Effects of Horizontal Geometrical Spreading on the Parameterization of Orographic Gravity Wave Drag. Part I: Numerical Transform Solutions

STEPHENV. ECKERMANN
Space Science Division, Naval Research Laboratory, Washington, D.C.

JUN MA AND DAVE BROUTMAN
Computational Physics, Inc., Springfield, Virginia

(Manuscript received 21 May 2014, in final form 12 March 2015)

ABSTRACT

Numerical transform solutions for hydrostatic gravity waves generated by both uniform and sheared flow over elliptical obstacles are used to quantify effects of horizontal geometrical spreading on amplitude evolution with height. Both vertical displacement and steepness amplitudes are considered because of their close connections to drag parameterizations in weather and climate models. Novel diagnostics quantify the location and value of the largest wavefield amplitudes most likely to break at each altitude. These horizontal locations do not stray far from the obstacle peak even at high altitudes. Resulting vertical profiles of wave amplitude are normalized to remove density and refraction effects, thereby quantifying the horizontal geometrical spreading contribution, currently absent from parameterizations. Horizontal geometrical spreading produces monotonic amplitude decreases with height through wave-action conservation as waves propagate into progressively larger horizontal areas. Accumulated amplitude reductions are appreciable for all but the most quasi-two-dimensional obstacles with long axes orthogonal to the flow, and even these are impacted appreciably if the obstacle is rotated by more than 20°–30°. Profiles are insensitive to the obstacle’s functional form but vary strongly in response to changes in its horizontal aspect ratio. Responses to background winds are captured by a vertical coordinate transformation that remaps profiles to a universal form for a given obstacle. These results show that horizontal geometrical spreading has comparable or larger effects on wave amplitudes as the refraction of vertical wavenumbers and thus is important for accurate parameterizations of wave breaking and drag.

1. Introduction

Horizontal Fourier transforms are employed routinely to derive three-dimensional orographic gravity wave (OGW) responses to low–Froude number (linear) flow over three-dimensional mountains. The advantages of a horizontal spectral formulation of this problem are reviewed by Broutman et al. (2004). They derive initially from incorporating arbitrary three-dimensional terrain elevations \( h(x, y) \) in their spectral form \( \hat{h}(k, l) \) to specify lower boundary forcing, where \( (x, y) \) are the horizontal spatial coordinates and \( (k, l) \) are the corresponding horizontal wavenumbers. This in turn permits a relatively straightforward one-dimensional treatment of vertical propagation and amplitude evolution of radiated wave activity at each wavenumber \( (k, l) \), which remains constant with height to a good approximation in many atmospheric environments. The resulting spatial wavefield solutions, derived by an inverse two-dimensional horizontal Fourier transform of the solutions at a given height \( z \), avoid the caustic singularities of stationary phase solutions and incorporate geophysical effects that are very difficult to capture using spatial ray methods. These include interference and diffraction, realistic lower boundary forcing, and influences of geometrical spreading on local wave amplitudes due to wave-action conservation.

Conversely, global numerical weather and climate prediction models [referred to collectively as general circulation models (GCMs)] parameterize drag due to
unresolved OGW activity using spatial rather than spectral wave solutions. There are many reasons for this. Spatial OGW solutions are expedient, computationally cheap, and interface directly to the spatial formulation of diabatic tendencies within GCMs. More generally, the primary requirement of subgrid-scale OGW drag (OGWD) parameterizations is to diagnose accurately the onset of wavefield instabilities that lead to OGW breaking and drag. Those dynamics are highly localized spatially and thus more naturally parameterized using spatial solution methods. The difficulty is that spatial OGW solutions require approximations that omit potentially important geophysical effects on wavefield evolution that are captured by spectral OGW solutions and may also be important for accurate parameterizations of OGWD.

Here we consider one such omitted process that can affect wave amplitude evolution with height: horizontal geometrical spreading. As OGWs propagate obliquely away from their parent obstacle in the form of three-dimensional ship waves (e.g., Smith 1980), they “spread out” into progressively larger spatial volumes. In the absence of dissipation, the total (volume integrated) wave action must be conserved, so this propagation into larger volumes should be accompanied by corresponding local reductions in wave-action densities $A(x, y, z)$ and, hence, local wave amplitudes. This can be quantified in the ray approximation using the continuity equation

$$\frac{\partial A}{\partial t} + \mathbf{V} \cdot (\mathbf{c}_g A) = 0,$$  

(1)

where

$$A = E_{tot}/\dot{\omega},$$

(2)

$E_{tot}$ is the total (kinetic plus potential) wave energy density, $\dot{\omega}$ is intrinsic frequency, $t$ is time, $\mathbf{V} = (\partial \dot{x}/\partial x, \partial \dot{y}/\partial y, \partial \dot{z}/\partial z)$, and $\mathbf{c}_g$ is the ground-based group velocity vector (Bretherton and Garrett 1968; Andrews and McIntyre 1978).

This equation is notoriously difficult to solve in spatial coordinates, requiring numerical methods to infer spatial gradients across a tube of neighboring ray trajectories (e.g., Lighthill 1978; Marks and Eckermann 1995). For steady-state OGWs, integrating action over the ray-tube volume, then applying the Gauss divergence theorem to the action flux propagating through a horizontal cross-sectional area of that ray tube, yields a ray-following solution to (1) of the form (e.g., Shutts 1998; Broutman et al. 2002)

$$c_g z J_h = \text{constant},$$

(3)

where $c_g$ is vertical group velocity. The Jacobian $J_h = \partial(x, y)/\partial(x_0, y_0)$ accounts for changes in the horizontal cross-sectional area of a ray tube as component rays propagate to new locations $(x, y)$ at some specified $z$, relative to their earlier locations $(x_0, y_0)$ at a reference height $z_0$. Closed form analytical solutions for $J_h$ valid for arbitrary vertical profiles of winds and stability, as would be required for OGWD parameterization, do not exist.

Thus, OGWD parameterizations simplify the problem by assuming that horizontal derivatives in (1) can be ignored, so that (1) and (3) each simplify as follows:

$$\frac{\partial}{\partial z} (c_g A) = 0 \iff J_h = 1 \iff c_g z A = \text{constant}. \quad (4)$$

Since the vertical flux of horizontal momentum for a plane two-dimensional gravity wave is equal to $(k, l)c_g A$, and $(k, l)$ values are also constant in the absence of horizontal gradients, then (4) is equivalent to the standard Eliassen and Palm (1961) result for two-dimensional OGWs.1 For three-dimensional OGWs, however, the vertical flux of horizontal momentum is conserved only when integrated over the entire horizontal domain (e.g., Vosper and Mobbs 1998), and local values can and do change because of horizontal geometrical spreading [i.e., $J_h \neq 1$; e.g., Fig. 5 of Eckermann et al. (2010)]. Since it is local rather than domain-averaged wave amplitudes that determine where OGWs break and how much momentum and energy they deposit locally, OGWD parameterizations should ideally be incorporating horizontal geometrical spreading effects on local wave amplitudes.

What remains unclear is whether horizontal geometrical spreading is an important or minor effect, with its omission from existing OGWD parameterizations currently suggesting the latter (i.e., $J_h \approx 1$). While some previous work has studied this issue for specific idealized OGW problems (e.g., Broad 1999), to date no study has provided detailed quantitative estimates of how horizontal geometrical spreading affects the vertical evolution of local OGW amplitudes across a range of obstacle shapes and upstream forcing profiles, as is required to assess the general relevance for OGWD parameterization.

The purpose of this paper is to quantify the horizontal geometrical spreading effect on OGW amplitudes as a function of height, obstacle aspect ratio, and upstream

---

1 This is the general connection between Eliassen–Palm flux and wave-action density for plane waves propagating in any horizontal direction, of which two-dimensional zonal relations such as (4A.11)–(4A.13) of Andrews et al. (1987) are a special case.
wind and stability profiles, using exact spectral transform solutions for OGWs. The paper is organized as follows. Section 2 presents background theory and modeling tools. First, section 2a reviews hydrostatic spatial OGW equations underpinning OGWD parameterizations, and then section 2b describes a Fourier-ray algorithm for deriving exact OGW solutions for linear flow over three-dimensional mountains. Section 2c describes our method for deriving wavefield amplitudes from the Fourier-ray solutions, and section 2d outlines how we isolate and quantify effects of horizontal geometrical spreading on those wavefield amplitudes. These diagnostics, when applied to the standard OGWD equations in section 2a, reveal that no horizontal geometrical spreading effects on wave amplitudes are currently incorporated into OGWD parameterizations. Conversely, on applying these diagnostics in sections 3 and 4 to Fourier-ray OGW solutions in uniform and vertically sheared flows, respectively, generally large impacts of horizontal geometrical spreading on wave amplitudes are revealed, which vary strongly with height, obstacle aspect ratio, and wind profiles. Section 5 summarizes our major findings, discusses their possible wider ramifications, and recommends future research avenues, some of which are taken up in the companion paper of Eckermann et al. (2015, hereafter Part II).

2. Theory and models

a. OGWD parameterization relations

Within a GCM grid box, terrain elevations can be expressed as

$$h(x, y) = \overline{H} + h(x, y),$$

where \(\overline{H}\) is the elevation of the GCM’s resolved orography and \(h(x, y)\) are elevations of the unresolved subgrid-scale orography (SSO). SSO is restricted here to horizontal scales \(\approx 1\) km, since shorter scales yield turbulent form drag that requires a separate parameterization.

Contemporary parameterizations of orographic mesoscale drag (OMD) typically characterize the mean elevation and anisotropy of SSO by computing within every grid box the mean variance of \(h\), \(\sigma_h^2\), and mean covariance terms involving the slope components \(h_x\) and \(h_y\). This process can be viewed as fitting the SSO to an effective three-dimensional elliptical obstacle, which in turn allows the OMD parameterizations to draw upon a rich modeling literature of both linear and nonlinear OMD responses to flow over idealized elliptical obstacles [see, e.g., discussion and references in the parameterization studies of Lott and Miller (1997), Gregory et al. (1998), Scinocca and McFarlane (2000), and Webster et al. (2003)].

The SSO’s effective peak obstacle height \(h_m = s_h \sigma_h\), where \(s_h\) is a constant that varies with the shape and variability of SSO: typical choices are \(s_h \approx 1.5–2.5\) (McFarlane 1987; Webster et al. 2003). The surface Froude number

$$Fr = \frac{U_0}{N_0 h_m}$$

defines mesoscale flow regimes (Smith 1989), where \(U_0\) and \(N_0^2\) are the upstream surface wind speed and stability. \(Fr \gtrsim 1\) yields a linear gravity wave response. For \(Fr \lesssim 1\), the flow splits about a dividing streamline at height \(z_d\) (\(0 \leq z_d < h_m\)), with flow above \((z > z_d)\) producing gravity waves and flow below \((z < z_d)\) diverted around the obstacle to produce nonlinear wake dynamics [see, e.g., Eckermann et al. (2010) and references therein].

Here we focus exclusively on the parameterized gravity wave response and, hence, only on the OGWD component of the parameterized OMD. The wave’s peak vertical displacement amplitude \(\eta_0\) at the ground is

$$\eta_0(z = 0) = \eta_0 = h_m - z_d.$$

In the linear regime, \(z_d = 0\) and thus \(\eta_0 = h_m\). Linear nonrotating hydrostatic wave theory is most commonly employed by parameterizations to propagate these waves away from the SSO to higher levels. By also assuming that vertical profiles of horizontal wind \(U_h(z) = (U, V) = U_h \cos \phi\) and stability \(N^2(z)\) are steady and have negligible horizontal gradients, then the spatial ray trajectory equations for a plane gravity wave can be solved analytically. These solutions yield a constant stationary horizontal phase speed \((c = 0)\) and constant horizontal wavenumber \(k_h = (k, l) = h_k \cos \phi\). The wave’s intrinsic frequency \(\omega = -k U - l V = -k_h U_{ij}\), where \(U_{ij} = U_h \cos(\phi - \varphi)\) is the component of \(\mathbf{U}\) parallel to \(k_h\). The vertical wave-number \(|m| = 2\pi/\lambda_z\) then follows from the hydrostatic nonrotating dispersion relation

$$m^2 = \frac{k^2 N^2}{\omega^2} = \frac{N^2}{U_{ij}^2}.$$

As reviewed in section 1, in the absence of dissipation, plane hydrostatic gravity waves conserve their vertical fluxes of horizontal momentum \((\rho u w\mathbf{w})\), where \(\rho\) is atmospheric density, and \(u_{ij}\) and \(w\) are, respectively, wave-induced perturbations of horizontal and vertical velocity, the former aligned parallel to \(k_h\). Overbars denote averages over one horizontal wavelength.
The simplest derivation for the onset of static instability (overturning) of a hydrostatic gravity wave yields $\alpha = 1$ (Fritts 1984).

Following Lindzen (1981), at a breaking height $z_b$ where the wave is unstable ($\eta_a > \eta_b$), the wave amplitude is scaled back to its breaking threshold (i.e., $\eta_a \rightarrow \eta_b$), so that the reduced momentum flux that propagates to the next vertical level is $\rho U^2 \left( \frac{\eta_b}{\eta_a} \right)$, and the amount of flux deposited at $z_b$ is $\rho U^2 \left( 1 - \frac{\eta_b^2}{\eta_a^2} \right)$. On progressing upward through all model levels $z$, a modified vertical profile of momentum flux is generated, the vertical gradients of which quantify the parameterized OGW, which decelerates the resolved GCM flow as

$$\frac{\partial U}{\partial t} = -\epsilon \frac{\partial \rho U^2}{\partial z}.$$ (18)

Linear plane-wave theory yields $\epsilon = 1$ in (18), whereas some parameterizations adopt a tunable value of $\epsilon < 1$ to capture the net effect of intermittent or inefficient OGW breaking (e.g., Kiehl et al. 1998). Since in general $U$ and $k_b$ are not coaligned, (18) does not produce pure drag forces, owing to an additional “lift” component orthogonal to $U$. Additional wave-induced heating and turbulent mixing of the GCM flow can also be quantified and applied. These additional forcing terms generally only become significant above the stratopause.

b. Numerical Fourier-ray solutions

As in the parameterization problem, we consider upstream flow profiles $U(z)$ and $N(z)$ that impinge upon a three-dimensional obstacle of elevation $h(x, y)$. To focus solely on gravity waves, we consider the linear limit $Fr \gg 1$. The resulting three-dimensional stationary gravity waves can be derived numerically in the long time limit ($t \rightarrow \infty$) using the Fourier-ray (FR) solution method of Broutman et al. (2002). Our FR model used here incorporates a flexible form of the gravity wave dispersion relation given by

$$m(k, l, z) = \sqrt{\left( k^2 + l^2 \right)^2 - \varepsilon \omega^2 \left( \frac{k^2 + l^2}{\omega^2} + \frac{f^2}{\omega^2} \right)^{1/2}},$$ (19)

where $f$ is the Coriolis parameter. Since the spectral representation of horizontal variables in the FR solution (see below) equipartitions wave amplitude between the $(k, l)$ and $(-k, -l)$ wavenumbers, then the sign parameter

$$s = -\text{sgn}(\omega(k, l, 0))$$ (20)
in (19) ensures a consistent sign for the vertical group velocity

\[
e_{s}(k,l,z) = \frac{\partial \omega}{\partial m} = \frac{-m(\alpha^2 - \epsilon_f^2)}{\omega[k^2 + l^2 + m^2]}. \tag{21}
\]

Hydrostatic or nonhydrostatic solutions result by setting \(\epsilon_f = 0\) or 1, respectively, while rotational modifications due to the Coriolis force can be turned on or off by setting \(\epsilon_f = 0\) or 1, respectively.

The FR solution for vertical displacement takes the form

\[
\hat{\eta}(k,l,z) = \hat{h}(k,l) \left[ \frac{c_{s}(k,l,0) \omega(k,l,z) \rho(0) P(0) N^2(0)}{c_{s}(k,l,z) \omega(k,l,0) \rho(0) P(0) N^2(z)} \right]^{1/2} e^{i \int_{m(k,l,z)} d\tau}, \tag{22}
\]

where \(\hat{h}(k, l)\) is the Fourier transform of \(h(x, y)\). The horizontal wavefield at each height \(z\) is recovered via the inverse Fourier transform

\[
\eta(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\eta}(k,l,z) e^{i(kx+ly)} dk dl. \tag{23}
\]

The polarization factor

\[
P(z) = \frac{1 + \epsilon_f f^2 / \alpha^2}(1 - \epsilon_f f^2 / \alpha^2) + (1 + \epsilon_n \omega^2/N^2) \tag{24}
\]

follows from conservation of wave action, via the action density

\[
A = \frac{1}{4} \rho(z) P(z) N^2 \eta_0^2 \frac{1}{|\omega|} \tag{25}
\]

[an equivalent derivation in terms of horizontal-velocity amplitude is given in appendix B of Marks and Eckermann (1995)]. Note that the ratio of \(c_{s}(k,l,z) / \omega P\) terms in (22) arises from an analysis based on wave action rather than momentum flux (Broutman et al. 2002): in the hydrostatic nonrotating limit (\(\epsilon_n = \epsilon_f = 0\)), these terms reduce to the same ratio of \(m\) values given in (13).

We compute solutions for other wave parameters using linear polarization relations. For example, FR solutions for steepness \(\eta_s\) follow by multiplying the integrand in (23) by \(m\) and solving in the same way.

c. Peak wave amplitude solutions

Once FR solutions \(X(k,l,z)\) for a given state parameter \(X\) are derived, we derive corresponding spatial wavefields via the inverse Fourier transform

\[
X_s(x,y,z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(k,l) X(k,l,z) e^{i(kx+ly)} dk dl. \tag{26}
\]

We consider two simple forms for \(S\). The first, \(S = 1\), yields the geophysical wavefield \(X(x,y,z)\) as in (23),

\[
X_A(x,y,z) = |\hat{X}(x,y,z)| = [\hat{X}(x,y,z) \hat{X}^*(x,y,z)]^{1/2}. \tag{29}
\]

Figure 1 compares \(\eta(x,y,z)\) and corresponding \(\eta_A(x,y,z)\) solutions, with the density term \([\rho_i / \rho(z)]^{1/2}\) in
factored out, for unsheared zonal flow of $U = 10 \text{ m s}^{-1}$ and $N = 0.01 \text{ s}^{-1}$ impinging upon an axisymmetric obstacle of $a = b = 10 \text{ km}$ and $h_m = 100 \text{ m}$ located at $x = y = 0$ [see green contour in (c) and (d)]: (a) $x$–$z$ cross section at $y = 0$ and (c) $x$–$y$ cross section at $z = 5 \text{ km}$. (b),(d) Corresponding cross sections of wave amplitude $\eta_A(x, y, z)$ derived from the complex FR solution $\eta(x, y, z)$ using (29). Units are meters; see color bars in each panel for dynamic range.

**Fig. 1.** (a),(c) FR solutions for $\eta(x, y, z)$, scaled by $[\rho(z)/\rho_0]^{1/2}$ to remove amplitude growth due to the density effect, for unsheared zonal flow of $U = 10 \text{ m s}^{-1}$ and $N = 0.01 \text{ s}^{-1}$ impinging upon an axisymmetric obstacle of $a = b = 10 \text{ km}$ and $h_m = 100 \text{ m}$ located at $x = y = 0$ [see green contour in (c) and (d)]: (a) $x$–$z$ cross section at $y = 0$ and (c) $x$–$y$ cross section at $z = 5 \text{ km}$. (b),(d) Corresponding cross sections of wave amplitude $\eta_A(x, y, z)$ derived from the complex FR solution $\eta(x, y, z)$ using (29). Units are meters; see color bars in each panel for dynamic range.

**,Quantification of horizontal geometrical spreading effects on local wave amplitudes**

Horizontal geometrical spreading effects on wave amplitudes are not explicit in the FR solutions $\hat{X}(k, l, z)$ and appear only after they are Fourier transformed into spatial coordinates using (26). For example, Fig. 1d reveals a maximum $\eta_A(x, y, z)$ at $z = 5 \text{ km}$ [after removal of the density term $(\rho_0/\rho)^{1/2}$ in (22)] of approximately $35 \text{ m}$, considerably less than the peak surface value of $h_m = 100 \text{ m}$. This reduction is the effect of horizontal geometrical spreading on the local wavefield amplitudes.

As evident from Fig. 1d, the solution $X_A(x, y, z)$ contains a range of local amplitudes at each $z$, whereas OGWD parameterization relations (11)–(13) consider a single parameterized wave amplitude at each height. For OGWD parameterization, only the largest wavefield amplitudes are relevant for wave breaking according to the inequalities (15)–(17). So, at each $z$ we locate and store the maximum value $X_A^{\text{max}}(z) = \max[X_A(x, y, z)]$. (30)

We do this initially for vertical displacements $\eta_A(x, y, z)$, since $\eta$ solutions have the most direct connection to OGWD parameterization equations (see section 2a). However, the local wavefield amplitudes most relevant for wave breaking are steepnesses $\eta_A(x, y, z)$, since wave breaking is associated with steepness amplitudes exceeding a single constant threshold $\alpha$ given by (17), whereas the breaking threshold $\eta_b$ in (16) is $m$ dependent and thus varies spatially across the wavefield. We will show that, apart from some interesting deviations near the ground, resulting $\eta_A^{\text{max}}(z)$ and $\eta_A^{\text{max}}(z)$ profiles have essentially identical characteristics.
To quantify the geometrical spreading contribution to $X_A^{\max}(z)$, all other sources of vertical variation must be quantified and removed. These are contained in the FR solution $X(k, l, z)$: for vertical displacement, these terms are contained within the square brackets in (22). In many cases we can accurately represent these terms as an altitude profile $G_X(z)$ that has no spectral wavelength dependence: for example, for constant $U$ and $N$, the terms in parentheses in (22) simplify to 

$$G_h(z) = \frac{h^2}{C_20^2} \left[\frac{\rho(0)}{\rho(z)}\right]^{1/2}.$$  

(31)

Given a surface $\eta$ amplitude of $h_m$ in the linear limit [see (7)], then we can define a normalized vertical displacement amplitude 

$$a_{\eta}(z) = \frac{\eta^{\max}(z)}{h_m G_{\eta}(z)},$$  

(32)

which will be unity at all heights in the absence of horizontal geometrical spreading. Deviations from unity then quantify an additional height-dependent multiplicative influence on wave amplitudes due to horizontal geometrical spreading. For example, note that (31) and (32) applied to the OGWD parameterization relation [(13)] yields $a_{\eta}(z) = 1$, illustrating that current OGWD parameterizations incorporate no horizontal geometrical spreading influence on wave amplitudes.

If our algorithm selected the same ray tube from a region of a three-dimensional wavefield where the stationary phase approximation was valid, then from (3), (32) would yield $a_{\eta}(z) = [I_{\eta}(z)/I_{\eta}(0)]^{1/2}$. However, our algorithm is not ray following and generally locates maximum amplitudes from different ray groups at each altitude. Moreover, it focuses on wavefields derived from exact transform solutions rather than from their stationary-phase approximations that are typically invalid close to the mountain (e.g., Shutts 1998). Thus, (32) can also potentially include interference and diffraction effects. Our algorithm is a bulk calculation performed over the entire wavefield rather than a ray-tube calculation focused on one portion of that wavefield and, as such, is potentially more applicable to bulk parameterizations of OGWD.

3. Hydrostatic waves in uniform flow

Here we consider height-invariant upstream profiles $U(z) = (U_0, 0)$ and $N(z) = N_0$, impinging upon an elliptical mountain of the generalized bell-shaped form (Phillips 1984)

$$h(x, y) = \frac{h_m}{[1 + (x/a)^2 + (y/b)^2]^{1/2}},$$  

(33)
where \( h_m \) is the peak elevation at \( x = y = 0 \). The length scales \( a \) and \( b \), aligned parallel and orthogonal to the flow \( U_0 \), respectively, quantify the obstacle’s ellipticity via the aspect ratio

\[
\beta = \frac{b}{a}.
\]  

(34)

We force the minimum value of \( a \) or \( b \) to be 10 km, then assign the remaining value based on a specified \( \beta \). Thus \( \beta = 3 \) yields \( a = 10 \) km and \( b = 30 \) km, while \( \beta = \frac{1}{3} \) yields \( a = 30 \) km and \( b = 10 \) km. Since OGWD parameterizations are generally hydrostatic and omit Coriolis effects, we set \( \epsilon_\theta = \epsilon_\tau = 0 \), so that (19) simplifies to

\[
m = s \frac{N_0}{U_0} \left( 1 + \frac{\beta^2}{k^2} \right).
\]  

(35)

We set \( h_m = 100 \) m although none of the results to follow depend in any way on the choice for \( h_m \) beyond a requirement to satisfy the linear limit \( Fr \gg 1 \), whereupon \( \eta_\theta(0) = h_m \) from (7). Further details of the solution numerics are given in the appendix.

As discussed in section 2d, under these simplifications, the normalized peak amplitude \( a_\theta(z) \) in (31), with \( G_\theta(z) \) given by (32), quantifies amplitude variations with height due to horizontal geometrical spreading. Figure 2 plots \( a_\theta(z) \) on a logarithmic height scale for \( N_0 = 0.01 \) s\(^{-1} \), \( U_0 = 10 \) m s\(^{-1} \), and \( p = \frac{3}{2} \) based on numerical FR results for \( \beta \) values ranging from \( \frac{1}{10} \) to 10. The curves show that horizontal geometrical spreading produces monotonic amplitude decreases with height for all \( \beta \). As expected, these reductions with height are smallest for \( \beta \gg 1 \) obstacles that approximate two-dimensional ridges generating plane waves that do not spread horizontally. More pronounced reductions with height occur as \( \beta \) decreases and the obstacle becomes progressively more three-dimensional with respect to the flow, leading
to three-dimensional “ship wave” responses that spread horizontally. As a check of the method, note that for \( \beta = 1 \), the yellow curve in Fig. 2 yields a value of approximately 0.35 at \( z = 5 \) km, in agreement with the maximum wave-field amplitude in Fig. 1d divided by \( h_m = 100 \) m.

Figure 3 plots the three-dimensional locations of these largest wavefield amplitudes. For \( \beta \gg 1 \) (not shown) the points remain close to the mountain peak at \( x = y = 0 \) at all \( z \), consistent with nearly plane two-dimensional hydrostatic OGWs that propagate purely vertically. Even as \( \beta \) reduces to axisymmetry (\( \beta = 1 \)) in Fig. 3a, these points still remain close to the obstacle peak at all heights. As \( \beta \) reduces still further, Figs. 3b–d show that, at very low altitudes, points move downstream and laterally along the diverging wings of a broad three-dimensional ship-wave response (e.g., Fig. 1d) but then reverse direction in \( x \) and return closer to the mountain as \( z \) increases. We interpret this reversal as horizontal spreading progressively reducing wave amplitudes in the diverging wake, whereupon largest amplitudes revert to wave groups located closer to the mountain that have not undergone appreciable amplitude reductions due to horizontal geometrical spreading.

For steepness \( \eta \), we first define a characteristic \( m \) over the obstacle peak where \( l = 0 \) and thus \( |m| = N_0/U_0 \) from (35). Then we can define a normalized steepness profile

\[
a_{\eta}(z) = \frac{\eta_{\max}(z)}{h_m G_{\eta}(z)},
\]

where \( G_{\eta}(z) = G_{\eta}(z) \) in (31) and

\[
h_m = \frac{h_m N_0}{|U_0|} = Fr^{-1}
\]

is the so-called normalized obstacle height (Eckermann et al. 2010), equivalent to both the peak wave steepness amplitude at the surface and the inverse surface Froude number [see (6)].
Figure 4 compares $a_H(z)$ and $a_{Hz}(z)$ profiles for a range of different $\beta$. At heights well above the mountain, the two curves lie on top of each other for all $\beta$. Closer to the obstacle, the $a_{Hz}(z)$ curves diverge to values exceeding unity. This latter behavior is studied in greater depth in Knight et al. (2015) and Part II, where it is shown to result from some unusual properties of the complex steepness solution near the ground.

The excellent agreement between the $a_H(z)$ and $a_{Hz}(z)$ curves at all far-field heights above the mountain indicates that both solutions identify the same common largest amplitudes that are most apt to break and are thus of primary relevance for bulk OGWD parameterization.

a. Sensitivity to functional form for $h(x, y)$

Figure 5 plots six different functional forms for $h(x, y)$ for which we computed FR solutions and horizontal geometrical spreading profiles $a_H(z)$. Different colored curves in Fig. 6a plot $a_H(z)$ for a $\beta = 1$ obstacle for each of these different $h(x, y)$ functions. For a given $U_0$ and $N_0$ all the colored curves for different obstacle shapes overlay, indicating that the vertical profile of amplitude reductions due to horizontal geometrical spreading has very little sensitivity to details in the obstacle’s shape.

b. Sensitivity to $|U_0|$ and $N_0$ values

Figure 6a reveals, however, that the $a_H(z)$ curves separate as $|U_0|$ is varied, indicating a sensitivity to background winds. However, on replotting all these curves in Fig. 6b using a normalized height coordinate

$$z' = \frac{N_0 z}{|U_0|},$$

we see that all the profiles collapse onto a common universal curve.

This is studied in more depth in Fig. 7. Normalized amplitude profiles of both vertical displacement and steepness for different values of $\beta$, $U_0$, and obstacle shape are plotted as a function of $z$ in the top row and as a function of $z'$ in the bottom row. Again, this change of height variable from $z$ to $z'$ causes both the $\eta$ and $\eta_z$ curves to collapse onto common universal curves for each parameter, asymptotically approaching a single universal form at large $z'$. This remarkable finding is investigated theoretically in Part II.

c. Variation with obstacle rotation

Next we study how $a_H(z)$ responds as the obstacle (33) is rotated by some horizontal azimuth angle $\phi$, such that
where

\[ h(x, y) = \frac{h_m}{\left[1 + \left(x' / a\right)^2 + \left(y' / b\right)^2\right]^{3/2}}, \quad (39) \]

\[ x' = x \cos \phi + y \sin \phi, \quad y' = -x \sin \phi + y \cos \phi. \quad (40) \]

Figure 8 plots \( a_n(z) \) profiles for \( |U_0| = 10 \text{ m s}^{-1} \) and \( N_0 = 0.01 \text{ s}^{-1} \) impinging on terrain (39) of \( p = \frac{3}{2} \) with an initial \( \beta = \beta_0 > 1 \). As this obstacle is successively rotated in 15° steps, numerical FR solutions are computed and used to derive an \( a_n(z) \) profile in each case. At \( \phi = 90° \), the opposing symmetric limit of \( \beta = \beta_0^{-1} < 1 \) is attained. Various panels show results for different \( \beta_0 \), as indicated by colored obstacle shapes in each panel.

The results in Fig. 8 reveal a smooth and monotonic change in \( a_n(z) \) values at every height between the limiting symmetric profiles values at \( \beta_0 \) and \( \beta_0^{-1} \). However, the shape of the curves also changes as \( \phi \) varies. For intermediate \( \phi \) values, the symmetry of the wave forcing...
problem is lost and the flow across these tilted elliptical obstacles forces an asymmetric wave pattern with respect to \( y = 0 \) that can send wave groups well downstream and laterally from the obstacle peak. As a result the profiles have different properties to the limiting symmetrical forms, and despite extensive tests we were not able to find any simple analytical fit or coordinate transformations that could reproduce a common universal form found earlier for the symmetrical configuration.

Figure 9 plots contours of \( a_h \) as a function of \( u \) and \( \log b \) at a series of six different heights, ranging from close to the ground (Fig. 9a) up to \( z = 80 \) km (Fig. 9f). Note the symmetries evident in these results due to redundancies in these elliptical obstacle rotations, such that \( a_h \) values at \( b = 90^\circ \) and \( 270^\circ \) are always identical to those at \( b = 180^\circ \) and \( 270^\circ \). At heights of about 40–100 km, Figs. 9d–f show that wavefields from the vast majority of obstacle aspect ratios and orientation angles yield \( a_h(z) \), indicating order of magnitude or more reductions in wave amplitude due to horizontal geometrical spreading. Only in the far bottom-right and top-left corners of these plots are significant \( a_h \) values retained, indicative of the smallest impacts of horizontal geometrical spreading on wavefield amplitudes. These corner regions correspond to large-\( \beta \) ridgeline obstacles with long axes nearly orthogonal to the incident flow. But note from Figs. 9d–f that rotating even these high-\( \beta \) obstacles by as little as \( \phi = 30^\circ \) leads to large reductions in \( a_h(z) \) values owing to asymmetries that trigger three-dimensional propagation of wavefields away from the obstacle peak and, thus, horizontal geometrical spreading that reduces local wave amplitudes. It is clear from Fig. 9 that horizontal geometrical spreading is not just nonnegligible but has a large effect on wavefields generated by obstacles across most of this \( b - \phi \) parameter space.

4. Hydrostatic waves in uniform shear

Here we retain constant \( N_0 \) but add a constant vertical shear \( C \), such that

\[
U(z) = U_0 + Cz.
\]

From (22), on defining a dominant hydrostatic \( m \) over the mountain, where \( l = 0 \), of \( |m| = N_0/U_0 \) from (35), we get a new normalization factor for \( a_h(z) \) in (32) of

\[
G_h(z) = (1 + Cz/U_0)^{1/2} \left[ \frac{\rho(0)}{\rho(z)} \right]^{1/2} = g_h(z) \left[ \frac{\rho(0)}{\rho(z)} \right]^{1/2},
\]

which is identical to the corresponding OGWD relation [(13)] under the current approximations. Similar arguments
yield a corresponding normalization factor for steepness amplitudes $a_{h_{m}}(z)$ in (36) of

$$G_{h_{m}}(z) = (1 + Cz/U_{0})^{3/2}\left[\frac{\rho(0)}{\rho(z)}\right]^{1/2} = g_{h_{m}}(z)\left[\frac{\rho(0)}{\rho(z)}\right]^{1/2},$$

(43)

where, as before, $h_{m}$ is given by (37). The dependence in (43) is identical to that resulting from the OGWD steepness relation in (14). We refer to $g_{h}(z)$ and $g_{h_{m}}(z)$ above as the refraction terms, since they quantify how the refraction of OGW $m$ values in response to background wind variations with height alters wave amplitude so as to conserve wave action. These $g_{h}(z)$ and $g_{h_{m}}(z)$ profiles replicate the $m/m_{0}$ terms in the OGWD relations in (13) and (14).

Figure 10 shows four different $U(z)$ profiles resulting from $C$ values indicated in the figure and $U_{0} = 10$ m s$^{-1}$. The first value (solid line) contains a weak negative shear that reduces winds to 5 m s$^{-1}$ at $z = 100$ km. The second value (dotted line) has no shear and thus should reproduce earlier unsheared results of section 3. The third and fourth values contain weak and strong positive shear, respectively, increasing winds at 100 km to 20 and 110 m s$^{-1}$, respectively.

The top row of Fig. 11 plots the resulting $a_{h}(z)$ and $a_{h_{m}}(z)$ profiles for $\beta$ values of 1/3, 1, and 3 using each of the four shear profiles in Fig. 10. As noted in the un-sheared case, changes in background winds give rise to changes in profile shape and value, but again at large $z$ we see that $a_{h}(z)$ and $a_{h_{m}}(z)$ curves superimpose for all four wind profiles.

By analogy to our earlier vertical coordinate transformation [(38)] for unsheared cases, we generalize $z'$ here to a sheared wind profile as

$$z' = \int_{0}^{z} \frac{N_{0}}{U(z)} d\tilde{z},$$

(44)

which, given (41), can be evaluated analytically, yielding the result

$$z' = \frac{N_{0}}{C} \log\left(1 + \frac{Cz}{U_{0}}\right).$$

(45)

The middle row shows the same collection of curves above replotted using this transformed $z'$ in (45). As for
the unsheared case, this $z'$ transformation collapses all the
curves onto a common curve for $h$ and for $h z$, and these
two curves merge into one universal curve at large $z$.

The bottom row shows the refraction terms $g_h(z)$ and
$g_{h z}(z)$. Since OGWD parameterizations include these
terms, it is interesting to compare their magnitudes rel-
itive to the corresponding $a_h(z)$ and $a_{h z}(z)$ profiles in
the top row of Fig. 11. In almost every case, the changes
with height of $a_h(z)$ and $a_{h z}(z)$ are as large, and in many
cases larger, than those of the corresponding $g_h(z)$ and
$g_{h z}(z)$. This strongly suggests that the horizontal geo-
metrical spreading terms are at least as relevant as the
refraction terms to an accurate parameterization of
OGWD in GCMs.

5. Summary and discussion

Our results show that OGW amplitude reductions
with height due to horizontal geometrical spreading are
generally appreciable for three-dimensional OGWs. For
example, at heights well above the surface, Fig. 9 reveals
order-of-magnitude amplitude reductions over most of
the $\beta-\varphi$ parameter space defining elliptical obstacle
forcing of OGWs, while Fig. 11 shows effects as large or
larger than refraction for OGWs in sheared wind profiles.

![Fig. 9. Contours of $a_h(z)$ for $N_0 = 0.01 \text{ s}^{-1}$ and $U_0 = 10 \text{ m s}^{-1}$ as a function of $\varphi$ and $\beta$, plotted at (a) $z = 2$,
(b) $z = 10$, (c) $z = 20$, (d) $z = 40$, (e) $z = 60$, and (f) $z = 80$ km. Note the symmetries in these plots implied by the
obstacle properties depicted in Fig. 8, such that $a_h(\beta, \varphi) = a_h(\beta^{-1}, 90^{\circ} - \varphi)$. The y-axis scale is logarithmic to
highlight these symmetries about $\beta = 1$.](Fig9.png)

![Fig. 10. $U(z)$ profiles for indicated shear values C, which are
used to compute FR solutions and resulting $a_h(z)$ and $a_{h z}(z)$ pro-
files in Fig. 11.](Fig10.png)
As demonstrated in section 2d, existing OGWD parameterizations do not include horizontal geometrical spreading effects. Figure 12 illustrates the errors that can result from their omission. The colored curves reproduce \( a_n(z) \) profiles from Fig. 2 for the indicated obstacle aspect ratio \( \beta \) but scaled as \( \left[ r_0/[\rho_0/U] \right] \) to produce geophysical amplitude variations with height. The gray curve shows the profile that results from the OGWD equation in (13): that is, \( a_n(z) = 1 \) and thus no horizontal geometrical spreading. The black vertical line denotes a threshold for wave breaking as given by (16): the actual value depends on choices for \( h_m \) and \( \alpha \) and is immaterial to the arguments that follow. Where the profiles intercept this vertical line determines the lowest wave-breaking height \( z_b \). For current OGWD parameterizations with \( a_n(z) = 1 \), the gray and black curves intersect in Fig. 12 at \( z_b \approx 40 \text{ km} \). The yellow curve in Fig. 12 for an axisymmetric mountain \( (\beta = 1) \) intersects the black line at \( z_b \approx 80 \text{ km} \). Thus, the omission of horizontal geometrical spreading in this case leads to a factor-of-2 error in breaking height. The ensuing errors in drag and mean-flow acceleration are also large (see Part II). Even moderately two-dimensional obstacles (e.g., \( \beta = 3 \)) show differences in breaking heights relative to current parameterization predictions of about 10–20 km. Note that horizontal geometrical spreading reduces amplitudes and so increases \( z_b \).
FIG. 12. Profiles of $a_h(z)$ from Fig. 2 scaled by $[\rho/\rho(z)]^{1/2}$ using a nominal density scale height of 7 km. Gray curve shows current OGWD approximation of no horizontal geometrical spreading [$a_h(z) = 1$]. Vertical black line shows nominal threshold amplitude for wave breaking.

Despite apparently positive impacts of parameterized OGWD in GCMs initially (e.g., Palmer et al. 1986), later studies diagnosed serious GCM errors due to excessive parameterized OGWD in the lower stratosphere (Klinker and Sardeshmukh 1992; Hogan and Brody 1993; Milton and Wilson 1996). Later inclusion of the dividing streamline $z_d$ and resultant reduction of OGW amplitudes and fluxes via (7) (Lott and Miller 1997) reduced but did not eliminate this problem. Modelers have since used various ad hoc methods to reduce these OGWD errors: deactivating OGWD above the tropopause (Hogan and Brody 1993; Norton and Thuburn 1999), increasing $z_d$ to reduce OGW flux via (7) (Webster et al. 2003; Scinocca et al. 2008), and tuning the efficiency $\epsilon$ in (18) to small values (Kiehl et al. 1998). All of these changes seek to reduce or eliminate OGWD breaking at lower levels and push breaking heights higher. We contend that these changes are (at least in part) a proxy for the missing physics of horizontal geometrical spreading, which, when included as in Fig. 12, produces reductions of local wave amplitudes and increases in breaking heights that seem to be needed to reduce the OGWD-induced GCM errors cited above.

An interesting consequence of the FR OGW solution [(22)] [and, indeed, of the spatial ray approximation in (3)] is a clean separation of the refraction effect [$g_\eta(z)$, $g_{\eta_1}(z)$] and of the horizontal geometrical spreading effect [$a_\eta(z)$, $a_{\eta_1}(z)$]. This allows us, as in Fig. 12, to add horizontal geometrical spreading physics to existing OGWD WKB relations in a similarly straightforward separable way: for example, by simply scaling (13) by $a_\eta(z)$ to yield the new form

$$
\eta^{\text{new}}(z) = a_\eta(z)\eta_0(z) = \eta_0(z)[m(z)\rho(0)N_0^2(z)]^{1/2}
$$

(46)

Note in particular that the $a_\eta(z)$ scaling in (46) provides a physical rationale for why constant ad hoc efficiency factors of $\epsilon \approx 0.1$ are needed in some OGWD schemes (e.g., Kiehl et al. 1998).

A remaining challenge is to specify $a_\eta(z)$ in simple, accurate, and general forms so as to augment OGWD parameterizations as in (46). Our results reveal properties of these solutions that are encouraging in this regard, particularly the insensitivity of the profiles to fine details of the mountain shape (see Figs. 5-7), and how variations in response to wind profiles are removed, to yield a common universal shape to these profiles, using a transformed height coordinate $z'$, given in its most general form by (44); see Figs. 7 and 11. Part II is devoted to a theoretical study of those topics, motivated by the numerical findings reported here.

Acknowledgments. This research was supported by the Office of Naval Research (ONR) through the NRL 6.1 Accelerated Research Initiative “The Boundary Paradox,” by ONR’s Departmental Research Initiative “Unified Physical Parameterizations for Seasonal Prediction,” and in part by a grant of computer time from the DoD High Performance Computing Modernization Program at the Navy DoD Supercomputing Resource Center.

APPENDIX

Details of Numerical FR Calculations

Without a sufficiently wide horizontal domain, the periodic boundary conditions in (23) cause wave solutions to wrap around horizontally, leading to spurious interference that contaminates wave amplitude estimates. The issue requires greatest attention for small $N$ obstacles whose wavefields disperse long horizontal distances and is a bigger issue for steepness rather than vertical displacement solutions because of the relatively larger amplitudes of large-$N$ wave groups. Our approach to the wraparound problem varied among applications.

For the $\eta$-based results in Fig. 2, we mitigated wraparound by using a very large horizontal $x$-$y$ domain of $49.152 \times 24.576$ points, with $\Delta x = 1$ km and $\Delta y = \beta\Delta x$ for
\[ \beta \geq 1, \text{ and } \Delta y = 1 \text{ km and } \Delta x = \beta^{-1} \Delta y \text{ for } \beta < 1. \] For the asymmetric obstacle orientations \( \phi \) in Fig. 8, we set \( \Delta x = \Delta y = 1 \text{ km.} \) Two separate simulations with different vertical grids were used in each case: a low-altitude range from 0 to 2 km of 400 points with \( \Delta z = 5 \text{ m, } a \) high-altitude range from 0 to 100 km of 100 points with \( \Delta z = 1000 \text{ m.} \) These simulations were performed using a parallelized version of the FR code run on multiple processors.

This “brute force” approach is computationally infeasible using single-processor versions of the FR code. For these FR integrations, we selectively suppressed energetically insignificant waves near the lateral boundaries using a physically based viscous damping rate of the form

\[ \tau^{-1} = \varepsilon(\omega) \nu (k^2 + l^2 + m^2), \quad (A1) \]

where \( \nu \) is the viscosity coefficient and \( \varepsilon(\omega) \) is a frequency-dependent polarization factor near unity [the full expression is given in (18)–(19) of Eckermann et al. (2011)]. This damping is applied by adding an imaginary component \( m_i = (\nu \varepsilon)^{-1} \) to the vertical wavenumber in (22).

The results in Figs. 4, 7, and 11 were derived using a single-processor FR code with \( \nu = 0.2 \text{ m}^2 \text{ s}^{-1} \). This level of viscosity is much less than typical values owing to turbulent diffusive mixing in the troposphere, upper stratosphere, and mesosphere, and is roughly equal to the minimum background levels of turbulent vertical diffusion found in the lower stratosphere [see, e.g., Fig. 7 of Marks and Eckermann (1995)]. Thus, this small amount of viscosity removes only those waves likely to be dissipated in the real atmosphere by turbulent diffusive damping. These serial FR runs used an 8192 \( \times \) 20484 \( x-y \) domain with \( \Delta x = \Delta y = 2 \text{ km} \) and an irregular 62-point height grid equipped in logarithmic height increments between 0.1 and 100 km, yielding 20 grid points per height decade plus the boundary values at 0 and 100 km. For the shear case in Fig. 11, the phase integral on this irregular grid was calculated using the analytical solution

\[ \phi(z) = \int_{0}^{z} m(z) \, dz = \frac{N_0^C}{C} \log \frac{U(z)}{U_0}. \quad (A2) \]

The most sensitive results to FR formulation are the three-dimensional loci in Fig. 3, since wraparound easily contaminates these results, calling for wide horizontal domains. Yet the maxima are concentrated close to the hill, calling for a narrower domain with increased horizontal resolution near the hill. These results used an FR integration on a 49152 \( \times \) 24,576 horizontal mesh with \( \Delta x = \Delta y = 1 \text{ km} \) and \( \nu = 0.25 \text{ m}^2 \text{ s}^{-1}. \)

We cross checked our solutions in various ways. For example, the problems in sections 3 and 4 have exact analytical eigenfunction solutions (Smith 1989)

\[ \tilde{n}_{89}(k, l, z) = \hat{h}(k, l) \left( \frac{\rho(0)}{\rho(z)} \right)^{1/2} \left( -1 - \frac{C \varepsilon_k}{U_0} \right)^a, \quad (A3) \]

where

\[ a = \frac{1}{2} - i \left[ \text{Re} \left( \frac{k^2 + l^2}{k^2} \right) - \frac{1}{4} \right]^{1/2}, \quad (A4) \]

where \( \text{Re} = N^2/C^2 \) is the gradient Richardson number. For unsheared flow (\( C = 0 \)) these solutions assume the simpler form (Smith 1989)

\[ \tilde{n}_{89}(k, l, z)|_{C=0} = \hat{h}(k, l) \left( \frac{\rho(0)}{\rho(z)} \right)^{1/2} e^{m_{89}^\prime z}, \quad (A5) \]

where

\[ m_{89}(k, l) = -\left[ \frac{N^2(k^2 + l^2)}{k^2 U_0^2} \right]^{1/2}. \quad (A6) \]

We regenerated results using these analytical relations and found identical horizontal geometrical spreading profiles for \( a_s(z) \).

REFERENCES


——, J. Ma, and X. Zhu, 2011: Scale-dependent infrared radiative damping rates on Mars and their role in the deposition of


