Dynamics of Upper-Level Frontogenesis in Baroclinic Waves

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(Manuscript received 26 August 2015, in final form 5 April 2016)

ABSTRACT

This paper reports a diagnosis of the structure and dynamics of upper-level fronts (ULFs) simulated with a high-resolution Weather Research and Forecasting Model with diabatic heating versus one without diabatic heating. The ULFs of both simulations develop in about 6 days as integral parts of intensifying baroclinic waves. Each has a curvilinear structure along the southern edge of a relatively narrow long tongue of high potential vorticity in which stratospheric air is subducted to different tropospheric levels by synoptic-scale subsidence. It resembles a veil in the sky of varying thickness across the midsection upstream of the trough of the baroclinic wave.

The 3D frontogenetical function $F_3$ is shown to be a necessary and sufficient metric for quantifying the rate of development of ULFs. Its value is mostly associated with the contribution of the 3D ageostrophic velocity component. Upper-level frontogenesis is attributable to the joint direct influence of the vortex-stretching process and the deformation property of the 3D ageostrophic flow component. The model also generates a spectrum of vertically propagating mesoscale gravity waves. The ULFs simulated with and without diabatic heating processes are qualitatively similar. The ULF is considerably more intense when there is heating. The heating, however, does not make a significant direct contribution to $F_3$ but indirectly does so through its impacts on the subsidence field of the baroclinic wave.

1. Introduction

Unlike surface fronts, upper-level fronts (ULFs) do not seem to separate polar and tropical air (Reed 1955). Their signature feature is an associated tropopause folding through which significant exchange of stratospheric and tropospheric air takes place (Reed and Danielsen 1958; Danielsen 1968). There are also reports of clear-air turbulence in ULFs presumably associated with breaking gravity waves. Potential vorticity is expected to be a suitable dynamical tracer of ULFs (Reed and Sanders 1953). The 3D gradient of potential temperature in ULFs is characteristically large with significant shear and static stability. Making use of research aircraft data together with radiosonde data, Shapiro (1978) resolved the 100-km cross-front scale of a ULF.

His focus was mostly on the evolution of the relatively short upper-level jet streaks that migrate through a background synoptic-scale baroclinic wave. He attributed the frontal structure to the forcing of an indirect transverse circulation by the geostrophic flow component in the upper-level jet streak (Shapiro 1981; Shapiro et al. 1984). However, ULFs seem to be a part of the synoptic-scale baroclinic waves in general (Reed 1955; Newton 1958; Nieman et al. 1998).

A 2D semigeostrophic model was used by Hoskins (1972) to demonstrate the formation of ULFs driven by an imposed vertically uniform confluent flow. The dynamics of frontogenesis was interpreted as a feedback process in the context of a developing transverse circulation deduced from the Sawyer–Elissien equation. The positive feedback in such a frontal system stems from the dependence of the coefficients in that equation upon the evolving part of the geostrophic vorticity and stratification. Keyser and Pecnick (1985a,b) extended such investigation with a 2D primitive equation model with a nonmodal initial disturbance and an external forcing. It
was found that, when the prescribed geostrophic flow has cold advection on the cross-front plane, the simulated ULF is more intense because the secondary circulation is more skewed than the counterpart in the case of warm advection.

ULFs can be also simulated with a zonally uniform baroclinic jet as a forcing and a corresponding unstable normal mode as an initial disturbance. Heckley and Hoskins (1982) applied this approach to a 3D semigeostrophic model in geostrophic coordinates. The formation of a ULF in that model was demonstrated to be an integral part of an intensifying unstable baroclinic wave, albeit the model front was rather weak. It is noteworthy that only a geostrophic frontogenesis function defined in terms of the horizontal thermal gradient was diagnosed in that investigation. There were also frontal investigations using models in isentropic coordinates (Buzzi et al. 1977, 1981) as well as low-resolution 3D primitive equation models (Mudrick 1974; Hines and Mechoso 1991). Keyser and Shapiro (1986) wrote a review of the literature of ULFs preceding 1986.

While the transverse circulation of a 2D front in a semigeostrophic model is governed by the Sawyer–Eliassen equation that in a 3D semigeostrophic model in physical coordinates is governed by two coupled equations for two scalar functions (Mak 2014). Those two equations are a form of generalization of the Sawyer–Eliassen equation. Such a solution highlights the fact that the effects of the three components of the transverse circulation should be viewed as integral parts of one mechanism underlying the 3D structure.

In recent years, there have been a number of investigations of fronts using 3D primitive equation models (Rotunno et al. 1994; Zhang 2004) and nonhydrostatic models (Plougonven and Snyder 2007; Wei and Zhang 2014) of increasingly fine resolution. The more recent investigations primarily focus on the generation of mesoscale gravity wave modes in a jet–front system. The sensitivity studies suggest that it would require grid resolution of 25 km or less to adequately resolve the properties of such gravity waves. Rotunno et al. (1994) investigated the dynamics of ULFs with a model of 100-km resolution. The forcing was a baroclinic jet straddling a tropopause. The initial disturbance was an unstable normal mode. It was found that it would be necessary to use a sufficiently strong baroclinic jet (60–80 m s\(^{-1}\)) to simulate a realistic ULF with an upper-level cutoff low. Their model ULF was “a nearly continuous feature going from ridge to trough” (Rotunno et al. 1994, p. 3373) Those authors diagnosed the model ULF with the traditional frontogenetical function \(\frac{D}{Dt} \nabla^2 \theta\), where \(\nabla^2 \theta\) is the horizontal potential temperature gradient and \(\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla\right)\), with \(\mathbf{V}\) the 3D velocity operator and \(\nabla\) the 3D velocity. Furthermore, the basic point of reference in their diagnosis is quasigeostrophic theory in the interest of “accuracy and clarity.”

Since surface fronts are characterized by a strong horizontal thermal gradient, \(\frac{D}{Dt} \nabla^2 \theta\) can be expected to be an appropriate metric for quantifying their development. This is, however, not the case for ULFs. The observational studies unmistakably indicate \(\nabla \theta \gg \nabla^2 \theta\) in ULFs. It does not seem likely that one could adequately quantify the evolution of ULFs without taking into account the change in \(\theta\). There is a brief discussion of \(\frac{D}{Dt} \nabla \theta\) in the text by Bluestein (1993, their section 2.3.2). This metric apparently has never been diagnosed in a self-contained frontal study either with observational data or model data. It is incumbent upon us to ascertain if \(\frac{D}{Dt} \nabla \theta\) would be a necessary and sufficient metric for measuring the rate of development of ULFs.

The model used in this investigation with and without diabatic heating and the experimental setup are described in section 2. The structure of the model ULF in the dry run is analyzed in section 3 and that in the moist run is analyzed in section 6. The formulas related to \(\frac{D}{Dt} \nabla \theta\) are extensively explained in sections 4a and 4c. Focusing on the dry simulation, we present the distributions of \(\frac{D}{Dt} \nabla \theta\) on several complementary cross sections and compare the key result with that of \(\frac{D}{Dt} \nabla^2 \theta\) in section 4b. Our diagnosis is not restricted to a quasigeostrophic framework, since all variables are self-consistent to the same degree in the model. We will see that our diagnosis indeed is capable of identifying the folding of the model tropopause as a key signature of ULFs. Furthermore, we will use the diagnosis of \(\frac{D}{Dt} \nabla \theta\) as a quantitative basis for delineating the relative importance of the different physical processes underlying the development of ULFs. The model is shown to also adequately generate mesoscale gravity waves in section 5. The role of diabatic heating in ULFs is discussed in section 6. The paper ends with brief concluding remarks in section 7.

2. Model

a. Model description

The Advanced Research version of the WRF Model (ARW) is used for making illustrative simulations of atmospheric fronts in this study. The ARW core is based on an Eulerian solver for the fully compressible nonhydrostatic equations, cast in flux (conservative) form, in terrain-following eta coordinates. Prognostic
variables for this solver are column mass of dry air $\mu$, horizontal velocities $u$ and $v$, vertical velocity $w$, potential temperature, and geopotential. Nonconserved variables (e.g., temperature, pressure, and density) are diagnosed from the conserved prognostic variables. The solver uses a third-order Runge–Kutta time-integration scheme coupled with a split-explicit second-order time-integration scheme for the acoustic and gravity wave modes. Fifth-order upwind-biased advection operators are used in the fully conservative flux divergence integration. It conserves all the fundamental properties, such as total mass, energy, and potential vorticity. The model horizontal domain is 4005 km $\times$ 8010 km depicted by $263 \times 534$ grid points so that the grid resolution is $dx = dy = 15$ km. The top of the model is at about $z = 16$ km, corresponding to about 150 hPa. The horizontal grid arrangement is Arakawa C-grid staggering. The time step is 90 s with 22.5 s for acoustic waves. The east–west boundary conditions are periodic. The north–south boundary conditions are symmetric. The bottom boundary is a flat surface through which there are momentum, sensible heat, and moisture transfers. The top boundary is a constant pressure surface. No damping is applied near the top of the domain. Full Coriolis terms are included. There are various physics packages to choose from for depicting the subgrid processes and atmospheric radiation. We make two simulations and diagnose the model output in the context of frontal dynamics. All diabatic heating processes are switched off in one simulation, except for the boundary layer dissipation. This is referred to as the “dry run.” The other simulation incorporates all diabatic heating processes and is referred to as the “control run.” The model physics schemes in this run are switched on with the use of the following parameters:

(i) mp_physics = 10 [Morrison et al. 2005 (two-moment scheme)];
(ii) cu_physics = 14 [new GFS simplified Arakawa–Schubert scheme from Yonsei University (YSU)];
(iii) bl_pbl_physics = 1 (YSU PBL scheme);
(iv) sf_sfclay_physics = 1 [MM5 Monin–Obukhov scheme (similarity theory)]; and
(v) ra_lw_physics = 4 [Rapid Radiative Transfer Model (RRTMG) longwave scheme] and ra_sw_physics = 4 (shortwave radiation).

b. Initial condition

The initial state of our model atmosphere is constructed in two steps. The first step is to construct a zonally symmetric component with a code of the 2D potential vorticity (PV) inversion used in Plougonven and Snyder (2007). The code is based on an overrelaxation method. The PV value in the model troposphere and stratosphere is assigned as 0.4 and 5.0 PV units (1 PVU $= 10^{-6}$ K kg $^{-1}$ m$^2$ s$^{-1}$), respectively. The algorithm of PV inversion is relatively slow to converge because the matrix associated with the high resolution of the model is very large. When the iterative procedure is eventually stopped, the change of the field from the last iteration is deemed acceptably small although there remain small variations in the PV field within the model troposphere and stratosphere. A built-in code with a number of adjustable parameters enables us to additionally prescribe the structure, location, and strength of a baroclinic jet with a sloping tropopause. We apply the kick-off procedure in the WRF source code designed for the specific purpose of simulating 3D baroclinic waves. When the integration starts, a thermal bubble (perturbation $\theta$ with maximum of 1 K) is automatically added at the center of domain. Its characteristic length scale is 4000, 2000, and 8 km in the $x$, $y$, and $z$ directions, respectively. We reduce the initial noisy oscillations arising from the imbalance in such an initial condition by integrating the model for two inertial periods (37 h in this study) and time averaging all fields. This procedure is repeated one more time to obtain a model-consistent balanced initial state. Time is then set to zero when we start the actual simulations. The resulting initial state of the model is referred to as the “reference state,” with an average surface pressure of 1014 hPa. The zonal-mean component of such state is shown in Fig. 1. It is characterized by a baroclinic jet with a maximum speed of 60 m s$^{-1}$ centering at about 350 hPa over the center of the meridional domain. The potential temperature field has a stable stratification

FIG. 1. Structure of the reference state in terms of a zonally uniform zonal velocity (m s$^{-1}$) and potential temperature (K); domain is 8010 km in the $y$ direction.
everywhere with much larger values above the level of the model tropopause.

The corresponding distributions of the potential vorticity and potential temperature of this reference state are shown in Fig. 2. The 1.5- and 3.0-PVU contours are highlighted. The former may be regarded as a meaningful indicator of the tropopause by convention. The initial model tropopause has realistic features: quite high in the southern portion of the domain at about 275 hPa, sharply decreasing through a middle region, and tapering off to a considerably lower level (~500 hPa) in the northern portion of the domain. The baroclinic jet is centered at the middle of the sloping tropopause. We have not fine-tuned this reference state.

3. Results

a. Overall evolution of the flow

The disturbance quickly intensifies, and it is an unstable baroclinic wave. As expected, surface fronts and upper-level fronts form in a matter of days, resulting from the nonlinear dynamics of the wave when it has become sufficiently strong. A feel for the overall development of the flow can be first gained by examining how the minimum surface pressure of the baroclinic wave evolves (Fig. 3). It is found that the evolution in the first 4.5 days in both runs is essentially identical because the diabatic processes are too weak to make a measurable impact up to this point. The minimum pressure in the dry run evolves toward a value of about 980 hPa, whereas that in the control run asymptotically approaches about 970 hPa. In other words, the surface pressure of the disturbance without diabatic heating has deepened by about 35 hPa in 6.5 days. The one with diabatic heating has deepened by about 45 hPa in 7 days. The surface pressure field on day 6.5 of the control run will be shown later (Fig. 15). We are now in a position to closely diagnose the nature of upper-level frontogenesis.

b. Structure of the model upper-level front in the dry run

Upper-level frontogenesis in our dry run can be most succinctly diagnosed from the perspective of PV, for it is largely a material conservative property. The generic definition of potential vorticity is

\[ Q = \frac{1}{\rho} (f \nabla \times \mathbf{V}) \cdot \mathbf{v}, \]

where \( \mathbf{k} \) is the vertical unit vector and the other notations have standard meanings. It should be noted that the three components in the expression of the vorticity vector written in pressure coordinate,

\[ \mathbf{V} = (u, v, w), \]

have mixed units. For example, \( v \) is in meters per second per pascal whereas \( u \) is in pascals per meter per second.

In computing such a quantity with data on pressure surfaces, it would be necessary to use a coordinate system that has the same unit in all three directions. Let us denote such vertical coordinates by \( \tilde{z} \) in meters. In these \((x, y, \tilde{z})\) coordinates, we would have

\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \tilde{z}} \right) \]

and

\[ \mathbf{V} = (u, v, \tilde{w}) \]

with \( \tilde{w} = D\tilde{z}/Dt \) in units of meters per second. An appropriate choice of \( \tilde{z} \) can be

\[ \tilde{z} = z_{\text{top}} \left( 1 - \frac{p}{g \rho_o} \right) \]

and

\[ \tilde{w} = \frac{\omega}{g \rho_o} + \frac{p}{g} \frac{D}{Dt} \left( \frac{1}{\rho_o} \right), \]

where \( z_{\text{top}} \) is the top of the model domain and \( \rho_o \) is density, which is a prescribed function of \( p \) compatible with the observed atmospheric mean density. Our choice of coordinate is compatible with a mean atmosphere.
approximately in hydrostatic balance. This is an alternative to the pseudoheight coordinate used by Hoskins and Bretherton (1972), which is compatible with a mean atmosphere with an approximately constant lapse rate. In particular, we use
\[ \rho_o = \rho_{\text{bot}} \exp \left( \frac{p - p_{\text{bot}}}{p_o} \right), \]  
(2)
where the subscript "bot" refers to the bottom surface level. We choose \( p_o \) to be a suitable constant, say \( p_o = 2p_{\text{bot}} \), which is close to a scale height of the density distribution measured in units of pressure.

The potential vorticity will be computed as
\[ Q = \frac{1}{\rho_o} \left[ (\bar{w} - v_z) \theta_x + (u_z - \bar{w}) \theta_y + (f + v_x - u_y) \theta_z \right] \]  
(3)
in units of \( \text{m}^2 \text{s}^{-1} \text{K kg}^{-1} \) or in PVU. The explicit formulas for \( \bar{w}, \bar{w}_x, \bar{w}_y, \bar{w}_z, \theta_x, u_z, \) and \( v_z \) can be readily obtained on the basis of (1). For example, one would get \( \partial \bar{w} / \partial \bar{z} = \omega \partial \rho / \partial p - \omega (p_o - p) (2 - p/p_o) \). Equations (1)–(3) are used in our computation.

We first focus on the development of an upper-level front in the dry run. The immediate task is to detect the existence of the model ULF and delineate its structure. Since it is a three-dimensional disturbance, the most concise and effective way of doing so is to construct judiciously chosen horizontal and vertical cross sections of the PV field in the model. We show in Fig. 4 the height field and PV field of the flow at two interior levels, 500 and 700 hPa, on day 6.5 of the dry run. It is seen in Fig. 4a that this synoptic midtropospheric wave is quite intense, with the lowest contour depicting a cutoff low. The range of the height field on 500 hPa is from 5100 to 5800 m. Particularly informative is the PV field, which ranges from 0.4 to 6 PVU. There is a narrow tongue of large values of PV across the midsection upstream of the trough oriented roughly from a west-northwest–east-southeast direction. Notice that the brightest red color (and hence largest PV value) appears at the southern edge of this penetrating tongue. These large PV values could not simply have arisen from horizontal advection of PV. The air parcels with greater than 1.5 PVU are of stratospheric origin and must have descent through the tropopause to this level. A corresponding tongue of relatively large though smaller PV values is also evident at the 700-hPa level (Fig. 4b). The maximum value here is about 3.5 PVU, implying that the air in this strip with this PV value is also of stratospheric origin. Taken together, these two panels reveal that an integral part of the wave development is to bring down some air parcels from certain stratospheric levels to mid- and even lower-tropospheric levels in a narrow strip of area across the upstream of the baroclinic wave trough. It is also noteworthy that the wave has already intensified to the maximum extent by this time since it has virtually no vertical tilt left. We have checked that the difference between the distributions of PV in the model troposphere obtained using our \( \tilde{z} \) coordinate and those obtained using the pseudoheight coordinate of Hoskins and Bretherton (1972) is negligible.

Next we show the vertical distributions of PV together with the potential temperature on two unbiased cross sections cutting through the upper-level cutoff low. These cross sections are the same as those in the diagnosis of Rotunno et al. (1994). One cross section is meridionally oriented passing through AA’, and the other is oriented 45° passing through BB’, indicated by dashed lines in Fig. 4. Such distributions are shown in
Fig. 5. Figure 5a clearly reveals an upper-level front, which manifests as a tongue of high PV values extending downward from the tropopause at a location about 1200 km north of the point A. It coincides with the steep sloping contours of potential temperature there. There are similar characteristics in the BB' cross section in Fig. 5b. This ULF extends down to about the 700-hPa level on the AA' cross section and to about the 600-hPa level on the BB' cross section. These two panels by themselves might misleadingly suggest that there exist three ULFs originating from different locations of the perturbed tropopause. They are, in fact, just three parts of one single ULF, which we happen to sample via the two vertical cross sections through the points AA' and BB'. Viewed in conjunction with Fig. 4, the three apparent ULFs in Fig. 5 tell us that they are three locations of one ULF lying along the southern edge of the narrow tongue of high PV. In passing, it is noteworthy that there is also a small patch of high PV values at the lower-tropospheric levels. That arises from the boundary layer turbulent heat flux.

Figure 6 shows that there is synoptic-scale descent in the region of this upper-level front on the cross sections along AA' and BB'. The structure of the vertical velocity field on the BB' cross section is particularly distinct. This descending motion reaches a maximum value of about 10 cm s$^{-1}$ at about the 570-hPa level. In passing, we also note that there is equally distinct synoptic-scale ascent on the eastern side of the trough, especially in the wrap-around region, as expected in a baroclinic wave.

We will diagnose the relative importance in the contributions of the geostrophic and ageostrophic components of the flow to the development of ULFs. So let us first take a look at a sample distribution of the geostrophic and ageostrophic velocity components of the 500-hPa flow together with the temperature field on day 6.5 of the dry run (Fig. 7). The geostrophic vectors are indeed parallel to the geopotential height contours, and the ageostrophic component is stronger where the curvature is larger. It is noteworthy that the horizontal thermal contrast associated with the ULF at this level is not nearly as pronounced as that of the surface fronts. In other words, the model ULF does not separate polar and tropical airs as first pointed out by Reed (1955). Another feature of interest is that the thermal advection by the ageostrophic velocity component is of opposite sign of that by the geostrophic velocity component.

4. Rate of frontal development

a. Frontogenetical functions

All diagnoses of fronts in the literature have focused on the evolution of the magnitude of the horizontal gradient of potential temperature $|\nabla \theta|$ as a quantitative measure of frontogenesis. This is a traditional convention based only on the observation of fronts near Earth’s surface. This bias is reflected in the AMS Glossary of Meteorology listing for “upper front”: “a front that is present in the upper air but typically does not extend to the ground” (American Meteorological Society 2012). It makes no distinction between surface fronts and ULFs from a structural and dynamical point of view. It would be relevant to emphasize that the variation of $|\nabla \theta|$ is considerably larger than that of $|\nabla \theta|$ in ULFs associated with tropopause folding. It would stand to reason that one should explore whether $(D/Dt)|\nabla \theta|$ is a necessary and sufficient metric for quantifying the development of ULFs.
One might question whether $(D/Dt)\nabla \theta$ is a suitable measure as a frontogenetical function for ULFs on the ground that a tropopause, which is characterized by a large vertical derivative of potential temperature alone, is generally not regarded as a front. It is relevant to note that a background tropopause and a ULF differ greatly in terms of their spatial and temporal scales as a result of entirely different physical processes. Two primary factors set the stage for the existence of a tropopause in middle and high latitudes: solar radiative heating at Earth’s surface and absorption of ultraviolet radiation by ozone in the stratosphere. The terrestrial radiative cooling within the troposphere and the vertical heat transports by numerous cycles of small-scale convection as well as baroclinic waves control the mean lapse rate and thereby the height of the tropopause in the extratropics. Such a resulting tropopause is essentially horizontal and has a planetary scale. In the tropics, the tropopause is distinctly higher because it largely reflects the vertical extent of the Hadley cell. The tropical mean tropopause then has a planetary scale in the zonal direction. The Hadley circulation stems from the meridional differential solar heating, longwave radiative cooling in the subtropical troposphere, and the self-induced condensational heating in the equatorial latitudes. The meridional extent of the Hadley cell in turn is dictated by the characteristics of its dynamical stability. A by-product of the Hadley circulation is a baroclinic jet just beyond the poleward side of the subtropics due to quasi conservation of angular momentum of the poleward-moving air parcels. It follows that the mean tropopause would slope steeply poleward over the subtropics and has a seasonal time scale. These factors are implicitly incorporated through a reference state of our model, as shown in Figs. 1 and 2. In contrast, the ULF is associated with
local and transient tropopause folding as a part of an
intensifying baroclinic wave. It follows that a ULF has
synoptic and subsynoptic scales in the departure field
from our reference state with a significant 3D gradient
of potential temperature. The latter is easily distin-
guishable from the largely seasonal and planetary
scales in the fluctuation of the extratropical tropo-
pause. Therefore, it would be reasonable to quantiﬁe
ULFs in terms of the 3D variation of the potential
temperature \(|\nabla \theta|\). We refer to the corresponding met-
ric \((D/Dt)|\nabla \theta|\) as the frontogogenous function in 3D
space \(F_3\).

We also use the coordinates \((x, y, z)\) elaborated in
section 3b so that all terms in \((D/Dt)|\nabla \theta|\) would have
the same unit. With the thermodynamic equation,
\((D\theta/Dt) = \partial H/\partial pT\), where \(H\) is the net heating rate
\((K\ s^{-1})\), it can be shown that the explicit form of \(F_3\)
\((K\ m^{-1}\ s^{-1})\) is

\[
\frac{D}{Dt}|\nabla \theta| = \frac{1}{|\nabla \theta|} \left( \theta \left[ \frac{1}{c_p} \left( \frac{p_0}{\rho} \right)^k \frac{\partial H}{\partial \mathbf{x}} - u_\theta \frac{\partial \theta}{\partial x} + v_\theta \frac{\partial \theta}{\partial y} - w_\theta \frac{\partial \theta}{\partial z} \right] + \right.
\]

\[
\left. + \theta \left[ \frac{1}{c_p} \left( \frac{p_0}{\rho} \right)^k \frac{\partial H}{\partial y} - u_\theta \frac{\partial \theta}{\partial x} + v_\theta \frac{\partial \theta}{\partial y} - w_\theta \frac{\partial \theta}{\partial z} \right] \right) .
\]

\hspace{1cm} (4)

It should be added that \(H\) not only includes the contri-
butions from all forms of diabatic heating processes but
also the conversion from kinetic to thermal energy due
to turbulent mixing. Furthermore, it would be in-
structive to decompose the horizontal velocity into a
geostrophic component \((\mathbf{u}, \mathbf{v})\) and an ageostrophic
component \((\bar{u}, \bar{v})\): \(u = \bar{u} + \bar{u}\) and \(v = \bar{v} + \bar{v}\). The ageo-
striphic component is expected to play a different but
important role in spite of being considerably weaker
than the geostrophic component. The inﬂuence of \(\bar{w}\) is
best viewed in conjunction with those of \((\bar{u}, \bar{v})\) as in-
tegral parts of the ageostrophic effect. We therefore
refer to \((\bar{u}, \bar{v}, \bar{w})\) as a single 3D ageostrophic velocity.

Let us introduce a 2D vector function involving the
geostrophic velocity \((\bar{u}, \bar{v})\) and \(\theta\),

\[
\mathbf{S} = (\nabla \bar{u} \cdot \nabla \theta, \nabla \bar{v} \cdot \nabla \theta),
\]

\hspace{1cm} (5)

and a 3D vector function involving the 3D ageostrophic
velocity \((\bar{u}, \bar{v}, \bar{w})\) and \(\theta\),

\[
\mathbf{A} = (\nabla \bar{u} \cdot \nabla \theta, \nabla \bar{v} \cdot \nabla \theta, \nabla \bar{w} \cdot \nabla \theta).
\]

\hspace{1cm} (6)

Then (4) can be rewritten concisely as

\[
\frac{D}{Dt} |\nabla \theta| = \frac{1}{c_p |\nabla \theta|} \nabla \theta \cdot \nabla \left( \frac{p_0}{\rho} \right)^k H
\]

\[
- \frac{1}{|\nabla \theta|} \nabla \theta \cdot \mathbf{S} - \frac{1}{|\nabla \theta|} \nabla \theta \cdot \mathbf{A}.
\]

\hspace{1cm} (7)

It should be emphasized that no physical or math-
ematical approximations have been invoked in getting (4)
and hence (7), although one would unavoidably in-
roduce some approximations when one evaluates \(H\).

Thus, there is no restriction to the validity of (7) in
principle. For convenience, we refer to (7) symbolically as

\[
F_{3T} = F_{3H} + F_{3G} + F_{3A},
\]

\hspace{1cm} (8)

where

\[
F_{3H} = \frac{1}{|\nabla \theta|} \nabla \theta \cdot \nabla \left( \frac{H}{c_p} \frac{p_0}{\rho} \right)^k,
\]

\[
F_{3G} = - \frac{1}{|\nabla \theta|} \nabla \theta \cdot \mathbf{S},
\]

\[
F_{3A} = - \frac{1}{|\nabla \theta|} \nabla \theta \cdot \mathbf{A}
\]

and \(F_{3T} \equiv F_3\).

Here \(F_{3H}\) represents the effect of diabatic heating on the
rate of frontogenesis when the latter is measured by
\((D/Dt)|\nabla \theta|\), and \(F_{3G}\) and \(F_{3A}\) represent the direct
dynamical effect of the geostrophic component and ageo-
striphic component of the flow on frontogenesis,
respectively.

We will compare the results of \(F_3\) with the counter-
parts of \((D/Dt)|\nabla \theta|\), which is referred to as the fronto-
genetical function on the \(x-y\) plane (represented with
\(F_2\)). It can be readily verified that

\[
\frac{D}{Dt} |\nabla \theta| = \frac{1}{c_p |\nabla \theta|} \nabla \theta \cdot \nabla \left( \frac{p_0}{\rho} \right)^k H
\]

\[
- \frac{1}{|\nabla \theta|} \nabla \theta \cdot \mathbf{Q} - \frac{1}{|\nabla \theta|} \nabla \theta \cdot \mathbf{B}.
\]

\hspace{1cm} (9)

Note that \(\mathbf{S}\) in (7) is reduced to the conventional \(\mathbf{Q}\)
vector:

\[
\mathbf{Q} = (\nabla \bar{u} \cdot \nabla \theta, \nabla \bar{v} \cdot \nabla \theta, \nabla \bar{w} \cdot \nabla \theta),
\]

\hspace{1cm} (10)
and $A$ in (7) is reduced to $B$:

$$B = (\nabla_2 \mathbf{u} \cdot \nabla_2 \theta, \nabla_2 \mathbf{w} \cdot \nabla_2 \theta, \nabla_2 \mathbf{v} \cdot \nabla_2 \theta).$$  

(11)

The symbolic form of (9) is

$$F_{2T} = F_{2H} + F_{2G} + F_{2A},$$  

(12)

where

$$F_{2H} = \frac{1}{|\nabla_2 \theta|} \nabla_2 \theta \left[ \frac{H C}{p} \left( \frac{p_0}{p} \right) \right], \quad F_{2G} = -\frac{1}{|\nabla_2 \theta|} \nabla_2 \theta \cdot \mathbf{Q},$$

$$F_{2A} = -\left(1/|\nabla_2 \theta|\right) \nabla \theta \cdot \mathbf{B},$$

and $F_{2T} = F_2$, with $\mathbf{Q}$ and $\mathbf{B}$ given by (10) and (11).

b. Results of the rate of frontal development

The distributions of $F_3$ and its parts on the north–south-oriented vertical cross section passing through $AA'$ on day 6.5 in the dry run are shown in Fig. 8. Note that we use different scales in the plots of $F_{3T}$, $F_{3H}$, $F_{3G}$, and $F_{3A}$ in order to bring out the detailed features in each. The values of $F_{3A}$ are so much larger than those of $F_{3H}$ and $F_{3G}$ that it is virtually equal to $F_{3T}$. We see that the large positive values of $F_{3T}$ collocate well with the ULF on this cross section. The maximum value is about $1000 \times 10^{-8}$ K s$^{-1}$ m$^{-1}$. The maximum value of $F_{3T}$ is located in a lower-tropospheric layer near the 700-hPa level and about 1200 km north of point A. Figure 8 reveals that $F_3$ is almost entirely attributable to the dynamical influence of the ageostrophic velocity. The much smaller values in the heating term of this dry run solely stem from the turbulent heat flux represented in the PBL scheme of the model. The direct influence of the geostrophic flow component is also small, although its indirect influence through the synoptic-scale transverse circulation of the intensifying baroclinic wave is significant. There is a hint of the presence of vertically propagating gravity waves at the stratospheric levels originated from the surface cold front. A closer diagnosis of this matter will be reported shortly.

The counterpart result of $F_3$ and potential temperature on the vertical cross section in a general orientation
passing through BB′ is shown in Fig. 9. It shows an even more definitive correlation between the positive values of \( F_{3T} \) with the location of the ULF. The significant positive values appear in the whole tropospheric layer from 900 hPa to the tropopause. It is also entirely attributable to the impact of the ageostrophic velocity component. There are equally conspicuous negative values in both Figs. 8 and 9. They are located near the center of the disturbance (~2000 km from point A and ~3000 km from point B). They too are associated with the impact of the ageostrophic flow component. They stand for a frontolytical effect throughout the troposphere over that location.

To see the horizontal coverage of \( F_3 \), we show in Fig. 10 its horizontal distribution together with the potential temperature at 700 hPa. Again, the locations of the large positive values of \( F_3 \) match very well with the location of the ULF almost entirely stemming from the ageostrophic velocity component. The values of \( F_{3A} \) are much larger than those of \( F_{3T} \) and \( F_{3G} \) at this level in the whole domain. In light of the results in Figs. 8–10, we may conclude that \( F_3 \) is an appropriate metric for quantifying the development of ULFs, and \( F_3 \) largely stems from \( F_{3A} \). It is of interest to note that the horizontal thermal contrast of the ULF at 700 hPa is more pronounced than that at 500 hPa (evident in a comparison of Figs. 7 and 10a) as well as at 850 hPa (the latter not shown for brevity).

As noted earlier, it would be instructive to compare the values of \( F_3 \) with the counterpart results of the traditional metric of frontogenetical function \( F_2 \) in this simulation of ULFs. Figure 11 shows that the four panels of \( F_2 \) have comparable values and are 1000 times smaller than those of \( F_{3T} \) largely stemming from \( \theta_z \gg |\nabla \theta| \). More importantly, the distribution of \( F_{2T} \) has poor correlation with the location of a ULF. Although \( F_{2G} \) has positive values and correlates fairly well with the lower part (between 650 and 800 hPa) of the ULF, \( F_{2A} \) has negative values of comparable magnitude. The values of \( F_{2A} \) cancel those of \( F_{2G} \) to a great degree, resulting in a poor correlation between the distribution of \( F_{2T} \) and the location of the ULF in Fig. 11a. We therefore conclude that \( F_2 \) is not an appropriate metric for the rate of development of a ULF.

Fig. 9. As in Fig. 8, but for the cross section BB′.
c. A closer examination of the frontogenetical function

It is instructive to rewrite the contributions of the geostrophic and ageostrophic velocity components in $F_3^G$ and $F_3^A$ in terms of their deformation properties. The stretching deformation and shearing deformation of a geostrophic velocity on the $x$–$y$ plane is $D_1 = \dot{u}_x - \dot{v}_y$ and $D_2 = \dot{u}_x + \dot{u}_y$, respectively. Then we can write

$$F_{3G} = -\frac{1}{|\nabla \theta|} \nabla \theta \cdot \mathbf{S}$$

$$= -\frac{1}{|\nabla \theta|} \left[ \theta_x \theta_y D_2 + \frac{1}{2} (\theta_x^2 - \theta_y^2) D_1 + \dot{u}_x \dot{u}_y + \dot{v}_x \dot{v}_y \right].$$  \hspace{1cm} (13)

Since hydrostatic balance holds very well as an approximation in the flow associated with the fronts, the thermal wind relation would apply with a high degree of accuracy and we would have $(\dot{u}_x \theta_x + \dot{v}_y \theta_y) \approx 0$. Thus, $F_{3G}$ entirely stems from the deformation property of the geostrophic velocity component $(\dot{u}, \dot{v})$. Recall that $F_{3G}$ only accounts for a small part of $F_3$.

In a similar consideration of $F_{3A}$, we define the stretching and shearing deformation of $(\dot{u}, \dot{v}, \dot{w})$ on the $x$–$y$ plane as $I_1 = \dot{u}_x - \dot{v}_y$ and $I_2 = \dot{u}_x + \dot{v}_y$, respectively; on the $x$–$z$ plane as $J_1 = \dot{u}_z - \dot{w}_z$ and $J_2 = \dot{u}_z + \dot{w}_z$; and on the $y$–$z$ plane as $K_1 = \dot{v}_z - \dot{w}_z$ and $K_2 = \dot{v}_z + \dot{w}_z$. Then $F_{3A}$ can be rewritten as

$$F_{3A} = -\frac{1}{|\nabla \theta|} \nabla \theta \cdot \mathbf{A} = M + N,$$  \hspace{1cm} (14)

where

$$M = -\frac{1}{|\nabla \theta|} \left( \theta_x \theta_y I_2 + \theta_y \theta_z J_2 + \theta_z \theta_x K_2 + \theta_x^2 I_1 + \theta_y^2 J_1 \right)$$

and

$$N = -|\nabla \theta| \dot{w}_z.$$

The computation reveals that the values of $M$ are considerably smaller than those of $N$ in the main part of the ULF and that the distribution of $N$ approximates well that of $F_{3A}$ (figures not shown for brevity). But $M$ is not entirely negligible. In other words, the joint influence of the vortex-stretching process and the deformation property of the 3D ageostrophic flow component are greatly...
responsible for the development of ULFs. To avoid misunderstanding, we should stress that the geostrophic component of the flow makes an important indirect and indispensable contribution through its central role in the intensification of the unstable baroclinic wave itself.

d. Transverse circulation

We next examine the projection of the three-dimensional ageostrophic velocity on the meridional vertical cross section $AA'$. The structure of such a $(v, w)$ field in Fig. 12 gives us a broad perspective of the transverse circulation. The figure reveals that this projected ageostrophic circulation has a cellular form spanning the whole troposphere. Its descending branch brings down stratospheric air to form the ULF, as remarked earlier.

5. Internal gravity waves in the model simulation

We have drawn attention to the relatively small-scale features in the distributions of $F_3$ and $F_2$ that extend through the troposphere into the stratosphere in the model (e.g., Fig. 11). These features suggest the existence of mesoscale gravity waves. It is not surprising to find such features. After all, other similar models of jet–wave–front systems with sufficiently fine resolution (e.g., Zhang 2004; Plougonven and Snyder 2007) had simulated different modes of this class of waves. Nevertheless, it calls for a closer diagnosis in order to verify that those small features are indeed gravity waves. We do so by examining the horizontal and vertical distributions of the horizontal divergence field. Figure 13 shows the horizontal distribution of that quantity together with the horizontal velocity field at the 500-hPa level on day 6.5 of the dry run. There is an extensive horizontal wavy pattern of convergence and divergence in the eastern half of the baroclinic wave trough. This clearly indicates mesoscale gravity wave motions in the midst of a synoptic-scale wave. Furthermore, there is also a large area of negative values slightly to the west of the trough in Fig. 13. It is an area of horizontal convergence and hence corresponds to

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**Fig. 11.** Vertical distributions of (a) $F_{2T}$ and its three components (b) $F_{2H}$, (c) $F_{2G}$, and (d) $F_{2A}$ (shading) passing through $AA'$ ($10^{-8}$ K s$^{-1}$ m$^{-1}$) and potential temperature (contours) on day 6.5 of the dry run.
the location of the ULF where there is synoptic-scale descent on the cross section BB‘ discussed earlier (Fig. 6b).

Figure 14 shows the vertical distribution of the horizontal divergence and the potential temperature on a cross section passing through AA‘ on day 6.5 of the dry run. The overall pattern can be viewed as a superposition of several upward-propagating trains of mesoscale gravity waves. In addition, there is a distinct column of negative values collocated with the ULF. It indicates a layer of horizontal convergence associated with the descending motion seen in Fig. 6a. Figures 13 and 14 together convincingly depict the presence of mesoscale gravity waves in our model.

6. Influence of diabatic heating on ULFs

The diabatic heating in the control run is mainly due to condensational heating because the radiative time scale is relatively long. The initial distribution of the mixing ratio is zonally uniform with a surface value of 16 g kg⁻¹ at the southern boundary decreasing northward to 1 g kg⁻¹ in the domain. Vertically it is mostly confined in the lower troposphere. While there is considerable precipitation along a surface cold front on day 6.5, particularly abundant precipitation falls in the part of a warm front region near the center of the disturbance. Figure 15 shows the model 6-h precipitation rate on day 6.5 in the control run together with the surface pressure field. The maximum value reaches 22 mm, which is a significant rate.

Not surprisingly, a similar ULF develops in the model simulation with diabatic heating. Although we have made a complete counterpart diagnosis, it would suffice to present a small subset of those figures. Figure 16 shows the PV field on day 6.5 in the control run. It is to be compared with Fig. 4 for the counterpart in the dry run. The baroclinic wave naturally intensifies more strongly in the control run, and, consequently, the related fronts are also stronger. We see that the baroclinic wave trough is deeper and the tongue of high PV values penetrates much farther south than in the dry run.

FIG. 12. Distribution of the ageostrophic circulation projected on the AA’ cross section (u, w), together with the potential temperature on day 6.5 of the dry run; magnitude of w is increased by 1000 times only for plotting of vectors.

FIG. 13. Distributions of the horizontal divergence (shading; 10⁻⁵ s⁻¹) and the horizontal velocity (vectors; m s⁻¹) at 500 hPa on day 6.5 of the dry run.

FIG. 14. Distributions of the horizontal divergence (shading; 10⁻⁵ s⁻¹) and potential temperature (contours) on the cross section passing through AA’ on day 6.5 of the dry run.
The counterpart results of Fig. 5 for the vertical distributions of PV are shown in Fig. 17. It clearly depicts the parts of the ULF on the AA’ and BB’ cross sections. The ULF lies along the leading edge of the narrow tongue of high PV values. Therefore, this ULF is more highly curvilinear. In addition, there is a more pronounced layer of high PV values in the lower troposphere near the center of the disturbance about 2000 km from point A as well as about 3000 km from point B. Diabatic heating, including the impact of turbulent mixing there, generates such high PV.

The counterpart $F_3$ values on the cross section passing through BB’ are shown in Fig. 18. The large values are collocated nicely with the presence of the ULF. We again use different scales for the plots of $F_{3T}$, $F_{3H}$, $F_{3G}$, and $F_{3A}$ in order to bring out the detailed features in each. The plots confirm that $F_3$ is an appropriate measure of the rate of frontal development of a ULF in general. It is noteworthy that the diabatic heating does not significantly contribute to $F_3$, because the heating mostly takes place in the lower half of the troposphere far to the east of the ULF. But the heating does significantly affect the ULF indirectly through its strong impact upon the ageostrophic velocity field of the intensifying baroclinic wave. The condensational heating also gives rise to stronger small-scale features to the east of the ULF.

7. Concluding remarks

An upper-level front is a disturbance in the upper troposphere characterized by developing a local large value of $|\nabla \theta|$ with both significant horizontal and vertical components. As a part of tropopause folding, it separates air of stratospheric origin and air of tropospheric origin. We adopt large $|\nabla \theta|$ as a definition for an upper-level front in this analysis and $F_3 = (D/Dt)|\nabla \theta|$ as the corresponding frontogenetical function. This definition is more general than that stated in the AMS Glossary of Meteorology. The term $|\nabla \theta|$ would be reduced to $|\nabla_2 \theta|$ as a special case by simply dropping $\theta$ from it. The corresponding $(D/Dt)|\nabla_2 \theta|$ would be a sufficient metric for diagnosing surface frontogenesis.

The structural properties of our model ULF simulated with diabatic heating versus that without and the different aspects of the distributions of the 3D frontogenetical function $F_3$ are summarized in the abstract. The model ULF has a curvilinear, relatively thin structure on the order of 1000 km in length extending down to different levels of the troposphere. The structural characteristics of the model ULF are compatible with
observations. The horizontal divergence field also reveals that a spectrum of mesoscale gravity waves is generated in the model jet–wave–front system. We have particularly demonstrated that $F_3$ is a necessary and sufficient metric for quantifying the development of ULFs. As such, the structural properties and the dynamics of ULFs are distinctly different from those of surface fronts. Vilhelm Bjerknes was said to have used

Fig. 17. Distributions of PV (shading; PVU) and potential temperature (contours) on the vertical cross section passing through (a) AA’ and (b) BB’ on day 6.5 of the control run.

FIG. 18. As in Fig. 8, but for cross section BB’ of the control run.
“wrinkles in the weather’s face” as a metaphor in his discussion of surface fronts (Jewell 1994). In the same spirit and for the reasons elaborated above, we think of the upper-level fronts metaphorically as “veils in the sky.” The detailed structure of a particular “veil” depends on the intensity of the specific baroclinic wave under consideration.

Acknowledgments. We wish to thank Dr. R. Plougonven for providing us with the code for performing PV inversion used in constructing the initial state. The extensive comments from the three reviewers are helpful and much appreciated. Yi Lu is supported by the NASA Earth and Space Science Fellowship under Grant NNX14AK77H, and Yi Deng is supported by the National Science Foundation under Grants AGS-1147601, AGS-1354402, and AGS-1445956. This project is simply a labor of love for MM.

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