a)–(49c)\) do coincide with (5.21)–(5.24) of DSWT03 if either \( \sigma \ll N_a \) or \( \delta_m \simeq 1 \). Hence, energy redistribution is small in the wavenumber region where the dispersion relationship \([(45c)]\) is accurate.

**e. Diagnostic equation for \( \lambda \)**

We finally perform a similar normal-mode analysis of the diagnostic equation for \( \lambda \) \([(37c)]\), \( \partial_w \). Equation \((37c)\) is already a linear problem, so that linearizing about a background flow is not necessary. We consider only the special case where the flow is horizontally homogeneous, isothermal, and at rest, so that it is characterized by a speed of sound \( c \) and a scale height \( H \).

Using first horizontal homogeneity \( \partial_x p_w = 0 \), \( \partial_x \rho = 0 \),

\[
\frac{1}{\rho} B^0 \lambda = (c^2 \partial_z - g) \partial_x \lambda \quad \text{and} \quad (50)
\]

Projecting \( \rho^{1/2} \lambda, \rho^{1/2} \partial_w \) onto the eigenmodes of \( \partial_z \) with Dirichlet boundary conditions:

\[
(\lambda, \partial_w) = \exp \frac{z}{2H} \sum_m (\hat{\lambda}_m, \hat{\partial}_m)(x) \sin(mx), \quad (56)
\]

Indeed, eliminating \( \hat{\partial} = -c^2 [m^2 + (1/4H^2)] \hat{\lambda} + \text{rhs leaves}

\[
(\Delta_H - L_m^{-2}) \hat{\lambda} = \text{rhs} \quad \text{where}
\]

\[
L_m^{-2} = \frac{m^2 + \frac{1}{4H^2}}{m^2 + \frac{1}{4H^2}} \geq \frac{4(1 - \kappa)}{H^2}. \quad (58)
\]

\[
B^0 \lambda = -(g + c^2 \partial_z) \rho (g - c^2 \partial_z) \Delta_H \lambda \quad \text{where}
\]

\[
\Delta_H \lambda = \partial_x (\partial_x \lambda). \quad (51)
\]

Using now the isothermal structure,

\[
H = \frac{RT}{g} = \text{const}, \quad c^2 = \frac{gH}{1 - \kappa} = \text{const},
\]

\[
\partial_z \rho = -\rho / H, \quad (52)
\]

so that for any field \( \Lambda \),

\[
\partial_x (\rho^{1/2} \Lambda) = \rho^{1/2} \partial_x \Lambda, \quad (g + c^2 \partial_z)(\rho \Lambda) = \rho c^2 \partial_x \Lambda, \quad \text{where}
\]

\[
\partial_x (\rho^{1/2} \Lambda) = \rho^{1/2} \partial_x \Lambda = \partial_x (\rho^{1/2} \Lambda), \quad (53)
\]

Hence,

\[
A = \rho c^2 D_1 D_0, \quad B^0 \lambda = \rho c^2 D_\lambda D_1 \Delta_H. \quad (54)
\]

Using \( D_{1 - a} = D_{1/2} - [1/4 - a(1 - a)] \) and \( D_{1/2}(\rho^{-1/2} \lambda) = \rho^{-1/2} \partial_z \lambda \), \((37c)\) becomes

\[
\left( \begin{array}{c}
\left( c^2 \left\{ m^2 + \frac{1}{4H^2} \right\} \frac{1}{H^2} \right) \Delta_H \frac{m^2 + \frac{1}{4H^2}}{1/c^2} \\
\partial_z = \frac{1}{4H^2} \\
\end{array} \right) \left( \begin{array}{c}
\hat{\lambda} \\
\hat{\partial}_w \\
\end{array} \right) = \text{rhs}. \quad (55)
\]

where \( m \) is quantized by \( m = n \pi / z_T, \ n \in \mathbb{N} \) yields a series of independent Helmholtz problems for \( \lambda_m \):

\[
\left( \begin{array}{c}
\left( c^2 \left\{ m^2 + \frac{1}{4H^2} \right\} \frac{1}{H^2} \right) \Delta_H \frac{m^2 + \frac{1}{4H^2}}{1/c^2} \\
\end{array} \right) \left( \begin{array}{c}
\hat{\lambda} \\
\hat{\partial}_w \\
\end{array} \right) = \text{rhs}. \quad (57)
\]

The significant point in \((58)\) is that \( L_m \leq O(H) \). Although the response of the flow to a Dirac forcing \( \text{rhs} \sim \delta(x - x_0) \) has presumably infinite support, so that information formally propagates instantaneously to infinity, \((58)\) implies that this response decays exponentially fast beyond a region of extent \( \sim H \). Therefore, in practice, information does not significantly
propagate to arbitrarily distant places of the fluid. Numerically, this limits the stiffness of the linear problem [(37d)].

6. Conclusions

Starting from Hamilton’s least-action principle, compressible nonhydrostatic equations of motion with density diagnosed from entropy (or potential temperature) through hydrostatic balance have been derived for an arbitrary equation of state. They possess a vector-invariant form [(23) and (26)] as well as a flux form [(32a)]. The analysis of linear normal modes and dispersion relationship for small departures from an iso-

thermal state of rest on $f$ and $\beta$ planes shows excellent accuracy from hydrostatic to nonhydrostatic scales, with the exception of the deep gravity wave regime, where both horizontal and vertical wave lengths are comparable to the scale height. Especially the Lamb wave and long Rossby waves are not distorted, unlike with anelastic or pseudoincompressible systems.

The advantage of the variational approach is the guarantee to obtain dynamically consistent equations. The difficulty lies in the physical interpretation of the Lagrange multiplier. To this end, the existence of a local momentum budget is of great help. Nevertheless, this budget includes, in addition to the gradient of a nonhydrostatic pressure perturbation, unexpected terms with a nonstraightforward interpretation. Providing this interpretation requires the notion of hydrostatic height, defined as the height to which air parcels in a given atmospheric column should move adiabatically in order to restore hydrostatic balance. This notion is, as far as we are aware, new and we expect that it can be useful in other contexts than the present work owing to the pre-

eminence of hydrostatic balance in geophysical flows. The unexpected terms are then found to be an apparent force resulting from the vertical coordinate not being the actual height of an air parcel but its hydrostatic height. Consequently, the Lagrange multiplier $\lambda$ is identified as the nonhydrostatic vertical displacement (i.e., the difference between the actual height of an air parcel and its hydrostatic height). By comparing the volume of air parcels before and after a hypothetical hydrostatic adjustment, the nonhydrostatic perturbations of density, pressure, and Exner pressure are found proportional to the vertical derivative of $\lambda$, assumed small. In addition to $\partial \lambda / \partial z \ll 1$, the condition $DA/Dt \ll w$ must be met for the Lagrangian [(29)] to be a good approximation of the exact one. It is this condition that fails in the deep gravity wave regime. Finally, $\lambda$ is naturally subject to Dirichlet boundary conditions for the rigid boundaries considered here.

Enforcing one holonomic constraint suppresses 2 degrees of freedom and results in the associated Lagrange multiplier satisfying a self-adjoint diagnostic equation. Here, slaving density to potential temperature suppresses the degrees of freedom supporting the propagation of acoustic waves and results in a soundproof system. As with hydrostatic primitive equations, vertical velocity is diagnosed through Richardson’s equation [(36a)]. The semihydrostatic system has therefore precisely the same degrees of freedom as the hydrostatic primitive equations. This suggests that the difference between hydrostatic and nonhydrostatic motion is essentially quantitative and not qualitative (i.e., any nonhydrostatic phenomenon should have a hydrostatic counterpart, albeit probably with quantitatively different amplitude, dispersion relationship, etc.). This conclusion is already suggested by the success of the anelastic and pseudoincompressible approximations, which, apart from those of the Lamb wave, have the same degrees of freedom as the hydrostatic equations.

In AK09, $z$ is interpreted as the full nonhydrostatic height (Konor 2014). Consequently, the terms due to the difference between the actual height of an air parcel and its hydrostatic height are omitted. In our case, these terms are crucial for the transparent derivation of a wellposed self-adjoint problem [(37d)] for $\lambda$, from which nonhydrostatic pressure derives. Generally speaking, the diagnostic equation satisfied by a Lagrange multiplier reflects the structure of the constraint that it imposes (e.g., Bernardet 1995). Our self-adjoint problem is vertically fourth order with four boundary conditions for $\lambda$, reflecting the fact that the hydrostatic constraint satisfied by density involves a vertical derivative, unlike the pointwise constraint at the core of the anelastic and pseudoincompressible approximations. Especially, it seems impossible to formulate a self-adjoint problem and boundary conditions directly for nonhydrostatic pressure, as attempted by AK09. If following AK09, the term $g\nabla\lambda$ is omitted from horizontal momentum balance [(26)], then a linear problem similar to (37c) is obtained, except that it loses its self-adjoint character.

It is natural to ask whether other soundproof approximations fare better than ours in the deep gravity wave regime $k \sim m \sim \Gamma$. Separating accurately gravity waves from acoustic waves seems problematic in this regime on physical grounds, since both branches have comparable frequencies $O(N_\ast)$. To discuss this issue we let $f = 0$ for simplicity. Then, in the deep gravity wave regime, the dispersion relationships [(46)] with $(\delta_V, \delta_A, \delta_U) \neq (1, 1, 1)$ differ at first order from the exact value given by $(\delta_V, \delta_A, \delta_U) = (1, 1, 1)$. This is all but unexpected since the general dispersion relationship is of the form $\sigma/N_\ast = \Sigma(kH_\ast, mH_\ast, \kappa)$. If both $kH_\ast$ and $mH_\ast$
are $O(1)$ and $\kappa$ is not small, there is no small dimensional parameter left for an asymptotic expansion. Any expression for $\sigma/N_\infty$ is then either exact or inaccurate. Furthermore, for any soundproof system $\sigma^2$ must be the ratio of two polynomials in $K_\infty$ and $M_\infty$. The expression for $\sigma^2$ solving (46) with $(\delta_1, \delta_2, \delta_3) \neq (1, 1, 1)$ involves a square root and cannot exactly coincide with a polynomial fraction. Hence, it seems impossible for a soundproof system to be accurate (in an asymptotic sense) in the deep gravity wave regime. To evaluate quantitatively the seriousness of this issue, we plot in Fig. 2 the relative difference between the exact and approximate value of $\sigma$, for $f = 0, \kappa = 2/\gamma$, as a function of normalized horizontal and vertical wavenumbers $K_\infty$ and $M_\infty$ for the present approximation (semihydrostatic), $\text{AK09}$ (unified), hydrostatic and pseudoincompressible approximations. The "inaccurate" region where $\Delta \geq 10^{-2}$ is shaded.

\begin{align*}
\text{semi-hydrostatic} & \quad \text{unified} \\
\text{hydrostatic} & \quad \text{pseudo-incompressible}
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Decimal logarithm of relative error $\Delta = |\sigma_{\text{exact}} - \sigma_{\text{approx}}|/\sigma_{\text{exact}}$ of the frequency of internal normal modes of a nonrotating isothermal atmosphere with $\kappa = 2/\gamma$, as a function of normalized horizontal and vertical wavenumbers $K_\infty$ and $M_\infty$, for the present approximation (semihydrostatic), AK09 (unified), hydrostatic and pseudoincompressible approximations. The "inaccurate" region where $\Delta \geq 10^{-2}$ is shaded.}
\end{figure}

However, the pseudoincompressible system is inaccurate for shallow gravity waves. Our approximate system is inaccurate for steep gravity waves, as discussed in section 5d, but corrects much of the errors of the hydrostatic system. Especially the relative error never exceeds $2\kappa (1 - 2\kappa)$, yielding a maximum relative error of $\frac{\gamma}{3}$ (133%). Although large, this is far from the complete breakdown of the hydrostatic equations for which relative frequency error is unbounded at small scale. The AK09 unified system has the smallest possible inaccurate domain, for deep gravity waves only, as discussed above. Hence, the term $g\alpha$ appearing in the horizontal momentum balance (26) restores desirable properties of the equations such as unambiguous local budgets of energy and momentum as well as an explicit self-adjoint diagnostic problem for $\lambda$ at the price of some accuracy. Nevertheless, the unified pseudoincompressible and semihydrostatic systems are quite accurate as soon as $M_\infty$ is of order 1–3 (see Fig. 3). The main difference is in the propagation of long Rossby waves, which is incorrect for the pseudoincompressible equations.

It is not clear at this point whether numerically integrating the semihydrostatic equations is more efficient than integrating the full Euler equations with a proper treatment of acoustic waves (Skamarock and Klemp 2008; Weller et al. 2013; Smolarkiewicz et al. 2014). The answer will depend primarily on the possibility to efficiently solve the self-adjoint problem for $\lambda$. Whatever the eventual outcome, one can think of several other
applications. From a theoretical point of view, the semihydrostatic system provides an explicit separation between the acoustic and, broadly speaking, vortical motion, be it hydrostatic or nonhydrostatic. Diagnosing the vertical velocity and nonhydrostatic displacement amounts to filtering out the acoustic part of motion. Variational data assimilation systems may benefit from such an accurate elimination of acoustic waves, reducing the optimization space. Similarly, it could be useful for initialization purposes, avoiding the transient generation of acoustic waves from an initial flow. It may also be helpful to occasionally perform this projection while integrating the fully compressible Euler equations, especially after physical parameterizations have acted, potentially triggering the transient emission of acoustic waves.

The present semihydrostatic theory suggests a number of follow-up works. Dubos and Tort (2014) provide the necessary formalism to obtain its Hamiltonian formulation in Eulerian and non-Eulerian coordinates, especially terrain-following coordinates (Kasahara 1974; Laprise 1992), which can be useful for numerical implementation as well as for theoretical purposes. Generalization to deep-atmosphere or spheroidal geometries is also straightforward using Tort and Dubos (2014b). Finally, the notion that air parcels are close to their hydrostatic height could be used to revisit and possibly simplify quasi-hydrostatic systems of equations as well (White and Bromley 1995; Tort and Dubos 2014a).

Acknowledgments. We thank C. S. Konor and R. Klein for their constructive reviews, which helped improve the accuracy and clarity of the manuscript.

REFERENCES


Dukowicz, J. K., 2013: Evaluation of various approximations in atmosphere and ocean modeling based on an exact treatment.


