An Improved Small-Angle Approximation for Forward Scattering and Its Use in a Fast Two-Component Radiative Transfer Method

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ABSTRACT

The vector radiative transfer equation is decomposed into two components: a forward component and a diffuse component. The forward component is analytically solved with a small-angle approximation. The solution of the forward component becomes the source for the diffuse component. In the present study, the diffuse component is solved using the successive order of scattering method. The strong anisotropy of the scattering of radiation by a medium is confined to the forward component for which a semianalytical solution is given; consequently, the diffuse component slowly varies as a function of scattering angle once the forward-scattering peak is removed. Moreover, the effect on the diffuse component induced by the forward component can be interpreted by including the low orders of the generalized spherical function expansion of the forward component or even replaced by the Dirac delta function. As a result, the computational effort can be significantly reduced. The present two-component method is validated using the benchmarks related to predefined aerosol and cloud layers with a totally absorbing underlying surface. As a canonical application, the optical properties of water clouds and ice clouds used for the Moderate Resolution Imaging Spectroradiometer (MODIS) Collection 6 cloud-property retrieval products are used for radiative transfer simulations under cloudy conditions.

1. Introduction

Radiative transfer is a sine qua non for ocean optics, the remote sensing of the atmosphere, and radiative forcing analyses involved in climate science. The radiative transfer equation has been extensively discussed (Chandrasekhar 1960; Liou 2002; Mobley 1994; Preisendorfer 1965; van de Hulst 1980). The vector radiative transfer equation (RTE) is an integral–differential equation that can be derived from statistical electromagnetics (Mishchenko 2002). Even though there is no analytical solution for the RTE, there are many numerically accurate methods to solve it, such as the adding–doubling method (de Haan et al. 1987; van de Hulst 1980), the matrix operator method (Grant and Hunt 1969a,b; Kattawar et al. 1973; Plass et al. 1973; Tanaka and Nakajima 1977), the Monte Carlo method (Marshak and Davis 2005; Plass and Kattawar 1968; Tynes et al. 2001; Zhai et al. 2008), the successive order of scattering method (Lenoble et al. 2007; Min and Duan 2004; Myneni et al. 1987; Zhai et al. 2009), the discrete-ordinate method (Liou 1973; Schulz et al. 1999; Stamnes et al. 1988, 2000), the spherical harmonics method (Garcia and Siewert 1986; Karp et al. 1980; Muldashev et al. 1999), the spherical harmonics discrete-ordinate method (Evans 1998), and the invariant imbedding method (Adams and Kattawar 1970; Chandrasekhar 1960; Liou 2002). Comprehensive discussions about the various computational procedures can be found in the literature (e.g., Karp et al. 1980; Lenoble 1985). In addition, numerous benchmarks have been reported (e.g., Coulson et al. 1960; Emde et al. 2015; Kokhanovsky et al. 2010; Natraj et al. 2009). Moreover, the solution to the RTE has also been

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comprehensively studied in the optically thick region—that is, the asymptotic regime (de Rooij 1985; Domke 1974; Kattawar and Plass 1976; King and Harshvardhan 1986; Sun et al. 2016; van de Hulst 1968, 1970).

Numerically accurate simulation of the radiative transfer process in a strongly forward-scattering medium requires a great deal of computer CPU time. To reduce the computational effort, approximate methods are necessary to obtain the solution of the RTE. Many techniques have been developed to deal with this computational challenge, including scaling the optical thickness by multiplying with \( \log \) (where \( \omega \) is the single-scattering albedo and \( g \) is the asymmetry factor of the corresponding phase function) in conjunction with replacing the original phase function in terms of an isotropic counterpart as introduced by van de Hulst and Grossman (1968), the use of the two-stream and four-stream approximations (Coakley and Chylek 1975; Liou 1974; Liou et al. 1988; Meador and Weaver 1980), the delta-function approximation (Potter 1970), the \( \delta \)-Eddington approximation (Joseph et al. 1976; Wiscombe 1977b), the \( \delta \)-M approximation (Wiscombe 1977a), the \( \delta \)-fit approximation (Hu et al. 2000), and the comparisons of different truncation techniques (e.g., Rozanov and Lyapustin 2010). The invariance of the scaling of the optical thickness and the phase function for the RTE has been reviewed by McKellar and Box (1981).

The gist of the above-mentioned delta approximations or the scaling approaches stems essentially from a simple two-component method, where the Dirac delta function \( \delta \) is employed to replace the forward-scattering peak, leading to the following approximate relationships:

\[
P \approx 2f\delta(1 - \chi) + (1 - f)P_s, \quad (1.1a)
\]

\[
\tau_s = \tau(1 - \omega f), \quad \text{and} \quad (1.1b)
\]

\[
\omega_s = \frac{1 - f}{1 - \omega f} \omega, \quad (1.1c)
\]

where \( P \), \( \tau \), and \( \omega \) are the original phase function, optical thickness, and single-scattering albedo, respectively; the subscript \( s \) represents the scaled quantities; \( f \) is the forward fraction; and \( \chi \) is the cosine value of the scattering angle—that is, the angle between the incident and the observed directions. The forward fraction in Eqs. (1a)–(1c) is restricted to keep a certain degree of accuracy while computational efficiency and accuracy are balanced.

The aforementioned simple two-component method can be improved by using a small-angle approximation (SAA) for the forward component and a numerically accurate method for the scaled component. SAA involves the computation of an angular distribution function \( F(\theta, \varphi) \), where \( \theta \) and \( \varphi \) are the zenith and azimuthal angles, respectively. In the approximation, \( \cos \theta \) is replaced by 1 and \( \sin \theta \) by \( \theta \) since the initial direction is assumed to be along the \( z \) axis of the relevant coordinate system for simplicity; moreover, the integral limits for the angles are extended from finite numbers to infinity based on the fact that the distribution function shall dramatically drop to zero for large angles; accordingly, the solution of the distribution function can be expressed as a polynomial summation or an integral of some special functions in terms of the Fourier transform. This approach has been extensively applied to the multiple scattering of charged particles and radiative transfer (Budak and Ilyushin 2010; Dolin 1966; Goudsmit and Sauderson 1940; Hirleman 1991; Irvine 1968; Ishimaru 1978; Kokhanovsky 1997; Kuga et al. 1986; Romanova 1962, 1963; Scott 1963; Sobolev 1975; Wang and Guth 1951; Yang 1951) and section 4.6 in Lenoble (1985). The forward peak is approximated in terms of the summation of Gaussian functions and the small-angle approximation is realized by the successive order of scattering method and the Hankel transform technique (Nakajima and Tanaka 1988; Nakajima et al. 1983; Weinman et al. 1975). However, a convergent summation or integral computation becomes time consuming for multiple scattering with a strong forward peak.

Another semianalytical SAA considers the cosine approximation of the zenith angle to high orders instead of 1; that is, the pathlength distribution is considered. Under this SAA, the integral–differential radiative transfer equation can be reduced to a differential equation, as shown in section 8.7 of Sobolev (1975). An analytical angular-spatial solution can be obtained for normal incidence (Dolin 1983; Remizovich et al. 1982) and for oblique incidence (Remizovich and Shekhmamet’ev 1990; Zege et al. 1987; Zege and Polonsky 1993). Note that detailed discussions in both the cases can be found in chapter 4 of Zege et al. (1991). The combination of the semianalytical SAA for the forward component and the adding–doubling method for the diffuse component, which is called the multicomponent method, is developed and employed to the atmosphere–ocean and the atmosphere–Earth systems (Chaikovskaya et al. 1999; Tynes et al. 2001; Zege and Chaikovskaya 1996, 2000). For the diffuse component, the source and the forward-scattering matrix are replaced by Dirac delta function, which can cause noticeable errors for a medium with strong scattering in the forward directions. In this study, a combination of the semianalytical SAA for the forward component and the successive order of scattering method
for the diffuse component is employed to give a highly accurate and fast solution to the vector RTE. Single scattering in the forward component is replaced by the analytical solution for numerical accuracy (Nakajima and Tanaka 1988; Zege et al. 1988).

A detailed derivation for the two-component radiative transfer method and the numerical realization are given in section 2. The validation of the present method against benchmarks and its numerical implementation in conjunction with the use of the MODIS Collection 6 cloud optical properties are given in section 3. The discussions of this study are given in section 4. The summary is given in section 5. Appendix A is a comprehensive derivation for the semianalytical small-angle approximation and the corresponding reductions. Appendix B presents the approximate expression of the Legendre polynomial for small angles.

2. Two-component radiative transfer method

The Stokes vector for polarized light in a plane-parallel scattering medium obeys the vector RTE:

$$\left( \frac{\partial}{\partial \tau} - 1 \right) \mathbf{I}(\tau, \mu, \varphi) = \frac{\omega(\tau)}{4\pi} \int \mathbf{P}(\tau, \mu', \varphi - \varphi') \mathbf{I}(\tau, \mu, \varphi) \, d\mu' \, d\varphi', $$

(2.1)

with the boundary conditions

$$\mathbf{I}(\tau = 0, \mu, \varphi, \varphi_0) = \delta(\mu - \mu_0) \delta(\varphi - \varphi_0) \mathbf{E}_0, \quad \mu \in [-1, 0] \quad \text{and} \quad \mathbf{I}(\tau = 0, \mu, \varphi, \varphi_0) = 0, \quad \mu \in [0, 1]. $$

(2.2a)

(2.2b)

The symbols in this equation are defined as follows:
- \( \mathbf{I} = (I \quad Q \quad U \quad V)^T \) is the Stokes vector, where the superscript \( T \) represents the transpose of a matrix or a vector, and the definition of the four elements, which are defined with respect to a meridian plane, can be found in Chandrasekhar (1960) and Mishchenko et al. (2006);
- \( \mathbf{E}_0 = (F_0 \quad 0 \quad 0 \quad 0)^T \), where \( F_0 \) is flux density (i.e., flux through a unit area perpendicular to the beam);
- \( \tau \) indicates the optical thickness; the optical thickness at the upper boundary is set to 0 and the optical thickness at the lower boundary is set to \( \tau_0 \);
- \( \mu = \cos \theta \) and \( \theta \) and \( \varphi \) are the zenith and azimuthal angles, respectively; the zenith angle is defined with respect to the upward zenith direction so that the zenith angle of an incoming solar radiation beam is larger than 90° and the zenith angles for the reflected light and for the transmitted light are 0°–90° and 90°–180°, respectively; without loss of generality, the solar azimuthal angle will be 0°;
- \( \omega \) is the single-scattering albedo;
- the phase matrix \( \mathbf{P} \) is defined in terms of the rotational matrix \( \mathbf{L} \) between the relevant meridian plane and the scattering plane and the scattering matrix \( \mathbf{F} \) of the medium:

$$\mathbf{P}(\tau, \mu, \mu', \varphi - \varphi') = \mathbf{L}(\pi - \xi_2) \mathbf{F}(\tau, \chi) \mathbf{F}(-\xi_1)$$

(2.3)

$$\mathbf{F} = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_4 \end{pmatrix}, $$

(2.4)

where \( \xi_1 \) and \( \xi_2 \) are the rotation angles between the corresponding scattering plane and the two meridional planes containing the incident and scattered beams and their definitions are consistent with those in Chandrasekhar (1960): \( \chi = \cos \Theta \), where \( \Theta \) is the scattering angle, and \( \chi = \mu \mu' + \sqrt{(1 - \mu^2)(1 - \mu'^2)} \cos(\varphi - \varphi') \); the arguments \( \chi \) and \( \tau \) in the scattering matrix \( \mathbf{F} \) have been suppressed.

In the present formalism, the \( \tau \) dependence of all quantities is suppressed for brevity unless necessary clarification is required.

The scattering matrix \( \mathbf{F} \) in the exact forward direction (i.e., \( \chi = 1 \)) can be written as (van de Hulst 1981)

$$\mathbf{F}(1) = \text{diag}[a_1(1), a_2(1), a_3(1), a_4(1)].$$

(2.5)

Accordingly, the scattering matrix \( \mathbf{F} \) in the forward direction or at small scattering angles is quasi diagonal, with small values for the off-diagonal elements relative to the diagonal elements (Zege and Chaikovskaya 2000).

a. Two-component method to solve the vector RTE

The phase function can be decomposed into a forward and a diffuse component:

$$a_i(\chi) = A_i a_i^f(\chi) + (1 - A_i)a_i^d(\chi),$$

(2.6)

where...
\[
\frac{1}{2} \int_{-1}^{1} \tilde{a}_i^d(\chi) \, d\chi = 1, \tag{2.7a}
\]

\[
a_i^d(\chi) = \frac{1}{A_i} \left\{ a_i(\chi) - a_i(\chi_c) \exp\left[ - (\chi - \chi_c)/n_{ph}(1 - \chi_c) \right] \right\}, \quad \chi \geq \chi_c, \tag{2.7b}
\]
\[
\chi < \chi_c, \quad \text{and}
\]
\[
a_i^d = \frac{1}{1 - A_i} \left\{ a_i(\chi) \right\}, \quad \chi \geq \chi_c, \tag{2.7c}
\]
\[
A_i = \frac{1}{2} \int_{\chi_c}^{1} \left\{ a_i(\chi) - a_i(\chi_c) \exp\left[ - (\chi - \chi_c)/n_{ph}(1 - \chi_c) \right] \right\} \, d\chi, \tag{2.7d}
\]

where \( \chi_c = \cos \Theta_c \) and \( \Theta_c \) is the truncation angle; the forward and the diffuse phase functions \( a_i^{f,d} \) are continuous at the truncation angle in Eqs. (2.7b) and (2.7c); and \( n_{ph} \) is a user-defined real positive number and the phase function definition when \( \chi \geq \chi_c \) can ensure that the derivative of \( a_i^d \) at the truncation angle is continuous, which has an advantage for the truncation of the phase matrix expansion based on the generalized spherical function (GSF). Correspondingly,

\[
F(\chi) = A_i F'(\chi) + (1 - A_i) F^d(\chi), \tag{2.8}
\]

\[
F^{fd}(\chi) = \frac{a_i^{fd}(\chi)}{a_i(\chi)} F(\chi), \quad \text{and} \tag{2.9}
\]

\[
P(\mu, \mu', \varphi - \varphi') = A_i P'(\mu, \mu', \varphi - \varphi') + (1 - A_i) P^d(\mu, \mu', \varphi - \varphi'). \tag{2.10}
\]

The definitions of the forward and the diffuse phase matrices \( P^{f,d} \) are the same as Eq. (2.3). Likewise, the Stokes vector can be decomposed into two components:

\[
I'(\mu, \varphi) = I'^f(\mu, \varphi) + I'^d(\mu, \varphi). \tag{2.11}
\]

Similarly, the original vector RTE can be decomposed into two vector RTEs based on Eqs. (2.8)–(2.11):

\[
\left( \frac{\partial}{\partial \tau} - 1 \right) I'(\mu, \varphi)
\]

\[
= -\frac{\omega}{4\pi} A_i \int P'(\mu, \mu', \varphi - \varphi') I'(\mu', \varphi') \, d\mu' \, d\varphi' \quad \text{and} \tag{2.12}
\]

\[
I'(\tau = 0, \mu, \mu_0, \varphi) = \delta(\mu - \mu_0) \delta(\varphi) E_0, \quad \mu \in [-1, 0], \tag{2.14a}
\]

\[
I'(\tau = \tau_0, \mu, \mu_0, \varphi) = 0, \quad \mu \in [0, 1], \tag{2.14b}
\]

\[
I'(\tau = 0, \mu, \mu_0, \varphi) = 0, \quad \mu \in [-1, 0], \quad \text{and} \tag{2.15a}
\]

\[
I'(\tau = \tau_0, \mu, \mu_0, \varphi) = 0, \quad \mu \in [0, 1]. \tag{2.15b}
\]

Equation (2.12) is referred to as the forward vector RTE while Eq. (2.13) is the diffuse vector RTE. The solution of the forward vector RTE becomes the source of the diffuse vector RTE in the last term of Eq. (2.13). The Stokes vector \( I' \) in the last term of Eq. (2.13) is referred to as the forward source in the diffuse vector RTE, and the phase matrix \( P^d \) included in \( P \) of Eq. (2.13) is referred to as the forward scattering in the diffuse component.

b. Small-angle approximation for the forward component

The forward-scattering matrix \( F^f(\chi) \) can be further decomposed into the leading (L) and nonleading (NL) matrices (Zege and Chaikovskaya 2000):

\[
F^f(\chi) = F^f_L(\chi) + F^f_N(\chi), \tag{2.16}
\]

with
where \( \mathbf{E} \) is a 4 \( \times \) 4 unit matrix. Consequently, the forward-phase matrix and Stokes vector can be further decomposed into the leading and nonleading components:

\[
P_f = P_L + P_{NL}, \quad I_f = I_L + I_{NL},
\]

and the forward vector RTE can be expressed in terms of the leading and nonleading RTEs:

\[
\begin{align*}
\left( \mu \frac{\partial}{\partial t} - 1 \right) I_L^f (\mu, \varphi) &= \frac{\omega A_1}{4 \pi} \int P_f^l (\mu, \mu', \varphi - \varphi') I_L^f (\mu', \varphi') \, d \mu' \, d \varphi' \quad \text{and} \\
\left( \mu \frac{\partial}{\partial t} - 1 \right) I_{NL}^f (\mu, \varphi) &= \frac{\omega A_1}{4 \pi} \int P_{NL}^l (\mu, \mu', \varphi - \varphi') I_{NL}^f (\mu', \varphi') \, d \mu' \, d \varphi'
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
I_L^f (\tau = 0, \mu, \mu_0, \varphi) &= \delta (\mu - \mu_0) \delta (\varphi) E_0, \quad \mu \in [-1, 0], \\
I_L^f (\tau = \tau_0, \mu, \mu_0, \varphi) &= 0, \quad \mu \in [0, 1], \\
I_{NL}^f (\tau = 0, \mu, \varphi) &= 0, \quad \mu \in [-1, 0], \quad \text{and} \\
I_{NL}^f (\tau = \tau_0, \mu, \varphi) &= 0, \quad \mu \in [0, 1].
\end{align*}
\]

The Stokes vector and the phase matrix are referenced with respect to the meridian plane in the vector RTE. In Eq. (2.20), they can be expressed with respect to the scattering plane for convenience in the form of

\[
P_L^f (\mu, \mu', \varphi - \varphi') I_L^f (\mu', \mu_0, \varphi') = \mathbf{L} (\sigma - \chi_b) a_l^f I_{NL}^f (\mu', \mu_0, \varphi') \quad \text{and} \\
I_L^f (\mu, \varphi) = \mathbf{L} (\sigma - \chi_b) I_{NL}^f (\mu', \mu_0, \varphi')
\]

The dependence on the solar incident direction of the Stokes vector is explicitly given in the equations and the subscript \( s \) denotes that the Stokes vector elements are specified with respect to the scattering plane; \( \zeta_b \) represents the rotation angle from the scattering plane spanned by the solar direction \( (\mu_0, 0) \) and the viewing direction \( (\mu, \varphi) \) to the meridian plane containing the viewing direction \( (\mu, \varphi) \). Equation (2.20) can thus be reduced to a scalar RTE equation:

\[
\left( \mu \frac{\partial}{\partial t} - 1 \right) I_L^f (\mu, \mu_0, \varphi) = -\frac{\omega A_1}{4 \pi} \int a_l^f (\mu, \mu', \varphi - \varphi') I_L^f (\mu', \mu_0, \varphi') \, d \mu' \, d \varphi',
\]

with the boundary condition

\[
I_L^f (\tau = 0, \mu, \mu_0, \varphi) = \delta (\mu - \mu_0) \delta (\varphi) F_0, \quad \mu \in [-1, 0] \quad \text{and} \\
I_L^f (\tau = \tau_0, \mu, \mu_0, \varphi) = 0, \quad \mu \in [0, 1].
\]

The last three elements of the Stokes vector are zero based on the boundary conditions. In terms of the small-angle approximation, where the details are given in the appendix A, an approximate analytical solution for radiance can be obtained as

\[
I_L^f = \frac{F}{\sqrt{V_{n_x} V_{n_y}}} \exp \left( -\frac{n_y^2}{2 V_{n_x}^2} - \frac{n_z^2}{2 V_{n_y}^2} \right)
\]

where

\[
F = \exp \left[ (1 - \omega A_1) \tau_{\text{eff}} - \gamma_b / 2 \right] \frac{\cos \theta_b F_0}{\cosh (s_{\text{sa}} \tau_{\text{eff}})}
\]

\[
V_{n_x} = \frac{s_{\text{sa}}}{P_{\text{sa}}} \sqrt{(1 - e^{-2}) (1 + \sin^2 \theta_b \sin^2 \theta_b) - 4 \gamma \sin^2 \theta_b / \cos^2 \theta_b}
\]

\[
V_{n_y} = \frac{s_{\text{sa}}}{P_{\text{sa}}} \tanh (s_{\text{sa}} \tau_{\text{eff}})
\]

\[
n_y = \sin \theta \cos \phi \cos \theta_b - \cos \theta \sin \phi, \quad n_y = \sin \theta \sin \phi, \quad n_z = \cos \theta \cos \phi
\]
In the quantities $\theta_b$ and $\gamma_b$, the subscript $b$ represents the quantities at the lower boundary. The $\theta_0$ and $\theta_b$ represent the incident direction and the maximum radiance direction at the lower boundary. The variable $\gamma$ represents the transition between the incident angle and the maximum radiance angle at a certain optical thickness. As for Eq. (2.31), a numerical solution for $\gamma_b$ can be obtained for a known $\tau_0$. Once $\gamma_b$ is known, all the other variables can be determined in a straightforward manner. The variances $V_{nx}$ and $V_{ny}$ are small values so that the radiation is confined to small scattering angles relative to the direction ($\mu_b = \cos \theta_b$, 0). The radiation flux at optical thickness $\tau_0$ can be obtained in the form of the following integral:

$$F\tau_0 = \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin \theta d\theta \cos \theta I_{L,s}^0 = \frac{\mu_0 F_0 \exp[-(1-\omega A_1)\tau_0 - \gamma_b/2]}{\cosh(\tau_{saa} \tau_0)},$$

where the approximation $\cos \theta \approx \cos \theta_b$ is used in Eq. (2.33) and

$$\sin \theta d\theta d\varphi = (1-n_x^b - n_y^b)^{-1/2} dn_x dn_y \approx dn_x dn_y.$$

Even though the integral limits for the integral elements $dn_x$ and $dn_y$ should satisfy $n_x^b + n_y^b \leq 1$, the integral limits can be extended to infinity without introducing significant errors as a dominant contribution to the integral corresponds to the small values of $n_x^b$ and $n_y^b$. This extension with respect to the integral limits has been extensively employed in the small-angle approximations (Ishimaru 1978; Remizovich and Shekhamet’ev 1990; Scott 1963; Zege et al. 1991). The Stokes vector of the leading component for the forward vector RTE can be written as

$$I_L = (I_{L,s}^0 \ 0 \ 0)^T.$$

The elements of the nonleading Stokes vector in Eq. (2.21) are small compared to the leading component (Zege and Chaikovskaya 2000) so that only single scattering needs to be taken into consideration.

c. Successive order of scattering for the diffuse component

The forward and diffuse components of Stokes vector and the phase matrix can be expanded in terms of a Fourier series of the azimuthal angle (de Haan et al. 1987; Hovenier and van der Mee 1983; Kuščer and Ribarić 1959; Lenoble et al. 2007; Min and Duan 2004; Zhai et al. 2009):

$$I(\mu, \varphi) = \sum_{m=0}^{\infty} (2-\delta_{m0})[\cos(m\varphi) I_{\text{cos}}^m(\mu) + \sin(m\varphi) I_{\text{sin}}^m(\mu)],$$

where

$$I_{\text{cos}}^m = (I^m Q^m 0 0)^T,$$

$$I_{\text{sin}}^m = (0 0 I^m V^m)^T,$$

and

$$P(\mu, \mu', \varphi - \varphi') = \sum_{m=0}^{\infty} (2-\delta_{m0}) [\cos(m(\varphi - \varphi')) P_{\text{cos}}^m(\mu, \mu') + \sin(m(\varphi - \varphi')) P_{\text{sin}}^m(\mu, \mu')],$$

where

$$P_{\text{cos}}^m = \begin{pmatrix} P_{11}^m & P_{12}^m & 0 & 0 \\ P_{21}^m & P_{22}^m & 0 & 0 \\ 0 & 0 & P_{33}^m & P_{34}^m \\ 0 & 0 & P_{43}^m & P_{44}^m \end{pmatrix}$$

and

$$P_{\text{sin}}^m = \begin{pmatrix} 0 & 0 & P_{13}^m & P_{14}^m \\ 0 & 0 & P_{23}^m & P_{24}^m \\ P_{31}^m & P_{32}^m & 0 & 0 \\ P_{41}^m & P_{42}^m & 0 & 0 \end{pmatrix}.$$
As a source of the diffuse vector RTE, $\mathbf{I}' \approx \mathbf{I}_L$ has been employed in Eq. (2.40) and the Fourier series $I_{L,m}^{m'}$ will be given in the following section. Moreover, $\mathbf{I}_{cos}^{m} = \mathbf{I}_{cos}^{m} + \mathbf{I}_{sin}^{m}$ and $\mathbf{P}^{(d),m} = \mathbf{P}^{(d),m} + \mathbf{P}^{(s),m} \Delta$, where $\Delta = \text{diag}(1 \ 1 \ -1 \ -1)$ and the $\mathbf{P}^{(d),m}$ can be concisely expressed in terms of the generalized spherical functions [e.g., Eqs. (66)–(82) in de Haan et al. (1987)].

In the successive order of scattering (SOS) method, the Stokes vector can be interpreted as a sum of contributions according to the scattering order (Lenoble et al. 2007; Min and Duan 2004; Zhai et al. 2009):

$$I_{n}^{m}(\mu) = \sum_{n=1}^{\infty} I_{n}^{m}(\mu),$$  \hspace{1cm} (2.41)

with

$$I_{n}^{m}(\tau, \mu < 0) = I_{n}^{m}(\tau = 0, \mu)e^{r/\mu}$$

$$- \int_{0}^{\tau} e^{(r' - \tau)/\mu} S_{n}^{m}(\tau', \mu) \, d\tau' / \mu$$  \hspace{1cm} (2.42a)

and

$$I_{n}^{m}(\tau, \mu > 0) = I_{n}^{m}(\tau = 0, \mu)e^{(\tau_0 - \tau)/\mu}$$

$$- \int_{0}^{\tau} e^{(r' - \tau)/\mu} S_{n}^{m}(\tau', \mu) \, d\tau' / \mu,$$  \hspace{1cm} (2.42b)

where

$$S_{n}^{m} = \frac{\omega(1 - A_{i})}{2} \int_{-1}^{1} \mathbf{P}^{m}(\mu, \mu') I_{L,m}^{m}(\mu') \, d\mu',$$  \hspace{1cm} (2.43a)

and

$$I_{n}^{m}(\tau = 0, \mu < 0) = 0$$  \hspace{1cm} (2.44a)

$$I_{n}^{m}(\tau = \tau_0, \mu > 0).$$  \hspace{1cm} (2.44b)

The integration in Eq. (2.42) is performed by discretizing the layer into many equal sublayers. The optical thickness of each sublayer has to be small to give an accurate approximation and the integration for each sublayer is solved through the exponential-linear approximation (Kylling and Stamnes 1992). Gaussian quadrature is used in integrating Eq. (2.43). The computational time depends strongly on the order of the Gaussian quadrature.

d. Numerical realization and optimization

1) DIRECT-TRANSMISSION, SINGLE-SCATTERING, AND MULTIPLE-SCATTERING SEPARATION FOR SAA

For computational efficiency, the small-angle variables $n_{x}$ and $n_{y}$ are transferred into $\theta_x$ and $\theta_y$ (Zege and Polonsky 1993):

$$n_{x} = \frac{\cos \theta_x}{\cos \theta_y} \sin (\theta_x - \theta_y) \approx (\theta_x - \theta_y)$$  \hspace{1cm} (2.45a)

$$n_{y} = \sin \theta_y \approx \theta_y,$$  \hspace{1cm} (2.45b)

where

$$\tan \theta_x = \tan \theta \cos \varphi$$  \hspace{1cm} (2.46a)

$$\sin \theta_y = \sin \theta \sin \varphi.$$  \hspace{1cm} (2.46b)

Subsequently, Eq. (2.27) can be written as

$$I_{L,s}^f = \frac{1}{2\pi} \frac{F}{\sqrt{V_{n_x} V_{n_y}}} \exp \left[ -\frac{(\theta_{x}-\theta_{x})^2}{2V_{n_x}} - \frac{(\theta_{y}-\theta_{y})^2}{2V_{n_y}} \right],$$  \hspace{1cm} (2.47)

with the integral properties (Remizovich and Shekhman‘ev 1990; Zege et al. 1988):

$$\langle I_{L,s}^f \rangle = F_{r},$$  \hspace{1cm} (2.48a)

$$\langle \theta_{x}^2 I_{L,s}^f \rangle = V_{n_x},$$  \hspace{1cm} (2.48b)

$$\langle \theta_{x} I_{L,s}^f \rangle = \theta_{b},$$  \hspace{1cm} (2.48c)

$$\langle \theta_{y}^2 I_{L,s}^f \rangle = V_{n_y} + \theta_{b}^2,$$  \hspace{1cm} (2.48d)

$$\langle \theta_{x} \theta_{y} I_{L,s}^f \rangle = V_{n_x} + \theta_{b} \approx 0,$$  \hspace{1cm} (2.48e)

where $F_{r}$ is the flux at the optical thickness $\tau$ and the same approximation as Eq. (2.33) has been used in the integration.

The accuracy can be effectively improved by using the exact solution for the single-scattering contribution in the small-angle approximation (Nakajima and Tanaka 1988; Zege et al. 1988). Furthermore, the direct transmission is also known and the second-order transmission can be obtained in small-angle scattering with little computational effort. Accordingly, the total transmission (tt) in the small-angle approximation can be decomposed into components associated with the direct-transmission (dt),...
single-scattering (ss), double-scattering, and multiple-scattering terms (the scattering order \( n > 2 \)) from SAA:

\[
I_{\text{li}} = I_{\text{di}} + I_{\text{si}} + I_{\text{ds}} + I_{\text{ms}},
\]

where \( I_{\text{di}} \) and \( I_{\text{si}} \) are accurate and \( I_{\text{ds}} \) is numerically accurate.

The accurate direct-transmission term and its corresponding flux and variances in SAA are

\[
I_{\text{ds}}(\tau, \mu, \mu_0, \phi) = \mu_0 F_0 \frac{(\omega A_1 \tau)^2}{2 \mu_0^2} \int_0^{2\pi} a_1(\mu, \mu', \phi - \phi') a_1(\mu', \mu_0, \phi') c_{\text{ds}}(\tau, \mu, \mu', \mu_0) d\mu' d\phi'.
\]

The accurate single-scattering term and its corresponding flux and variances in SAA are

\[
I_{\text{ss}} = I_0 \frac{\omega A_1 \tau a_1}{4\pi \mu} e^{\nu_0},
\]

where \( X \) represents \( \theta_{a1}^2 \), \( \theta_a^2 \), or \( \theta_2 \). Once the multiple-scattering flux and the variances are computed, the multiple-scattering radiance in the small-angle approximation can be obtained by substituting the flux and the variances into Eq. (2.47).

2) FORWARD-SOURCE AND -SCATTERING MATRIX APPROXIMATION FOR THE SOS

Because of the extremely small variances in Eq. (2.47), the forward source of the diffuse vector RTE can be approximated as

\[
I_{L,s} \approx I_{L,s}^f = \frac{F}{4\pi} \int_0^{2\pi} e^{-\psi/2V} d\psi.
\]

where

\[
F = F_0 \exp[(1 - \omega A_1) \tau/\mu_0 + s_{\text{saa}} \tau/\mu_0] \quad \text{and} \quad V = \frac{s_{\text{saa}}}{P_{\text{saa}}} \tanh(-s_{\text{saa}} \tau/\mu_0).
\]

The Fourier series of the source can be expressed as

\[
\vec{I}_{L,s}(\mu, \mu_0, \phi) = \sum_{m=0}^{\infty} (2 - \delta_{m0}) \cos(m \phi) \vec{I}^m_{L,s}(\mu, \mu_0),
\]

where
In the above equations, \( P_{\ell} \) and \( P_f \) are the generalized spherical functions and the Legendre polynomials, respectively. The direct integration of the expansion coefficient \( \alpha_i \) in Eq. (2.63) is laborious as the variance \( V \) is related to the optical thickness \( \tau \). To avoid the integral for every sublayer or any optical thickness, the Legendre polynomial in the small-angle approximation can be expanded as a series with respect to scattering angle \( \Theta \):

\[
P_f(\cos \Theta) = 1 - a_i \frac{\Theta^2}{2} + \left( \frac{d_i}{3} + b_i \right) \frac{\Theta^4}{2^2 \times 2!} - \left( \frac{a_i}{15} + b_i + c_i \right) \frac{\Theta^6}{2^3 \times 3!} + \left( \frac{a_i}{105} + \frac{3}{5} b_i + 2 c_i + d_i \right) \frac{\Theta^8}{2^4 \times 4!} - \left( \frac{a_i}{945} + \frac{17}{63} b_i + \frac{7}{3} c_i + \frac{10}{3} d_i + e_i \right) \frac{\Theta^{10}}{2^5 \times 5!} + \cdots.
\]

The recurrence relations for the expansion coefficients are given in Eqs. (B.5) in appendix B. These expansion coefficients are independent of scattering angle \( \Theta \); that is, they are fixed numbers so that they can be calculated and saved prior to solving the vector RTE. The derivation of the small-angle approximation for the Legendre polynomials is given in appendix B. The above expansion is only up to \( \Theta^6 \); however, the expansion can be extended to higher orders if necessary in terms of the binomial coefficient by using a mathematical software called Mathematica. The moments for scattering angle \( \Theta \) is

\[
\frac{1}{V} \int_0^\infty e^{-\Theta^2/2V} \Theta^{2l+1} d\Theta = (2l!) V^l.
\]

The expansion coefficients can be approximately written as a series of the variance \( V \):

\[
\alpha_{l} = -\frac{(2l + 1)}{V} \int_0^\infty e^{-\Theta^2/2V} P_f(\cos \Theta) \sin \Theta d\Theta.
\]

The expansion coefficient can be directly obtained in the order of variance \( V \) at any optical thickness. The computation can be more efficient by including only several low Legendre orders because of the small variance. In terms of the \( \delta-M \) technique (Wiscombe 1977a), the source can be rewritten as

\[
\tilde{I}_{L,x}^f = \frac{F_l}{4\pi} \left[ 2f_{sca} \delta(1 - \chi) + (1 - f) \sum_{l=0}^{N_{\text{sc}}} \alpha_{l}^s P_{l}(\chi) \right],
\]

(2.67)

\[
f_{sca} = \alpha_{N_{\text{sc}} + 1}^s \quad \text{and} \quad \alpha_{l}^s = \frac{\alpha_{l} - (2l + 1)f_{sca}}{1 - f_{sca}}.
\]

(2.68a)

(2.68b)

where subscript \( \text{sca} \) stands for the forward-scattering truncation.

The diffuse component of the vector RTE varies much more slowly than the original counterpart because the strong forward Stokes vector component has been approximately solved by using the small-angle approximation. Consequently, the forward-scattering matrix in the diffuse component can similarly be reduced by using the \( \delta-M \) technique:

\[
d_{l} = \sum_{l=0}^{N_{\text{sc}}} \alpha_{l}^s P_{l}(\chi) \approx 2f_{sca} \delta(1 - \chi) + (1 - f_{sca}) \sum_{l=0}^{N_{\text{sc}}} \alpha_{l}^s P_{l}(\chi),
\]

(2.69)

\[
f_{sca} = \alpha_{N_{\text{sc}} + 1}^s \quad \text{and} \quad \alpha_{l}^s = \frac{\alpha_{l} - (2l + 1)f_{sca}}{1 - f_{sca}}.
\]

(2.70a)

(2.70b)

where subscript \( \text{sca} \) indicates the forward-scattering matrix truncation.

After the forward-source and -scattering matrix are approximated, the diffuse vector RTE in Eq. (2.40) can be reduced as

\[
\left( \mu \frac{\partial}{\partial \mu} - 1 \right) \frac{\hat{V}}{A_\ell} = -\frac{\omega_d}{2} \int_{-1}^{1} \mathbf{P}_{\mu} \mathbf{M}_{\mu} \mathbf{V}_{L} \mu^2 d\mu,
\]

\[
-\frac{\omega_d}{2} \frac{1 - A_{\ell}}{1 - A_{f_{sca}}} \int_{-1}^{1} \mathbf{P}_{\mu} \mathbf{M}_{\mu} \mathbf{V}_{L} \mu^2 d\mu,
\]

(2.71)

where

\[
\tau_{d} = (1 - \omega A_{f_{sca}}) \tau,
\]

(2.72a)

\[
\omega_{d} = \frac{1 - A_{f_{sca}}}{1 - \omega A_{f_{sca}}} \omega,
\]

(2.72b)
\[ \tilde{\mathbf{P}}^m = \frac{A_1(1 - f_{sca})}{1 - A_1 f_{sca}} \mathbf{P}_t^m + \frac{1 - A_1}{1 - A_1 f_{sca}} \mathbf{P}^{d,m}, \quad \text{and} \]
\[ \mathbf{I}_L^m \approx (\tilde{\mathbf{P}}^m_{L,s} 0 0 0)^T, \]
(2.72c)
(2.72d)

where \( \tilde{\mathbf{P}}^m \) is the forward-phase matrix in conjunction with the \( \delta-M \) truncation (Wiscombe 1977a). The order \( N_{sca} \) plays a critical role in the calculation because it determines the scale of the diffuse vector RTE. Equation (2.43) for the source of every order can be rewritten as
\[ S_1^m = \frac{\omega_d}{2} \frac{1 - A}{1 - A f_{sca}} \int_{-\pi}^{\pi} \mathbf{P}^{d,m}(\mu, \mu') \mathbf{I}_L^m(\mu') d\mu' \quad \text{and} \]
\[ S_n^m = \frac{\omega_d}{2} \int_{-\pi}^{\pi} \tilde{\mathbf{P}}^m(\mu, \mu') \mathbf{I}_n^{m-1}(\mu') d\mu'. \]
(2.73a)
(2.73b)

The realization of the SOS is a standard procedure (e.g., Kylling and Stamnes 1992; Lenoble et al. 2007; Min and Duan 2004; Zhai et al. 2009) so that it is not captured here. After the Stokes vector in the diffuse vector RTE is obtained, the total Stokes vector in Eq. (2.11) can be computed in a straightforward manner.

3) Dirac Delta Function Reduction for Forward Scattering and Source

If either the total optical thickness \( \tau_0 \) or the forward-scattering fraction \( A_1 \) is small, the source and the forward-scattering contribution in the diffuse vector RTE can be further reduced as
\[ \frac{1}{V} e^{s_{\tau_0} n^2 V} \approx \lim_{\nu \to 0} \frac{1}{V} e^{-(1-\chi) V} = 2\delta(1-\chi) \quad \text{and} \]
\[ a_{\tau}^f \approx 2\delta(1-\chi). \]
(2.74a)
(2.74b)

The above equations lead to \( f_{sca} = f_{sca} = 1 \) in the Eqs. (2.67) and (2.69). Equation (2.71) can be further simplified as (Chaikovskaya et al. 1999; Tynes et al. 2001; Zege and Chaikovskaya 1996)
\[ \left( \mu \frac{\partial}{\partial \mu} - 1 \right) \mathbf{I}^{d,m} = -\frac{\omega_d}{2} \int_{-\pi}^{\pi} \mathbf{P}^{d,m}(\mu, \mu') \mathbf{I}^{d,m}(\mu') d\mu' \]
\[ -\frac{\omega_d}{4\pi} c_{\mu_0} \mathbf{P}^{d,m}(\mu, \mu_0) E_0, \]
(2.75)

where
\[ \tau_d = (1 - \omega A_1) \tau \quad \text{and} \]
\[ \omega_d = \frac{1 - A_1}{1 - \omega A_1}. \]
(2.76a)
(2.76b)

Equation (2.75) is actually a standard vector RTE with the effective optical thickness, single-scattering albedo, and phase matrix.

3. Numerical Results

The verification is performed by comparison with the benchmarks reported by Kokhanovsky et al. (2010). The straightforward adding–doubling method (de Haan et al. 1987; Huang et al. 2015) and the \( \delta-M \) technique in conjunction with the adding–doubling method have been employed to compare the results calculated by the two-component vector RTE method (SAA + SOS) that combines the small-angle approximation and the SOS method. In the adding–doubling method, the expansion of each function in a Fourier series and the GSF expansion of the scattering matrix are employed to avoid the integrations in azimuthal angle. Moreover, the zenith integration, which is from 0° to 90° in the adding–doubling [e.g., the Eqs. (35) and (36) in de Haan et al. (1987)] whereas it is from 0° to 180° in the SOS [e.g., Eq. (2.43)], is performed using Gaussian quadrature. The order of Gaussian quadrature used in the SOS is fixed at 72. The truncation angle \( \Theta_c \) and the variance constant \( v_{\text{ph}} \) in (SAA + SOS) are determined based on the phase function and is given in each case. Rayleigh scattering normally should be the first case used to validate a new method; however, the current method decomposes the original scattering matrix into the forward and the diffuse components and the forward component for Rayleigh scattering is so small that the fraction for the forward scattering is negligible. All the Stokes vectors are normalized as (Kokhanovsky et al. 2010)
\[ \mathbf{I} = \pi I_{\text{dif}}^f (-\mu_0 F_0). \]
(3.1)

Here the subscript dif represents the diffuse radiation. All the simulation parameters are listed in Tables 1 and 2. The optical thickness for each sublayer in the SOS is set to be around 0.05 except the aerosol layer (shown in Tables 1 and 2). The present simulations were conducted by using an Intel x86–64 Linux cluster with 2.5-GHz Intel processors. The FORTRAN compiler is IFORT with the default optimization level -O2 and only one processor is used.

In section 3a, the benchmarks of a scattering aerosol or cloud layer with a totally absorbing underlying surface from Kokhanovsky et al. (2010) are employed to validate the present simulations. In section 3b, applications to the radiative transfer in the case of water clouds and ice clouds based on the Moderate Resolution Imaging Spectroradiometer (MODIS) Collection.
6 cloud optical properties in the infrared and visible channels in conjunction with a totally absorbing underlying surface are illustrated. A straightforward adding–doubling code (SAD) developed by Huang et al. (2015) is used as a benchmark, and, furthermore, a combination the δ-M technique and the adding–doubling method (δ-M + ADM) in used for additional comparisons.

a. Results from the aerosol and cloud layers

The solar zenith angle \( \theta_0 \) is 120° relative to the upward zenith direction; that is, \( \mu_0 = -0.5 \) and the corresponding azimuthal angle is 0°. The numerical simulations are performed with an angular resolution of 1° in viewing zenith angle (VZA) in conjunction with three viewing azimuthal angles (VAA), 0°, 90°, and 180°. The single-scattering albedo is 1 for the aerosol and cloud layers. The scattering matrix in Fig. 1 and the benchmarks in Figs. 2–5 are taken from Kokhanovsky et al. (2010). The optical thickness values for the aerosol and the cloud layers are 0.3262 and 5, respectively. The scattering matrix is produced using the Lorenz–Mie theory at wavelength \( \lambda = 0.412 \mu m \) and a lognormal particle distribution. Accordingly, \( a_2 \) equals \( a_1 \) and \( a_4 \) is the same as \( a_1 \). The input parameters for (SAA + SOS) and computational wall time are listed in Table 1. The definitions for the relative and absolute differences are \( (I - I_{\text{Benchmark}})/I_{\text{Benchmark}} \) and \( (X - X_{\text{Benchmark}}) \) (\( X \) indicates \( Q, U, \) or \( V \), respectively). Figure 2 shows the Stokes vector elements and the corresponding differences in the reflected radiation in the case of an aerosol layer. The differences in Fig. 2a for the VAA = 0° and 90° are small whereas the difference for the VAA = 180° is relatively large. The VZA = 60° in VAA = 180° is the antisolar angle. Thus, there are spikes at this angle due to the glory at a scattering angle of 180°, as shown in the right inset of \( a_1 \) in Fig. 1. The truncation techniques cause an overestimate of the single-scattering contribution. The relatively large differences around VZA = 60° in VAA = 180° are from the single-scattering contribution. Another type of relatively large differences around VZA = 90° could be explained using the optical thickness \( \tau_\mu \), which becomes very large when the VZA is close to the horizontal direction, and, consequently, the computational error becomes pronounced. The simulation is based on the plane-parallel assumption instead of the real spherical shell atmosphere geometry. This simplification

| Table 2. Input parameters and CPU time for the simulations by SAA + SOS and δ-M + ADM simulations. The cloud optical properties are taken from the MODIS Collection 6 cloud optical property dataset. N/A is as in Table 1. |

<table>
<thead>
<tr>
<th>System</th>
<th>( n_f )</th>
<th>( \Delta \tau )</th>
<th>( \Theta_0 (\degree) )</th>
<th>( N_{\text{sca}} )</th>
<th>( N_{\text{sec}} )</th>
<th>( M )</th>
<th>GSFs</th>
<th>Time (s)</th>
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has larger errors when the VZA is roughly larger than 85° (Adams and Kattawar 1978).

Figure 3 shows the Stokes vector elements and the corresponding differences in the transmitted radiation in the case of an aerosol layer. The large differences are around VZA = 120° in VAA = 0° is the regime SAA dominates. The differences are from SAA. The differences around VZA = 90° are due to the same reason explained in conjunction with Fig. 2.

Figure 4 shows the Stokes vector elements and the corresponding differences in the reflected radiation in the case of a cloud layer. The differences around VZA = 60° in VAA = 180° are from the same mechanism revealed by Fig. 2. The large relative difference in the case of an aerosol layer is that a cloud layer has a stronger forward peak than an aerosol layer (shown in the left inset of a1 in Fig. 1), which may cause an overestimate of the single-scattering contribution. The differences around VZA = 23° in VAA = 180° are due to the rainbow around scattering angle 143° (shown in a1 if Fig. 1). The overestimated single scattering causes the differences in I and Q elements.

Figure 5 shows the Stokes vector elements and the corresponding differences in the transmitted radiation in the case of a cloud layer. The small truncation angle of 3° is used to reproduce the small-angle radiation. The differences around VZA = 120° in VAA = 0° are due to the same reason as Fig. 3. For an intermediate optical thickness, the multiple-scattering contribution dominates and SAA causes relatively large differences. The differences around VZA = 100° in VAA = 180° are from the peak of b1 around scattering angle 110° (shown in b1/a1 of Fig. 1). The mechanism for the differences is the same as the counterparts due to the glory and rainbow.

b. Results from the scattering matrices of the MODIS Collection 6

The MODIS Collection 6 optical properties of water clouds and ice clouds in the visible 0.65-μm channel and in the infrared 12-μm channel are used in the present simulations. A typical droplet diameter of 24 μm and optical thickness of 15 are employed for water clouds whereas a typical diameter of 60 μm and optical thickness of 5 are used for ice clouds. The refractive indices, computation methods, and the shapes used to generate the scattering matrices can be found in references (Downing and Williams 1975; Hale and Querry 1973; Platnick et al. 2016; Warren and Brandt 2008; Yang and Liou 1998; Yang et al. 2013). The solar zenith angle θ0 is 120° relative to the upward zenith direction, and the VAA = 0° in the transmitted radiation, which is SAA dominant regime, and the arbitrarily chosen VAA = 60° in the reflected radiation are evaluated. SAD has been used as benchmark and (δ-M + ADM) as comparison. For a fair comparison between the δ-M and SAA, quadrature order 36 (72 for the integration of μ from −1 to 1) has been used in (δ-M + ADM). Moreover, the Fourier order M for decoupling the azimuthal integration and the GSF term for expanding the scattering phase matrix are the same. The definition for relative difference is the same as the section 3a and the absolute error.
difference is defined as \((X - X_{SAD})\), where \(X\) indicates \(Q\), \(U\), or \(V\).

Figure 6 shows the scattering matrices of water clouds and ice clouds of MODIS Collection 6 in the visible and infrared channels. The phase functions \(a_1\) have strong peaks and the corresponding reduced elements \(a_2/a_1\), \(a_3/a_1\), and \(a_4/a_1\) are close to 1 for small scattering angles; however, the remaining reduced elements \(b_1/a_1\) and
$b_2/a_1$ for the small scattering angles are close to 0 and do not have significant contributions for small angles compared to the leading diagonal elements.

Figure 7 shows the Stokes vector elements and the differences of water clouds in the infrared channel calculated by SAD, (SAA + SOS), and (δ-M + ADM). The single-scattering albedo $\omega$ is 0.412240. The differences caused by (δ-M + ADM) are negligible compared to the counterparts caused by (SAA + SOS). The original scattering matrices for water clouds in the infrared channel are smooth and they can be expanded with tens of terms of the GSF. Truncation used in SAA instead increases the number of expansion terms, which, however, introduces the overall differences. (δ-M + ADM) in this case is basically an accurate solution whereas (SAA + SOS) is still an approximation. The differences around VZA = 90° have been explained in Fig. 2.

Fig. 3. As in Fig. 2, but for the transmitted radiation related to an aerosol layer.
Figure 8 shows the Stokes vector elements and the differences in the case of water clouds in the visible channel, calculated by SAD, (SAA + SOS), and (\(\delta\)-M + ADM). The single-scattering albedo \(\omega\) is 0.999996. The small-angle effect is negligible in the reflected radiation. Consequently, the reflected radiance in Fig. 8a has been underestimated. Other than the radiance in Fig. 8a, the differences are of the same order for (SAA + SOS) and (\(\delta\)-M + ADM). In Fig. 8b, (SAA + SOS) better reproduces the forward radiation than (\(\delta\)-M + ADM).

Figure 9 shows the Stokes vector elements and the differences of ice clouds in the infrared channel calculated by SAD, (SAA + SOS), and (\(\delta\)-M + ADM). The single-scattering albedo \(\omega\) is 0.523801. In Fig. 9a, the two approximations give similar differences except around the VZA = 90° in the radiance. In Fig. 9b,
(δ-M + ADM) causes larger differences in the small-angle regime than (SAA + SOS).

Figure 10 shows the Stokes vector elements and the differences of ice clouds in the visible channel, calculated by SAD, (SAA + SOS), and (δ-M + ADM). However, the reflected radiation calculated by (δ-M + ADM) has not been shown in Fig. 10a because of the extremely large differences with the same input parameters. The single-scattering albedo \( \omega \) is 0.9999999425.

For SAD, 3500 terms of GSFs, 250 Fourier series, and 800 Gaussian quadrature terms (1600 for the zenith region from \(-1\) to 1) are employed to obtain the benchmarks. The phase function of ice clouds in the visible channel has a strong forward peak; however, the peak is maintained in a small angular region. The truncation angle of 1° is used to reproduce the transmitted radiation in the small-angle regime. A large truncation angle of 10° is taken because the small-angle
effect is negligible in the reflection regime. Underestimated radiance and relatively large differences around VZA $= 90^\circ$ are reasonable in Fig. 10a. In the small-angle regime of Fig. 10b, $\delta$-$M$ causes large differences. The value in VZA $= 120^\circ$, the exact solar direction, from $(\delta$-$M + ADM)$ is not shown because of large deviations from the benchmark. The several-percent differences in the small-angle regime from (SAA $+$ SOS) have been explained in Fig. 5.

4. Discussion

The small scattering angles relative to the incident direction are dominated by the small-angle approximation whereas for other scattering angles, the small-angle effect is negligible. The small-angle approximation in this study is composed of the exact single scattering, double scattering, and the multiple scattering (scattering order larger than 2) in the small-angle approximation as shown in Eq. (2.49). In the case of a small optical thickness, the single and double scatterings are dominant contributions. For a large optical thickness, the multiple scattering dominates and approaches a Gaussian distribution. The small-angle approximation gives a good reproduction of the small-angle radiation. As for the intermediate optical thickness, the SAA can cause several-percent errors in the small-angle regime.

As for the truncation angle $\Theta_c$, it is a trade-off: the smaller $\Theta_c$ is, the more accurate the simulation is; however, a small $\Theta_c$ requires more computational effort. Normally, for a medium with a strong forward scattering, a small truncation angle (e.g., $\leq 3^\circ$) is necessary to reproduce the small-angle radiation. Otherwise, for radiation not in the small-angle regime or the radiation of a medium with a smooth forward scattering, a larger truncation (e.g., $6^\circ$–$10^\circ$) can be taken. The forward-scattering term $N_{sca}$ and the forward-source term $N_{src}$ represent the impact made by the truncated forward component on the diffuse component. The extra contribution from these forward components has a noticeable effect on the small-angle regime. Moreover, (SAA $+$ SOS) leads to an underestimate in comparison with the benchmarks due to the negligible effect of the SAA in the non-small-angle regime. The introduction of the extra forward terms improves the radiative transfer simulation. The number of terms or using Dirac delta function depends on the scattering phase matrix and the viewing configuration. If necessary, around 30 terms of $N_{sca}$ and 10 terms of $N_{src}$ are sufficient to give reasonable accuracy.

The computational time used by (SAA $+$ SOS) shown in Table 2 is mostly less than the one used by $(\delta$-$M + ADM)$. This is correct for cases with small or intermediate optical thickness. Even if the two methods use the same GSF order, (SAA $+$ SOS) includes a relatively smooth scattering matrix and, hence, a relatively quick convergence. However, for a case with extremely large optical thickness, SOS would be time consuming. After all, the computational time of
the SOS method is linearly proportional to the optical thickness while the one of the adding–doubling method is proportional to the binary logarithm of the optical thickness.

As for the differences caused by single scattering around the rainbow and glory angles shown in Figs. 2a and 4a, improving the computation of the single-scattering contribution to avoid introducing the
forward terms into the diffuse component requires further study. A similar study for the $\delta$-$M$ approximation is given by Nakajima and Tanaka (1988).

5. Summary

A two-component radiative transfer method is introduced in this paper. We divided the phase matrix and the Stokes vector into the forward and diffuse components; the original vector RTE can correspondingly be decomposed into two components: a forward and a diffuse vector RTE. The forward vector RTE can approximately be solved by a small-angle approximation. The diffuse vector RTE is solved in terms of the successive order of scattering method. The source of the diffuse vector component is from the solution of the forward component, which is a Gaussian distribution, while the scattering of the diffuse component is the
original scattering. For small optical thicknesses or small forward fraction, the forward source and the forward scattering for the diffuse component can be replaced by the Dirac delta function. However, for large optical thicknesses or a large forward fraction, the effect of the forward source and scattering on the diffuse component can be realized by including their lower orders into the simulation in terms of the GSFs and the $\delta$-$M$ technique. The simulation for the forward source is improved by expanding the Gaussian source using Legendre polynomials and their expansion coefficients are explicitly used as a series for the variance of the Gaussian source, thereby avoiding the laborious integration for each sublayer.

For the current two-component method, a large truncation angle can be chosen so that the strong forward scattering can be confined largely to the forward component. The forward component can be approximately solved by the small-angle approximation without much
computational effort. The remaining diffuse component is slowly varying. Moreover, the effect from the forward component can be implemented by including the low GSF orders or even be replaced by the Dirac delta function. Furthermore, the effective optical thickness can be significantly reduced. Ultimately, the computational time can be appreciably reduced, especially for media with a strong forward-scattering peak.

Homogeneous aerosol and cloud layers from benchmarks are employed to validate the two-component method. The reflected and transmitted radiations are shown for validation. Further transmission and reflection comparisons are shown for one homogeneous layer with the use of the MODIS Collection 6 cloud optical properties; good agreement has been shown in all cases.

**FIG. 10.** As in Fig. 7, but for an ice cloud layer in the visible channel and no \((\delta-M + \text{ADM})\) results in the reflected radiation.
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APPENDIX A

Small-Angle Approximation for the Scalar Radiative Transfer Equation

a. Small-angle approximation in oblique incidence

The scalar RTE for a narrow light beam in oblique incidence is

\[ \left( \mathbf{n} \cdot \nabla \right) I(\mathbf{r}, \mathbf{n}, \mathbf{n}_0) + \sigma_s I(\mathbf{r}, \mathbf{n}, \mathbf{n}_0) = \frac{\sigma_p}{4\pi} \int_{4\pi} P(\mathbf{n}, \mathbf{n'}) I(\mathbf{r}, \mathbf{n'}, \mathbf{n}_0) \, d\mathbf{n'}, \]

(A.1)

where \( \mathbf{r} \) and \( \mathbf{n} \) are the position and direction vectors, respectively, \( \sigma_s \) and \( \sigma_p \) are the scattering and extinction coefficients, respectively, and \( P \) is the scattering phase function of a homogenous medium. The scalar RTE can be represented in terms of dimensionless quantities:

\[ \left( n_x \frac{\partial}{\partial \tau_x} + n_y \frac{\partial}{\partial \tau_y} + n_z \frac{\partial}{\partial \tau_z} + 1 \right) I(\tau_x, \tau_y, \tau_z, \mathbf{n}, \mathbf{n}_0) \]

\[ = \frac{\omega}{4\pi} \int_{4\pi} P(\mathbf{n}, \mathbf{n'}) I(\tau_x, \tau_y, \tau_z, \mathbf{n'} \cdot \mathbf{n}_0) \, d\mathbf{n}', \]

(A.2)

where

\[ \tau_x = \beta \cdot x, \]

\[ \tau_y = \beta \cdot y, \]

\[ \tau_z = \beta \cdot z, \]

\[ \omega = \beta \cdot \beta. \]

(A.3)

In Fig. A1, the frame of reference of the medium is shown, where the \( Z \) axis is the downward normal to the boundary of the medium, and the origin is lying on the upper boundary. Note that for convenience the definition of the \( Z \) axis is opposite to the one in the paper; that is, these zenith angles are supplements of the ones used earlier. The incident zenith and azimuthal angles are \( (\theta_b > 0, 0) \). As the optical thickness increases, the photons will gradually lose memory of the direction from whence they came and the maximum radiance will tend to move closer to the normal direction; that is, \( 0 < \theta_b < \theta_b \) as the optical thickness increases. The azimuthal angle of the maximum radiance is still zero owing to mirror symmetry in the \( XOZ \) plane. The \( Z \) axis of the new local frame of reference is pointing in the direction of the maximum radiance, as shown in Fig. A1. In the new frame of reference \( O-X'Y'Z' \), the radiation is still small-angle scattering (Zege et al. 1991).

The \( x \) and \( z \) components of the direction vector \( \mathbf{n} \) in the local frames of reference are

\[ \begin{pmatrix} n_x' \\ n_z' \end{pmatrix} = \begin{pmatrix} \cos \theta_b & -\sin \theta_b \\ \sin \theta_b & \cos \theta_b \end{pmatrix} \begin{pmatrix} n_x \\ n_z \end{pmatrix}. \]

(A.4)

The \( y \) component \( n_y \) is the same in the two frames of reference. In the small-angle approximation, \( n_x' \) and \( n_z' \) are small quantities; subsequently,

\[ n_z' = \sqrt{1 - (n_x')^2 - (n_z')^2} \approx 1 - (n_x^2 + n_z^2)/2. \]

(A.5)

The left-hand side of the scalar RTE in Eq. (A.2) can be written as (Remizovich and Shekhamet’ev 1990; Zege et al. 1987).
In the first term, the dependence on the optical thickness \( \tau \) of the maximum radiance angle \( \theta_b \) has been explicitly expressed. In the right-hand side of Eq. (A.2), if the phase function has a strong forward peak, the radiance can be approximately replaced by its Taylor expansion. Limiting the expansion to the quadrature terms, the right-hand side of Eq. (A.2) turns into (Sobolev 1975)

\[
\text{rhs} = \omega I + \frac{\omega \nu}{4} \Delta_n I \quad \text{and} \quad \nu = \frac{1}{2} \int P \sin^3 \Theta \, d\Theta. \quad \text{(A.7a)}
\]

In Eqs. (A.6) and (A.7), the arguments have been suppressed for convenience. In the small-angle approximation, the scalar radiative transfer equation has been changed from an integral–differential equation into a differential equation:

\[
\begin{align*}
\text{lhs} &= \left[ \left( 1 - \frac{n_x^2 + n_y^2}{2} \right) \cos \theta_b - n_x \sin \theta_b \right] \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} \frac{\partial}{\partial n_x} I \\
&+ \left[ \left( 1 - \frac{n_x^2 + n_y^2}{2} \right) \sin \theta_b + n_x \cos \theta_b \right] \frac{\partial}{\partial \tau} I \\
&+ \left[ n_y \frac{\partial}{\partial \tau} + (1 - \omega) \right] I = \frac{\omega \nu}{4} \left( \frac{\partial^2}{\partial n_x^2} + \frac{\partial^2}{\partial n_y^2} \right) I.
\end{align*}
\quad \text{(A.8)}
\]

Then the differential equation can be rearranged as

\[
\begin{align*}
\left[ \left( 1 - \frac{n_x^2 + n_y^2}{2} \right) \cos \theta_b - n_x \sin \theta_b \right] \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau} \frac{\partial}{\partial n_x} I \\
&+ \left[ \left( 1 - \frac{n_x^2 + n_y^2}{2} \right) \sin \theta_b + n_x \cos \theta_b \right] \frac{\partial}{\partial \tau} I \\
&+ \left[ n_y \frac{\partial}{\partial \tau} + (1 - \omega) \right] I = \frac{\omega \nu}{4} \left( \frac{\partial^2}{\partial n_x^2} + \frac{\partial^2}{\partial n_y^2} \right) I.
\end{align*}
\quad \text{(A.9)}
\]

where

\[
\begin{align*}
I &= \exp\left[ -(1 - \omega) \tau_{\text{eff}} \right] I, \quad \text{(A.10)}
\end{align*}
\]

\[
\frac{d\tau_{\text{eff}}}{d\tau} = 1 / \cos \theta_b, \quad \text{(A.11a)}
\]

\[
P = 1 - \omega, \quad \text{and} \quad D = \frac{\omega \nu}{4}. \quad \text{(A.11b)}
\]

Accordingly, we seek a solution of Eq. (A.9) like this (Remizovich et al. 1982; Zege et al. 1987):

\[
\begin{align*}
\tilde{I} &= \frac{1}{4\pi^2} \tilde{F}(\tau) \exp \left[ - \frac{V_x n_x^2 - 2A_x (\tau_x - \tau_d)n_x} {2V_x} - \frac{V_y n_y^2 - 2A_y (\tau_y - \tau_d)n_y} {2V_y} \right].
\end{align*}
\quad \text{(A.12a)}
\]

\[
V_x = V_{\tau_x} n_x - A_x, \quad \text{and} \quad \text{(A.12b)}
\]

\[
V_y = V_{\tau_y} n_y - A_y, \quad \text{and} \quad \text{(A.12c)}
\]

with the condition

\[
\int d\tau_x d\tau_y dn_x dn_y \left[ \tilde{F}(\tau) \tilde{F}(\tau) \right] = 1. \quad \text{(A.13)}
\]

Equation (A.12) is actually a standard four-dimension spatial–angular Gaussian distribution. The \( \tau_d \) is the optical x-axis position of the maximum radiance at the optical thickness \( \tau \). Substituting Eq. (A.12) into Eq. (A.9) and equating the coefficients of the same power of the small quantities up to a quadrature order, one can get the equation for the zero order:

\[
\frac{1}{\tilde{F}} \frac{d\tilde{F}}{d\tau_{\text{eff}}} = \frac{1}{2V_x} \frac{dV_x}{d\tau_{\text{eff}}} - \frac{1}{2V_y} \frac{dV_y}{d\tau_{\text{eff}}} = -D \left( \frac{V_{\tau_x}}{V_x} + \frac{V_{\tau_y}}{V_y} \right). \quad \text{(A.14)}
\]
for $n_x \tau_y$:

$$\frac{d(A_y/2V_y)}{d\tau_{\text{eff}}} - \frac{V_{n_y}}{2V_y} = - D \frac{V_{A_y}^2}{V_y^2}; \quad (A.16)$$

for $\tau_y^2$:

$$\frac{d(V_{n_y}/2V_y)}{d\tau_{\text{eff}}} = D \frac{A_y^2}{V_y^2}; \quad (A.17)$$

for $n_y^2$:

$$\frac{d(V_{r_x}/2V_x)}{d\tau_{\text{eff}}} + \tan \theta_y \frac{A_{r_y}}{V_x} \frac{d\tau_y}{d\tau_{\text{eff}}} + \cos \theta_y \frac{A_y}{V_x} - \tan \theta_y \frac{d\theta_y}{d\tau_{\text{eff}}} \frac{V_{r_x}}{2V_x} + \frac{P}{2} = D \frac{V_{A_x}^2}{V_x^2}; \quad (A.18)$$

and

$$\frac{d(V_{n_y}/2V_y)}{d\tau_{\text{eff}}} = D \frac{A_y^2}{V_y^2}; \quad (A.20)$$

for $n_x$:

$$- \frac{A_x}{V_x} \frac{d\tau_y}{d\tau_{\text{eff}}} + \frac{d\theta}{d\tau_{\text{eff}}} \frac{V_{r_x}}{V_x} + \sin \theta_b \frac{A_b}{V_x} + P \tan \theta_b = 0; \quad (A.21)$$

for $(\tau_x - \tau_d)^2$:

$$\frac{d(V_{r_x}/2V_x)}{d\tau_{\text{eff}}} = D \frac{V_{A_x}^2}{V_x^2}; \quad (A.22)$$

For $y$-axis-related equations, the analytical solutions can be obtained owing to the $XOZ$ symmetry:

$$\frac{dV_{n_x}}{d\tau_{\text{eff}}} = 2D - PV_{n_x}^2; \quad (A.23a)$$

For clarity, a new variable $\gamma$ is defined as (Zege et al. 1987, 1991)

$$\frac{dy}{d\tau_{\text{eff}}} = PV_{n_y}. \quad (A.31)$$

Formerly, the variances $V_{n_y}$ and $A_x$, and the angle $\theta_b$ can be analytically represented as a function of $\gamma$:

$$\frac{dV_{n_y}}{dy} = \frac{2D - PV_{n_y}^2 (1 + 2 \tan^2 \theta_b)}{PV_{n_y}}; \quad (A.32a)$$

$$V_{n_y} = \frac{s \sqrt{(1 - e^{-2\gamma})(1 + \sin^2 \theta_b \sin^2 \theta_y) - 4 \gamma \sin^2 \theta_b \cos^2 \theta_b}}{\cos^2 \theta_y}; \quad (A.32b)$$

For $x$-axis-related equations, they can be simplified as

$$\frac{d\tau_y}{d\tau_{\text{eff}}} = \sin \theta_b - PA_x \tan \theta_b; \quad (A.27a)$$

$$\frac{dA_y}{d\tau_{\text{eff}}} = -PV_{n_y} \tan \theta_b; \quad (A.27b)$$

and

$$\frac{dV_{n_x}}{d\tau_{\text{eff}}} = 2D - PV_{n_x}^2 (1 + 2 \tan^2 \theta_b), \quad (A.28)$$

and

$$\frac{dA_x}{d\tau_{\text{eff}}} = \frac{V_{n_x}}{\cos \theta_b} - PV_{n_x} A_x (1 + 2 \tan^2 \theta_b), \quad (A.29)$$

Finally, the analytical solutions can be obtained owing to the $XOZ$ symmetry:

$$\frac{dA_x}{dy} = \frac{1}{P \cos \theta_b} - A_x (1 + 2 \tan^2 \theta_b), \quad (A.33a)$$
Subsequently, the variable $\gamma_b$ can be numerically obtained from the equation
\[
\tau = \int_0^{\gamma_b} \frac{\cos \theta_b}{PV_{n'}} d\gamma.
\] (A.35)

As long as $\gamma_b$ is known, all left $x$-axis-related quantities can be obtained in terms of numerical integration:
\[
\tau_{\text{eff}} = \int_0^{\gamma_b} \frac{1}{PV_{n'}} d\gamma,
\] (A.36)
\[
d\tau_d = \frac{\sin \theta_b - PA_b \tan \theta_b}{PV_{n'}},
\] (A.37a)
\[
\tau_d = \int_0^{\gamma_b} \left( \frac{\sin \theta_b - PA_b \tan \theta_b}{PV_{n'}} \right) d\gamma,
\] (A.37b)
\[
dV_{\tau_x} = \frac{2}{P \cos \theta_b} \frac{A_1}{V_{n'}} \frac{A_x^2}{V_{n'}} \left( 1 + 2 \tan^2 \theta_b \right), \quad \text{and}
\] (A.38a)
\[
V_{\tau_x} = \int_0^{\gamma_b} \left( \frac{2}{P \cos \theta_b} \frac{A_1}{V_{n'}} \frac{A_x^2}{V_{n'}} \left( 1 + 2 \tan^2 \theta_b \right) \right) d\gamma.
\] (A.38b)

The zero-order equation can be explicitly solved and gives the solution
\[
d\ln[1/(F \cos \theta_b)] = d(\gamma/2) + d\ln \sqrt{\cosh(\tau_{\text{eff}})}
\] (A.39a)
\[
\cos \theta_b F = \frac{\cos \theta_0 F_0}{e^{\gamma_0^2} \sqrt{\cosh(\tau_{\text{eff}})}},
\] (A.39b)
where $F_0$ is the solar incident flux.

b. Reduction for normal incidence

For normal incidence, the maximum radiance direction is still in the normal direction and the distribution has spatial and angular symmetries; that is, $\theta_b = 0$ and $\tau_d = 0$. The distribution coefficients can be explicitly given as a function of the optical thickness $\tau$. The spatial and angular dependences are set to be two vectors: $\tau_p = (\tau_x, \tau_y)$, $n_p = (n_x, n_y)$. Then the radiance for normal incidence can be simplified as
\[
I = \frac{1}{4\pi^2} \frac{F}{V} \exp \left( -\frac{V_{n'}^2 - 2A\tau_r + V_{r'}^2}{2V} \right)
\] (A.40a)
\[
V = V_r V_n - A^2,
\] (A.40b)
where
\[
F = \exp[-(1 - \omega)\tau]\cosh(\tau r),
\] (A.41a)
\[
V_n = \frac{s}{P} \tanh(\tau r),
\] (A.41b)
\[
A = \frac{1}{P} \left[ 1 - \frac{1}{\cosh(\tau r)} \right],
\] (A.41c)
\[
V_r = \frac{\tau}{P} \left[ 1 - \frac{\tanh(\tau r)}{s r} \right].
\] (A.41d)

c. Collimated source reduction

With a collimated incident radiation, the solution is only angularly dependent. Integrating the radiance distribution with respect to the spatial coordinates, the angular distribution is obtained:
\[
I = \frac{1}{2\pi} \frac{F}{\sqrt{V_{n'} V_{n'}}} \exp \left( -\frac{n_x^2}{2V_{n'}} - \frac{n_y^2}{2V_{n'}} \right)
\] (A.42)
\[
F = \exp[-(1 - \omega)\tau_{\text{eff}}] F.
\] (A.43)

Other parameters have been given in appendix A, section a.

The physical meaning of the small-angle approximation in Eq. (A.42) can be explained in two canonical cases. When the truncation angle approaches $0^\circ$, both the forward fraction $A_1$ and the variances are close to 0. The effective optical thickness $\tau_{\text{eff}}$ is close to $\tau/\mu_0$. Equation (A.42) is exactly Bouger–Lambert–Beer’s (BLB) law. The small-angle approximation is effectively a generalization of this law with a distributed Gaussian profile. Another limitation is that one can assume the variances are approaching 0 (the $\delta$ distribution approximation). The Eq. (A.42) is the BLB law in the scaled optical thickness $(1 - \omega)\tau$. Similarly, the scaled optical thickness is used in the $\delta$-$M$ or the $\delta$-$f$ techniques. Accordingly, the small-angle approximation can be treated as an improvement of forward peak solution other than the $\delta$ approximation of the forward peak used in the $\delta$-$M$ and $\delta$-$f$ techniques.
where \( \chi = \cos \Theta \). The \( \chi^k \) can be represented as a Taylor series in the angle \( \Theta \):

\[
P_l(\chi) = 2^l \sum_{k=0}^{l} \chi^k \left( \frac{l + k - 1}{2} \right), \quad (B.1)
\]

The Legendre polynomials can accordingly be represented as a series in the angle \( \Theta \):

\[
P_l(\chi) = 1 - \frac{\Theta^2}{2} + \frac{a_1}{3!} \frac{\Theta^4}{2^2 \times 2!} - \frac{a_2}{5!} \frac{\Theta^6}{2^3 \times 3!} + \frac{a_3}{7!} \frac{\Theta^8}{2^4 \times 4!} - \frac{a_4}{9!} \frac{\Theta^{10}}{2^5 \times 5!} + \frac{a_5}{11!} \frac{\Theta^{12}}{2^6 \times 6!} - \frac{a_6}{13!} \frac{\Theta^{14}}{2^7 \times 7!}, \quad (B.2)
\]

where \( a_i, b_i, \ldots, g_i \) represent the expansion coefficients of the Legendre polynomials. For high orders, these coefficients can be represented straightforwardly in terms of the binomial coefficients if necessary. The Legendre polynomials can accordingly be derived from Eq. (B.2), where \( \chi^k \) has been expanded to the fourteenth order in the angle \( \Theta \).
\[ e_i = (l + 4)(l + 2)(l - 2)(l - 4) + (5l^4 + 10l^3 - 50l^2 - 55l + 45)a_{i-5} + (5l + 10)d_{i-5} + e_{i-5}, \quad e_5 = 9!; \quad e_i = 0, \quad i < 5; \]

\[ f_i = (l + 5)(l + 3)(l + 1)(l - 1)(l - 3)(l - 5) + (6l^6 + 15l^4 - 120l^3 - 195l^2 + 384l + 225)a_{i-6} + (5l^6 + 12l^5 - 21l^4 - 66l^3 + 9l^2 + 10(l^2 + 9l^2 + l - 12)c_{i-6} + (5l^6 + 4l^5 + 2)d_{i-6} + (6l + 15)e_{i-6} + f_{i-6}, \quad f_6 = 11!; \quad f_i = 0, \quad i < 6; \quad \text{and} \]

\[ g_i = (l + 6)(l + 4)(l + 2)(l - 2)(l - 4)(l - 6) + (7l^6 + 21l^5 - 245l^4 - 525l^3 + 1813l^2 + 2079l - 1575)a_{i-7} + (21l^6 + 105l^5 - 315l^4 - 1365l^3 + 609l + 1575)b_{i-7} + (35l^6 + 210l^5 - 35l^4 - 1050l - 315)c_{i-7} + (35l^6 + 210l^5 + 175l - 210)d_{i-7} + (21l^6 + 105l^5 + 84)e_{i-7} + (7l + 21)f_{i-7} + g_{i-7}, \quad g_7 = 13!; \quad g_i = 0, \quad i < 7. \]

The recurrence relations can be obtained using mathematical software called Mathematica (Wolfram 1999).

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