Tropical Atmospheric Madden–Julian Oscillation: A Strongly Nonlinear Free Solitary Rossby Wave?

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ABSTRACT

The Madden–Julian oscillation (MJO), a planetary-scale eastward-propagating coherent structure with periods of 30–60 days, is a prominent manifestation of intraseasonal variability in the tropical atmosphere. It is widely presumed that small-scale moist cumulus convection is a critical part of its dynamics. However, the recent results from high-resolution modeling as well as data analysis suggest that the MJO may be understood by dry dynamics to a leading-order approximation. Simple, further theoretical considerations presented herein suggest that if it is to be understood by dry dynamics, the MJO is most likely a strongly nonlinear solitary Rossby wave. Under a global quasigeostrophic equivalent-barotropic formulation, modon theory provides such analytic solutions. Stability and the longevity of the modon solutions are investigated with a global shallow-water model. The preferred modon solutions with the greatest longevity compare well overall with the observed MJO in scale and phase velocity within the factors.

1. Introduction

The Madden–Julian oscillation (MJO) is an atmospheric enigma [Hartmann and Hendon 2007; see Zhang (2005) for a review]. Initiated over the Indian Ocean exhibiting growth of convective activity, an enhanced oceanwide horizontal convergence in the lower troposphere, and an upper-troposphere divergence, this planetary-scale system propagates eastward with a speed of 3–6 m s$^{-1}$ (Madden and Julian 1972). Accelerating its propagation speed as it crosses the date line with convective activity dying out, the MJO completes a full cycle traveling around the globe in about 30–60 days.

Its propagating structure consists of a well-defined pair of solitary vortices, as shown in Fig. 1 (cf. Yanai et al. 2000). Its longitudinal scale is about 3000 km, although the composite studies tend to suggest larger scales [e.g., see Figs. 1–5 of Hendon and Salby (1994), Fig. 2 of Kiladis et al. (2005), Fig. 6 of Adames et al. (2014), and Fig. 1 of Zagar and Franzke (2015)]. As a pair snapshot in Fig. 1 suggests, the phase relationship between convection (OLR field) and the vortex pair tends to fluctuate with time, which contribute to the smoothing effect of the vorticity field, when convection is used as a reference variable. Even in the latter, a core part of the vortex pair tends to represent a comparable longitudinal scale as the snapshots within a factor of 3 (e.g., Fig. 2 of Kiladis et al. 2005). Even a closer correspondence in scale is identified by a pair of the potential-vorticity anomalies in Fig. 9 of Zhang and Ling (2012). Thus, the present paper pursues the explanation of the MJO in terms of its vortex pair structure.

The major challenge of the MJO theories is to explain the slow eastward propagation of this structure. The Kelvin wave, identified as the best candidate under the dry linear limit, propagates too fast (50 m s$^{-1}$). Thus, a particular additional mechanism must be invoked. Since the tropical atmosphere displays an abundance of moist...
cumulus convective clouds, it is naturally identified as a prime candidate for providing such a slowdown mechanism. Because of this, vast investments have been made over more than three decades attempting to understand the MJO in terms of an interaction between the large-scale atmospheric flow and moist convection (cf. Wang 2005). The present study, alternatively, suggests that nonlinearity can equally be invoked for obtaining qualitatively an equivalent effect.

Theoretical efforts to explain the MJO in terms of an interaction between the large-scale atmospheric flow and moist convection are numerous to provide an exhaustive list herein, but they include Hayashi (1970), Lindzen (1974), Emanuel (1987), Yano and Emanuel (1991), Fuchs and Raymond (2007), Raymond and Fuchs (2007, 2009), Majda and Stechmann (2009), Yang and Ingersoll (2011, 2013), and Adames and Kim (2016). These theories usually treat the MJO as a linear wave. Unfortunately, these theories have difficulties in explaining both the scale and the propagation speed of MJO. A closer look at some successes often reveals extensive parameter tuning.

Earlier studies (Hayashi 1970; Lindzen 1974) simply fail to identify the MJO as a separate mode from a faster eastward-propagating Kelvin mode. The WISHE instability only predicts the phase speeds faster than 10 m s$^{-1}$ even with its most sophisticated version (Fig. 2 of Yano and Emanuel 1991). The moisture mode (Raymond and Fuchs 2007) is a promising possibility with substantially slower phase velocities. One of the most sophisticated versions predicts the phase velocity of about 6 m s$^{-1}$ (Fig. 2 of Fuchs and Raymond 2007) with the longitudinal wavenumber 2. However, it fails to explain the preferred scale of the MJO (i.e., wavenumber 2) with the growth rate approximately constant independent of the wavenumber. The skeleton model (Majda and Stechmann 2009; Thual and Majda 2016) is apparently successful in explaining the phase speed of 3–6 m s$^{-1}$ at the longitudinal wavenumber 2 (Fig. 2 of

![Fig. 1. The two phases of a Madden–Julian oscillation observed during 1992: the streamlines at 150-hPa level are overlaid on the OLR power (W m$^{-2}$; color shading). Averaged over (a) 25–29 Nov and (b) 11–15 Dec. Reproduction of a figure from Yanai et al. (2000) based on an updated ERA-Interim.]
Majda and Stechmann 2009). However, a close look at their formulation reveals that this is due to a careful tuning of their free parameter $Q$, which controls the gross moist stability (GMS: effective moist stability of the atmosphere): making the GMS sufficiently small but positive. This choice is rather limited, because observations suggest that GMS could often be negative (López Carrillo and Raymond 2005). Though clearly the convectively coupled wave theories continue to progress (e.g., Yang and Ingersoll 2011, 2013; Adames and Kim 2016), it is fair to say that those theories hardly suggest any tendency for converging.

All these theoretical studies are based on parameterized convection. From this perspective, the difficulties in simulating the MJO in global models (cf. Jiang et al. 2015) may also be attributed to the fact that moist convection is parameterized. Thus, if the horizontal resolution is high enough with mesh sizes less than approximately 50 km, convective parameterization could be turned off, and the MJO would more easily be simulated. A successful simulation of MJO in a high-resolution global model by Japanese Earth Simulator (Miura et al. 2007) supports this expectation.

However, similar efforts with comparable resolutions have turned out to be less successful (Gustafson and Weare 2004; Monier et al. 2010; Hagos et al. 2011; Ray et al. 2011; Holloway et al. 2013). For example, Fig. 1 of Holloway et al. (2013) shows that none of the six simulations reproduces eastward propagation of convective precipitation that is seen in TRMM satellite observation. In contrast, dry large-scale circulations associated with MJO are well simulated, as shown by their Figs. 6–8 for the zonal winds. In spite of their decaying tendencies, eastward propagation closely follows that of the observation in the phase space consisting of the two leading empirical principal components as shown by their Fig. 4. The same point is more succinctly presented by Fig. 2 of Monier et al. (2010), who compare the wavenumber–frequency spectra for the zonal wind at 850-hPa height and the top of the atmosphere outgoing long-wave (infrared) radiation intensity (OLR), as a measure of convective variability, between the observation and a simulation. The model spectrum for the zonal wind agrees well with the observation, whereas the model OLR spectrum is about 3 times weaker.

Thus, in spite of their difficulties simulating convective fields correctly, models somehow reproduce MJO-associated large-scale circulations rather well. It further suggests a possibility of considering the large-scale dynamical component of MJO in isolation without coupling it with convection, because it appears that these large-scale MJO circulations can be reproduced without properly simulating convection.

In fact, it may be argued that the tropical atmosphere is not as convective as commonly believed. Both theoretical (Charney 1963; Yano and Bonazzola 2009) and observational (Yano et al. 2009; Adames et al. 2014; Zagar and Franzke 2015) analyses suggest that the tropical large-scale circulation is rather nondivergent to a leading-order approximation; that is, the tropical large-scale horizontal flows are dominated by vorticity. The point is probably best made with the example of the tropical cyclone: its maintenance fundamentally hinges on moist convection. However, as far as the leading-order dynamics is concerned, it is well understood in terms of pure vortex dynamics (cf. Yanai 1964). Its purely dynamical aspects, such as the trajectory, can well be predicted without moist convection (e.g., Kasahara and Platzman 1963). The present paper will argue that the same is also true for the MJO (cf. Adames et al. 2014; Zagar and Franzke 2015): its maintenance may still fundamentally hinge on moist convection. However, as far as the leading-order dynamics is concerned, the main feature to be explained is the eastward-propagating solitary vortex pair as shown in Fig. 1, or composite fields shown in references cited above, without moist convection.

Our argument above, to which we also follow in the present study, does not intend to dismiss another facet that some of the studies do produce the MJO well by adopting improved or better-tuned convection parameterizations. Especially, the improvement of the MJO simulations in GCMs by introducing superparameterization is dramatic (e.g., Benedict and Randall 2009; Pritchard and Bretherton 2014; Arnold and Randall 2015). However, its success has hardly led to any theoretical consensus nor does it negate the premise of dry dynamics playing the leading role. Thus, there is clearly scope for pursuing a completely different theory for the MJO. The successes of the superparameterization models and of the Earth Simulator hardly explain why so many other high-resolution explicit-convection models have difficulties for simulating the convective component of MJO in spite of relative success in simulating its “dry” component. The present paper presents an explanation for this general tendency.

An attempt to explain the MJO solely in terms of the dry dynamics itself is not an original idea, as this was already pursued by, for example, Straus and Lindzen (2000), Moncrieff (2004), Lin et al. (2007), and Wedi and Smolarkiewicz (2010). Among the aforementioned studies, Wedi and Smolarkiewicz (2010) is closest to the present study in its approach. Moncrieff (2004) proposes a somewhat similar idea, but from a point of view of considering the MJO as an organized convective archetype. Despite invoking the notion of “convection” here, Moncrieff’s paper does not consider convective heating effect explicitly.
In the next section, we will argue from a theoretical basis that the MJO must be considered a nonlinear Rossby wave, if it is to be understood in terms of the dry dynamics. The “modon” theory (Tribbia 1984a; Verkley 1984), identified as a candidate for the theory, is formulated in section 3. Numerical experiments of a global shallow-water system are presented in section 4 to verify the structural stability and the longevity of the modon solutions. The paper concludes with further discussion in section 5.

2. Theoretical considerations

How can we, then, explain the MJO in terms the dry dynamics? Under the linear limit, the large-scale tropical atmospheric dynamics consists of a set of the equatorial waves (Matsuno 1966; Yano and Bonazzola 2009). Thus, if the MJO is to be explained in terms of the linear dynamics, it must be explained in terms of the slowest-propagating equatorial planetary-scale waves: the Kelvin or the Rossby wave.

At first sight, the Kelvin wave appears to be more attractive, because it propagates eastward, as does the MJO. However, regrettably, the Kelvin wave is non-dispersive, and its propagation speed is fixed by its vertical structure to about 50 m s$^{-1}$. That is far too fast for explaining the MJO.

Thus, we need to turn to the Rossby wave. The Rossby wave presents more flexibility owing to its dispersive-ness, and its propagation speed can be adjusted by changing the horizontal wavenumber.

Equatorial Rossby waves also take the form of a vortex pair trapped along the equator as suggested in Fig. 1, which is fairly nondivergent (Delayen and Yano 2009). An observational study (Zagar and Franzke 2015) more directly shows by a mode decomposition that the MJO is dominated by Rossby modes. Here, note that explaining the equatorially trapped structure of the MJO is much less challenging than its propagation speed, because the former is already consistent with the dry linear equatorial Rossby wave structure.

The required propagation speed $c_p$ gives an estimate of the total horizontal wavenumber by the dispersion relationship of the Rossby wave:

$$c_p = -\beta K^{\prime-2},$$

with $\beta$ the latitudinal gradient of the Coriolis parameter, $K^{\prime 2} = K^2 + (1/L_R)^2$ the effective wavenumber defined by $K$ the horizontal total wavenumber, and $L_R = (gH)^{1/2}/f_0 \sim 10^3$ km the Rossby deformation radius. Here, $g \approx 10$ m s$^{-2}$ is the acceleration of the gravity, $H = 10$ km is the mean depth (equivalent depth) of the atmosphere, and $f_0 \approx 10^{-4}$ s$^{-1}$ the Coriolis parameter at a reference latitude.

A simple conclusion following from this deduction is that the square of the horizontal wavenumber $K^2$ must have a sufficiently large negative value in order for a Rossby wave to propagate eastward. More quantitatively,

$$-K^{\prime-2} \approx \frac{c_p}{\beta} \approx 10^{12} \text{ m}^2,$$

assuming $\beta \approx 10^{-11}$ (m s$^{-1}$)$^{-1}$. This leads to an estimate, $iK^{\prime-1} \sim 10^3$ km $\sim L_R$, where $i$ is the imaginary unit. Thus, the required imaginary wavelength is compared to the Rossby deformation radius (10$^3$ km). Note that this imaginary scale is much shorter than the typical horizontal scale expected for the MJO, that is, on the order of 10$^4$ km.

This rather odd conclusion can be understood in a straightforward manner by basic knowledge of the wave theories: the nonlinear wave must be identified in the wavenumber–frequency phase space outside the range of the linear-wave dispersion (cf. Grimshaw and Iooss 2003). Thus, the above result simply suggests that the Rossby waves must be nonlinear in order to propagate eastward.

In more physical terms, the above argument suggests that the linear Rossby wave must be evanescent in order to propagate eastward. An immediate application of the general theoretical statement just above is that henceforth, an eastward-propagating Rossby wave must be nonlinear, if it is to be nonevanescent at least in some part. In the following, we are going to apply this conclusion to the wavenumbers both in latitudinal and longitudinal directions; thus, the above rather abstract conclusion from wave theories can be better understood in physical terms. Here, keep in mind that although the following analysis is strictly limited to the linear dispersion relation in Eq. (1), the conclusion always leads to a nonlinear wave based on the general theoretical principle just presented.

We first apply this argument in the latitudinal direction, because it is often argued that the results of these aforementioned high-resolution simulations (Monier et al. 2010; Hagos et al. 2011; Holloway et al. 2013) are rather trivial, simply being forced from the lateral latitudinal walls of the model channel domain typically placed at 30°–45° latitude. Once the lateral forcing is turned off, the planetary-scale MJO circulations disappear (Ray et al. 2009; Vitart and Jung 2010; Ray and Li 2013). Gustafson and Weare (2004), Ray and Zhang (2010), Ray and Li (2013), and Zhao et al. (2013) suggest that the eastward-propagating Rossby wave train arriving from higher latitudes induces the MJO.
However, the above argument suggests that any eastward-moving structure with a longitudinal wavenumber comparable to that of the MJO would only have an evanescent response away from the forcing wall with a decaying scale of 10^3 km. So long as a linear theory is concerned, no response would be found at the equator by forcing at 30°–45° latitude. Thus, although the latitudinal sidewall forcing may help to maintain a simulated MJO-like structure, the sole possibility for obtaining a MJO-like feature centered at the equator as its response is by a nonlinearity (cf. Wedi and Smolarkiewicz 2010).

An analogous argument can also be applied in the longitudinal direction. In this case, we prescribe a typical latitudinal scale associated with the MJO (as a real wavenumber) and estimate a required longitudinal wavenumber for obtaining the MJO propagation speed by a linear Rossby wave. This estimate suggests that a linear Rossby wave consistent with the MJO must be evanescent with a decaying scale of about 10^3 km in the longitudinal direction. Here, it is slightly odd to apply the concept of the evanescent wave to a periodic domain like in the longitudinal direction, because this concept is valid only locally, and only in the decaying direction. In the opposite direction, the evanescent wave actually grows exponentially, and at a certain point it is no longer linear; thus, nonlinearity must be invoked. So, if the MJO is to have nonevanescent structure somewhere along its propagation, it must again be a nonlinear wave.

These two applications of the result from the Rossby wave dispersion in Eq. (1) suggest that (i) the MJO must be a nonlinear wave and (ii) the MJO contains exponentially decaying tails longitudinally. As Yano and Flierl (1994) suggest, a nonlinear wave with exponential tails typically consists of a solitary structure: the main high-amplitude part, which is nonlinear, is isolated in longitude, and the wave consists of exponentially decaying linear tails away from the high-amplitude center. Thus, the most consistent manner for interpreting the MJO by dry dynamics is as a nonlinear solitary Rossby wave.

As for the inherent nonlinearity of the MJO, even a simpler but rather convincing argument can be provided: the observed order of the magnitude (0.2) of horizontal divergence relative to that of vorticity associated with MJO (Yano et al. 2009) is much larger than the values expected for linear Rossby waves of the similar scale, which are 0.03 and 0.09 for the zonal wavenumbers 1 and 2, respectively (Delayen and Yano 2009). The nonlinear nature of the MJO is also emphasized in some of the recent modeling and theoretical studies (e.g., Hagos et al. 2014; Pritchard and Bretherton 2014; Adames and Kim 2016). A scale analysis in appendix A based on the observational data analysis by Zhang and Ling (2012) also suggests strong nonlinearity of MJO.

Here, strong nonlinearity is essential: though the Rossby soliton theory (Boyd 1980) under weak nonlinearity is an attractive alternative explanation, $K^2$ is only weakly negative, that makes $K^2$ smaller than linear waves, but remains positive, and thus it propagates only westward, but faster than the linear waves under the dispersion relation in Eq. (1). For this reason, an earlier study based on the weakly nonlinear theory by Wedi and Smolarkiewicz (2010) had to add some background zonal winds in order to obtain eastward propagation. On the other hand, by proceeding to a strongly nonlinear regime, we can construct an eastward-propagating wave solution without adding a background zonal wind. As a strongly nonlinear theory, Tribbia (1984a) and Verkley (1984) provide solutions for a solitary Rossby–vortex pair called a modon constructed under a global quasigeostrophic equivalent-barotropic formulation. They predict eastward propagation also for the solutions placed right at the equator. We review this theory next.

### 3. Modon formulation

The strongly nonlinear modon Rossby–vortex pair solution is constructed on the global equivalent-barotropic quasigeostrophic system by Tribbia (1984a) and Verkley (1984). For its construction, they seek a steadily propagating solution with the phase velocity $c_p$ (positive eastward). Thus, the equivalent-barotropic quasigeostrophic system reduces to

$$J(\psi + c_p u/a, q) = 0,$$

where $\psi$ is the streamfunction; $u = \sin\phi$ is the sine of latitude $\phi$, $a$ is the planetary radius; $J(A, B) = A_1B_2 - A_2B_1$ is the Jacobian in spherical coordinates with $\lambda$ the longitude; and

$$q = \left(\nabla^2 - \frac{1}{L_R^2}\right)\psi + 2\Omega u$$

is the potential vorticity with $\Omega$ the planetary angular velocity. Equation (2) states that the total potential vorticity vanishes under a coordinate system moving with the phase velocity $c_p$. The scale analysis in appendix A shows that Eq. (2) under the approximation $c_p \approx 0$ is consistent with the dominant observed balance with MJO. Zhang and Ling (2012), to which we refer for the scale analysis, also suggest an importance of the nonlinear advection in determining the MJO propagation tendency.
The first step to construct an exact nonlinear solution to Eq. (2) is to note that this equation is equivalent to

\[ q = F(\psi + c_p \mu/a) \]

with an arbitrary functional \( F \). For facilitating an analytic progress, the most convenient choice would be to assume a linear functional, but with different forms for the inner and the exterior parts: the inner part represents an internal wave locally characterized by a real wavenumber \( \kappa_{in} \) and the exterior part constitutes an exponentially decaying wave with an evanescent rate \( \kappa_{ext} \), defined for the limit of \( 1/L_R \to 0 \). This is consistent with the general requirements for a solitonic nonlinear wave solution discussed in the last section. Thus,

\[ q = \begin{cases} -\kappa_{in}^2(\psi + c_p \mu/a) + C_{in} & \text{for the interior} \\ \kappa_{ext}^2(\psi + c_p \mu/a) + C_{ext} & \text{for the exterior} \end{cases} \]

where we have further added constant spatially homogeneous potential-vorticity anomalies \( C_{in} \) and \( C_{out} \) to the interior and the exterior, respectively, to the potential vorticity \( q \).

For further simplification, they divide the inner and the outer regions by a circular boundary with the angular radius \( \phi_0 \) centered at longitude \( \lambda_0 \) and latitude \( \phi_0 \). Keep in mind that this “radius” does not measure the actual modon vortex size. The latter further extends exponentially beyond the transitional angular radius.

We may set \( \lambda_0 = 0 \) without loss of generality. By seeking a MJO-like equatorially trapped solution, we also set \( \phi_0 = 0 \) in the following. By definition, the wave solutions are to be internal and external in the inner and the outer regions, respectively. Thus, \( \kappa_{in}^2 - 1/L_R^2 > 0 \) and \( \kappa_{ext}^2 + 1/L_R^2 > 0 \). We also require to assume \( C_{ext} = 0 \), and, for less obvious reasons, \( c_p \kappa_{ext}/a = 1 \) in order to ensure the purely evanescent solution at the exterior. An analytic solution is constructed by assuming the continuity of the streamfunction up to the second derivatives in the normal direction at the transition boundary of a circle with angular radius \( \phi_0 \). A complete form of the analytic solution is given in terms of the Gauss hypergeometric functions, as presented in appendix B.

Note that there are four parameters to be determined in the problem: two constants \( (\kappa_{in}, C_{in}) \) introduced by Eq. (3) and the two further parameters, \( c_p \) and \( \phi_0 \), for characterizing the solution. The continuity over the transition boundary leads to two constraints to these four parameters; thus we are left with the two free parameters to the problem. Note that for the case with \( \phi_0 = 0 \), we obtain \( C_{in} = 0 \). For characterizing the solution, we choose \( c_p \) and \( \phi_0 \) as the free parameters in the following.

4. Numerical experiments

The solitary Rossby–vortex pair modon (Tribbia 1984a; Verkley 1984) is introduced in the last section as a theory for the MJO. The present section tests this theory numerically.

Numerical experiments were performed using a global shallow-water model in order to demonstrate the relevance of this analytic theory. Recall that the primitive equation system reduces to a shallow-water system under a separation of the variables when only the horizontal structure of the flows is in concern. Under this reduction, as detailed in appendix C, a vertical structure of the flow in concern is characterized by a mean depth of the shallow-water system (i.e., equivalent depth).

In choosing the value for the mean depth, note the equivalent barotropic nature of the total flows associated with the MJO, as most carefully elucidated by Nishi (1989), but also as seen in Fig. 2 of Yano et al. (2009), Figs. 6–8 of Holloway et al. (2013), Fig. 8a of Zhang and Ling (2012), and Fig. 2 of Zagar and Franzke (2015). All these figures show that the signals associated with the MJO has a single sign in the vertical direction throughout the troposphere with an opposite sign above the tropopause level, a typical structure of the equivalent barotropic mode. Note the importance of considering the total-flow field owing to a strongly nonlinear nature of the MJO expected here.

Figure 2 of Kasahara and Puri (1981) and Fig. 3 of Fulton and Schubert (1985) show the vertical structure of the modes under the given mean depth for the shallow-water system. As seen therein, the mode presents an equivalent barotropic structure for the mean depth of 1–10 km. In the following, we take the mean depth of 10 km as the standard case. The same sets of experiments are also performed with the mean depths of 1 and 3 km, and the results are discussed later in the section. Note that all the other parameters are set to those of Earth’s with the reference Coriolis parameter defined by \( f_0 = 2\Omega \sin \phi_0 \). Thus, the mean depth \( H \), or equivalently the deformation radius \( L_R \), constitutes the sole free parameter in the present global shallow-water system.

For the initialization, an analytic modon solution constructed under quasigeostrophic theory is translated into a shallow-water model by using a nonlinear normal-mode balance condition (Bauer and Tribbia 1977; Kasahara 1982; Tribbia 1984b). More specifically, first, the two horizontal components of the velocity are directly obtained from the streamfunction \( \psi \) of the modon solution under the quasigeostrophic approximation. The height field is initially estimated by \( h = f^4 \psi/g \), where \( f = 2\Omega \sin \phi \) is the Coriolis parameter. Afterward,
a nonlinear normal-mode balance condition is applied to the three variables \((u, v, h)\) by expanding them in terms of the Hough functions (Baer and Tribbia 1977). Here, only the first-order initialization is applied, which consists of two steps: first, all the expansion coefficients for the inertia–gravity (IG) wave modes (including the Kelvin mode) are set to zero. Then these coefficients are redefined (i.e., initialized and balanced) in such a manner that the IG modes remain stationary under nonlinear forcing due to Rossby modes (the remaining modes). An example of the full modon solution constructed in this manner, and to be used as initial conditions for the experiments described below, is shown in Fig. 2: it constitutes an idealization of the observed vortex-pair structure (Fig. 1 and the references in the introduction).

Here, the nonlinear normal-mode balance condition not only suppresses the initial tendency for the radiation of gravity waves but also, at least partially, alleviates the wedge instability (Neven 2001) arising from the discontinuity in the gradient of the potential vorticity in the analytic modon solution.

The shallow-water model code used for the runs is also based on the Hough-function expansion (Kasahara 1977; Salby et al. 1990). All the linear computations are performed in Hough-function space. Nonlinearities are evaluated in gridpoint space and converted back to Hough-function space (i.e., transformed spectrum method) and added as an extra forcing term for the evaluations of the temporal tendencies. Linear-wave tendencies are treated analytically using exponentials in the time integration, which allows a longer time step than otherwise. Time stepping is based on a leapfrog technique with an averaging procedure every 100 time steps. Otherwise, the code contains no space–time filtering, dissipation (friction), or forcing either physically or numerically. Thus, the model investigates the pure “free” dynamics.

Computations are performed with R40 truncation in Hough-function space, being truncated at the longitudinal wavenumber \(k = 40\). As a result, 41 (including the zonal mean) and 40 modes are considered, respectively, for longitudinal and latitudinal directions each for eastward-propagating and westward-propagating IGs and Rossby waves. Thus, the total number of modes retained in the latitudinal direction is \(40 \times 3 = 120\). The given Hough-function expansion is projected onto \(128 \times 128\) grid points spanning the longitudinal and latitudinal directions. The model is run for 100 days with the time step of 0.05 h.

As reviewed in the last section, the modon solution contains the two free parameters: the phase speed \(c_p\) and a size parameter \(\phi_0\). A series of numerical experiments was performed by changing both of them.

Here, the key question asked is the stability and the longevity of the modon solution, initialized under the nonlinear balance condition as described above, and compare it with the observed MJO behavior. The results are summarized in the phase space of \(c_p\) and \(\phi_0\) in terms of the total duration of a stable propagation of the initial modon solution by Fig. 3: the modon solution is most stable for the range of the propagation speeds of 8–18 m s\(^{-1}\). Under this range, the solution remains stable for more than 80 days. These values compare somewhat favorably with the observed MJO propagation speeds of 3–6 m s\(^{-1}\), especially considering the overall simplicity of the theory. Even the slower-propagating solutions last more than 20 days so long as the phase speeds remain larger than 4 m s\(^{-1}\). This duration is still long enough for crossing the Indian Ocean. Note that obtained preferred phase speeds compare well with those obtained by the existing convectively coupled wave theories reviewed in the introduction.

The most stable size parameter of \(\phi_0 = 3^\circ\) (corresponding to a horizontal scale of 3000 km; cf. Fig. 2) also roughly agrees with the observed core size of the MJO vortex (cf. Fig. 1). The obtained preferred scale is still smaller than those obtained by most of the existing
composite studies (Hendon and Salby 1994; Kiladis et al. 2005; Adames et al. 2014; Zagar and Franzke 2015), but with the exception of Zhang and Ling (2012). Whether the discrepancy is due to the smoothing effect in composite, as we speculate, or a defect of the present theory is still to be determined by a future data analysis.

Time evolution with the initial condition given in Fig. 2, as an example of steadily eastward propagation of the modon solution, is shown in Fig. 4.

In general, a typical numerical time evolution of a modon solution may be divided into three stages: 1) steady eastward propagation of the solution as predicted by the analytic solution; 2) the modon propagation stalls, and sometimes, begins to propagate backward (westward); and 3) breakdown of the modon solution coherency, either by wave radiation (Flierl and Haines 1994) or local nonlinearity (Neven 2001). Durations until the end of the first stage are shown in Fig. 3. Durations until the end of the last two stages are shown in Fig. 5. Here, the end of each stage is judged by eye with increments of 2.5 days. Note that the transition from stage 1 to stage 3 is relatively short compared to the overall durations. The overall agreement between these three distributions suggests that the results shown in Fig. 3 are robust.

The stability characteristics of the modon solutions obtained by Fig. 3 may be qualitatively understood in the following manner. According to Eq. (1), the modon propagates slower with the larger $-K^2$ with the associated larger amplitudes, and vice versa. When the propagation speed is too fast, the modon amplitude is not large enough to maintain the structure, but disperses by wave radiation (Flierl and Haines 1994). When the propagation speed is too slow, the modon amplitude is too large, and nonlinear steepening makes it hard to maintain (Neven 2001). The modon stability dependence on the size can be understood similarly: the larger modons tend to be less stable because of wave radiation; the smaller modons tend to be unstable because of nonlinear steepening.

Finally, to see a sensitivity of the results to the choice of the mean depth, the same series of runs is repeated for the cases with mean depths of 1 and 3 km. The obtained duration diagram in the same format as in Fig. 3 but with a mean depth of 1 km is shown in Fig. 6. Arguably, the case with the 1-km depth predicts the observed phase speed better with the preferred phase speeds reducing to 2–4 m s$^{-1}$, but also with a result of a reduced stability and longevity. Also, the preferred stable scale reduces to $\phi_0 = 2^\circ$ (a horizontal scale of 2000 km). The evolution of the case with $\phi_0 = 2^\circ$ and $c_p = 4$ m s$^{-1}$ is shown in Fig. 7: the modon solution relatively stably and steadily propagates for 20 days over 60° longitude, then begins to propagate backward (westward), and after a few days suddenly breaks down. Both the duration and the distance are just enough for crossing the Indian Ocean. The results with the mean depth of 3 km shown in Fig. 8 are essentially intermediate between those with 1 and 10 km.

5. Further discussion

The present study suggests that moist convection may not be even an essential part of the dynamics of the MJO. It is shown that a simple nonlinear free Rossby wave solution can generally explain the basic characteristics of the MJO, especially the slow propagation speed, but also the observed dipolar-vortex structure, in terms of the stability of the theoretical solutions. The nonlinear nature of the MJO is already emphasized by, for example, Hagos et al. (2014), Pritchard and Bretherton (2014), and Adames and Kim (2016) in terms of moisture advection. The present study argues, in turn, that the strong nonlinearity is sufficient for explaining the essential feature of the slow eastward propagation of the MJO as well as its preferred scale. Both agreements, within a factor of 3 with the observations, are rather promising with the simplest theory adopted herein.

As discussed in section 2, the success of simulating the dry component of MJO by high-resolution models under...
channel configurations is likely due to the observed forcing provided at the latitudinal walls. Combination of observational evidences and modeling (Gustafson and Weare 2004; Ray and Zhang 2010; Ray and Li 2013; Zhao et al. 2013) suggest that the eastward-propagating Rossby wave train arriving from higher latitudes induces the MJO. Based on this picture, an earlier nonlinear experiment by Wedi and Smolarkiewicz (2010) explicitly adds the lateral forcing. On the other hand, the present result suggests that the overall feature of the MJO may be understood as a free nonlinear wave without any forcing from the higher latitudes to a leading-order approximation.

Moist convection is clearly another element that is totally neglected in the present study. Here, the present study does not suggest that moist convection is not important for the MJO. We rather simply argue that very basic overall features of MJO can be understood even without moist convection as a leading-order approximation to the full problem. As a whole, it is difficult to overemphasize the importance of such a much-simplified “prototype” model for the MJO: it would provide a solid

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**Fig. 4.** Time evolution of the equatorial modon solution with $c_p = 12 \text{ m s}^{-1}$, $\phi_0 = 3^\circ$, and the mean depth of 10 km: a case in which a steady eastward propagation is obtained. (top left) The initial condition, and time progresses with an increment of 12.5 days downward the left column and then down the right column. The perturbation height field normalized by the total depth (10 km) is shown. It is seen that the initial field, defined as in Fig. 2, simply shifts eastward with time as predicted by the theoretical modon solution.
basis for further theoretical studies as well as for better interpreting full global model results. It is furthermore a true test for actually improving the prediction of the MJO by global models. The basic purpose of the present study is to provide such a prototype.

In this respect, our proposal of the MJO as a “dry” nonlinear free solitary vortex pair may be compared to the well-established dry baroclinic instability theory for explaining midlatitude cyclogenesis. Both phenomena are clearly associated with extensive convective activities and cloud microphysics. Thus, both explanations of these phenomena using dry dynamics are counterintuitive. Indeed, there was a rather long time lapse for baroclinic instability theory to be fully accepted in the meteorological community. For example, the chapter on the synoptic-scale dynamics in Ludlam (1980) has no reference to baroclinic instability.

The present paper suggests that the MJO must first of all be considered from the perspective of dry planetary-scale dynamics, rather than that of convective forcing to planetary-scale motions. This suggestion is based on the recent results by high-resolution global channel-domain simulations as well as data analysis. Rather simple theoretical considerations herein further suggest the importance of the inherently nonlinear nature of the MJO dynamics. The scale analysis presented in appendix A further support this view in a more concrete manner. A more general tropical scale analysis (Yano and Bonazzola 2009) suggests that the same could be true for the whole tropical planetary-scale circulations.

Such a new perspective may have an immediate impact on global model initialization strategy over the tropics: the current basic strategy is an initialization based on a linear equatorial wave decomposition (cf. Zagar et al. 2005). The proposed MJO–modon theory suggests a nonlinear balance initialization (Baer and Tribbia 1977; Kasahara 1982; Tribbia 1984b) as the key for a successful MJO forecast. From a perspective of the present theory, a key for the successful MJO simulation would rather reside on preparing an initial condition that is consistent with the nonlinear balance expected from the modon theory, rather than improving (or tuning) a convection parameterization.
It is commonly considered that a planetary-scale moist convective coherency is the core of the MJO in the same sense as an intensive rainband looks like a core of midlatitude synoptic-scale weather. The present study, in turn, suggests that the core of the MJO is rather a planetary-scale solitary Rossby–vortex pair in the same sense as the core of midlatitude synoptic-scale weather is baroclinic-instability waves. As a case for the latter, it clearly becomes important to analyze the potential-vorticity budget in MJO studies both based on observations and modeling. Specifically, a key for demonstrating the validity of the modon theory would be to identify the nonlinear balance [cf. Eq. (2)] in the potential-vorticity budget as a manifestation of modon dynamics. A preliminary but positive result is already found by the scale analysis presented in appendix A. The analysis in the Hough-function space (cf. Zagar and Franzke 2015) would likely to be helpful for further elucidating the nonlinear balance in a more lucid manner.

For a lucid theoretical demonstration, various higher-order effects are neglected in the present study. An obvious next step is to move to higher-order analyses in asymptotic expansion in the MJO problem. Such an analysis would be likely to identify a necessary constraint on moist convection in order to ensure a stability of the leading-order modon solution, analogous to the solvability condition in weakly nonlinear wave dynamics. Latitudinal lateral forcing may also be included as a higher-order effect. In this manner, the present theory...
provides a solid basis for constructing a full theoretical picture of the MJO in stepwise manner, which can furthermore be used for interpreting and improving the forecasts by numerical modeling.

Further data analyses of the MJO are also clearly required from the point of view of the potential-vorticity dynamics. Especially, the discrepancy between the vortex-pair size in snapshots, as well as predicted by the present theory and those obtained by previous composite studies, is still to be resolved. We speculate that the discrepancy is due to 1) the smoothing effect of composite by taking convection as a reference variable 2) the fact that a relatively heavy spatial filter is applied in some composite studies (e.g., Hendon and Salby 1994). However, this speculation must be verified by performing an MJO composite by taking the vorticity field as a reference variable.

Moreover, the characteristic equivalent depth (mean depth in the shallow-water system) for the "dynamical" MJO must still be identified observationally. This uncertainty leads to a major uncertainty in interpreting the results of the present study. With the standard parameter with a mean depth of 10 km, the identified preferred propagation speeds, 8–18 m s\(^{-1}\), are too fast compared to the observation (3–6 m s\(^{-1}\)). However, simply by reducing the mean depth to 1 km, the preferred propagation speeds can easily be reduced to a range (2–4 m s\(^{-1}\)) that is closer to the observation. At the same time, the scale analysis in appendix A suggests that the propagation speed of MJO sensitively depends on diabatic heating associated with moist convection. These issues need to be investigated further.

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APPENDIX A

MJO Scale Analysis

The scale analysis of the MJO is presented in this appendix in support of the main text claiming that the MJO is a strongly nonlinear free wave to a leading-order approximation. Note that with only the orders of magnitude as a concern, the rounded numbers are used for the estimates throughout this appendix.

Being consistent with the basic approach in the main text, we focus on the quasigeostrophic potential-vorticity system:

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) q = f \frac{\partial}{\partial p} \left[ \frac{\left( \frac{\partial q}{\partial p} \right)}{\left( \frac{\partial q}{\partial p} \right) + \frac{\partial q}{\partial s}} \right] = \left( \frac{\partial q}{\partial t} \right)_{\text{diabatic}},
\]

(A1)

taking the pressure \( p \) as the vertical coordinate (Holton 2004, chapter 6). Here, \( \mathbf{v} \) is the horizontal velocity; \( \bar{\theta} \) and \( \bar{T} \) are the potential temperature and the temperature, respectively, for the reference state; \( C_p \) is the heat capacity at the constant pressure; and \( Q \) is the diabatic heating.

We refer to the analysis by Zhang and Ling (2012, hereafter ZL12) as an observational reference. As a minor complication, ZL12 use Ertel’s full definition of the potential vorticity, which has a different dimension than the quasigeostrophic potential vorticity. To convert the values from their analysis (potential vorticity units; 1 PVU = \( 10^{-6} \text{Kg}^{-1} \text{m}^2\text{s}^{-1} \)) to the quasigeostrophic value (s\(^{-1}\)), we divide the former by \( g \left( \frac{\partial \bar{\theta}}{\partial p} \right) \sim 10^{-2} \text{ms}^{-2} \text{KPa}^{-1} \), assuming the acceleration of the gravity \( g \sim 10 \text{m s}^{-2} \), the typical increase of the potential temperature over the troposphere \( \Delta \theta \sim 10^5 \text{K} \), and the pressure depth of the troposphere \( \Delta p \sim 10^5 \text{Pa} \). This leads to a translation rule: 1 PVU \( \sim 10^{-3} \text{s}^{-1} \).

A typical potential-vorticity value found by ZL12 associated with the MJO, which may also be estimated by relative vorticity \( \zeta \) to a good approximation according to ZL12, is \( \zeta \sim 10 \text{PVU} \sim 10^{-5} \text{s}^{-1} \). The fact that potential vorticity can be approximated well by relative vorticity suggests that the contribution of the term with the deformation radius is less important, as is the case for any flow that is close to barotropic.

Their major finding is that the local potential-vorticity tendency, \( \frac{\partial q}{\partial t} \sim \frac{\partial \zeta}{\partial \tau} \), is of the same order of magnitude as the diabatic heating [right-hand side
of Eq. (A1)], \((\partial q/\partial t)_{\text{diabatic}}\) with a magnitude of 
\(\partial \zeta/\partial t \sim (\partial q/\partial t)_{\text{diabatic}} \sim 10^{-3} \text{ PVU day}^{-1} \sim 10^{-11} \text{ s}^{-2}\) assuming 1 day \(\sim 10^5 \text{ s}\). A characteristic time scale associated with this variability is 
\(\Delta t \sim \xi/(\partial \zeta/\partial t) \sim 10^6 \text{ s} \sim 10 \text{ days}\).

Among the all the terms contributing to the potential-vorticity budget, the largest contribution arises from the beta-twisting term \(\nu \beta\) as a part of the total advection tendency \(\mathbf{v} \cdot \nabla q\). Here, \(\nu\) is the meridional velocity. Assuming a typical wind velocity to be \(U \geq 1 \text{ m s}^{-1}\), this term is estimated as \(\nu \beta \sim U \beta \geq 10^{-11} \text{ s}^{-2}\), and we also expect that the total advection tendency is also likely to be the same order of magnitude; that is,

\[
\mathbf{v} \cdot \nabla q \sim \nu \beta \sim U \beta \geq 10^{-11} \text{ s}^{-2}. \tag{A2}
\]

On the other hand, ZL12 find that the total advection \(\mathbf{v} \cdot \nabla q\) does not contribute to leading order (cf. their Figs. 6e and 9e–h). Thus,

\[
\mathbf{v} \cdot \nabla q \ll \partial \zeta/\partial t \sim 10^{-11} \text{ s}^{-2}. \tag{A3}
\]

Unfortunately, this clearly contradicts with the estimate in Eq. (A2).

The only way of avoiding this contradiction and to maintain the small total advection tendency, as suggested by Eq. (A3), is to balance the large beta-twisting term with other advection terms, notably with the advection of the relative vorticity. Thus,

\[
\mathbf{v} \cdot \nabla \zeta \sim \nu \beta
\]

or

\[
U^2 / L^2 \sim U \beta,
\]

noting that the relative vorticity scales as \(\zeta \sim U / L\) with the horizontal scale \(L\). Rearranging this order-balance condition, we obtain an estimate of the horizontal velocity as

\[
U \sim \beta L^2 \sim 10 \text{ m s}^{-1},
\]

assuming \(L \sim 10^3 \text{ km}\). Though the choice of the horizontal scale appears to be rather small, note that the larger horizontal scale leads to increasingly larger velocity scale; thus, this is probably the most observationally consistent choice. Also note that as a result, we recover the magnitude of the relative vorticity diagnosed by ZL12, \(\zeta \sim U / L \sim 10^{-5} \text{ s}^{-1}\), with the above choice.

In the main text, we have already estimated the deformation radius to be \(L_R \sim 10^3 \text{ km}\). Thus, the relative vorticity and the deformation term have the same order of magnitude of contribution to the advection. As a result, we conclude that the advection terms cancel each other to leading order and

\[
(\mathbf{v} \cdot \nabla) \left( \frac{1}{L_R} \right) \psi + \nu \beta \approx 0 \tag{A4}
\]

by following Eq. (A3). In other words, the balance

\[
J(\psi, q) = 0 \tag{A5}
\]

is satisfied for the MJO to a good approximation. This balance is exactly consistent with the basic modon equation [Eq. (2)] presented in the main text, with \(c_p = 0\).

In this manner, we demonstrate by scale analysis that the nonlinear self-advection balance is the key in the MJO dynamics, as we also assume is the main ingredient of the modon dynamics in the main text. An important corollary from the scale analysis here is that to a leading-order approximation, the MJO may be considered stationary (i.e., \(c_p \approx 0\)), and the propagation speed of the MJO is sensitively influenced by the diabatic heating according to the observational analysis by ZL12.

### APPENDIX B

#### Full Modon Solution

The modon solution outlined in section 3 is given in terms of the Gauss hypergeometric function \(H\) (cf. Abramowitz and Stegun 1965, chapter 15) by

\[
\psi_\text{m} = A_0 + A_1 H \left( a_1 + 1, 2 - a_1, 2, \frac{1 - \mu}{2} \right) \cos \phi \cos \lambda \tag{B1a}
\]

and

\[
\psi_\text{ext} = A_2 H \left( a_2 + 1, 2 - a_2, 2, \frac{1 + \mu}{2} \right) \cos \phi \cos \lambda \tag{B1b}
\]

for the interior (inside the circle of the angular radius \(\phi_0\)) and exterior, respectively.

Here, the coefficients \(A_0, A_1,\) and \(A_2\) are determined from the conditions of the continuity of the streamfunction over the boundary (a circle with the angular radius \(\phi_0\) centered at the equator) for the internal and external solutions and are given by

\[
A_0 = \frac{1 + c_p \kappa_m^2 / a}{\gamma^2 - 2}, \tag{B2a}
\]

\[
A_1 = \frac{1 + (2 + 1/L_R^2) c_p / a}{\gamma^2 - 2}
\]

\[
\times H^{-1} \left( a_1 + 1, 2 - a_1, 2, \frac{1 - \sin \phi_0}{2} \right), \quad \text{and} \tag{B2b}
\]
To further satisfy the continuity of the streamfunction over the transition boundary, \( \gamma^2 \) (henceforth \( \kappa_{in} \)) must also satisfy

\[
H \left( a_1 + 2, 3 - a_1, 3, \frac{1 - \sin \phi_0}{2} \right) \frac{H(a_1 + 2, 2 - a_1, 2, \frac{1 - \sin \phi_0}{2})}{H(a_1 + 2, 3 - a_1, 3, \frac{1 + \sin \phi_0}{2})} = \frac{H(a_2 + 2, 3 - a_2, 3, \frac{1 + \sin \phi_0}{2})}{H(a_2 + 1, 2 - a_2, 2, \frac{1 + \sin \phi_0}{2})} \quad \text{(B3)}
\]

under the given phase velocity \( c_p \) and the angular radius \( \phi_0 \) of the transition boundary.

Various additional parameters have been introduced for simplifying the expressions in this appendix. Those are

\[
a_1 = \frac{1}{2} \left[ 1 + (1 + 4\gamma^2)^{1/2} \right],
\]

\[
a_2 = \frac{1}{2} \left( 1 + (4\kappa_R^2)^{1/2} \right),
\]

and

\[
\gamma^2 = \kappa_{in}^2 - 1/L_R^2 \quad \text{and} \quad \kappa_R^2 = 1/L_R^2 + \frac{c_p}{a}.
\]

Here, the full solution assuming the center latitude at the equator is presented. The general solution with \( \phi_0 \neq 0 \) refers to the original work (Tribbia 1984a).

**APPENDIX C**

**Derivation of the Shallow-Water System from the Primitive Equation System**

The primitive equation system under the linearization reduces to the shallow-water system by applying a separation of the variables to the former (Kasahara and Puri 1981; Fulton and Schubert 1985).

For this derivation, here, we consider the primitive equation system without external forcing or diabatic heating, being consistent with the main text; thus, after the linearization by taking the pressure \( p \) as the vertical coordinate [e.g., Eq. (1) of Yano et al. 2009]. Here, \( \xi \) is the geopotential, \( k \) is the normal vector in the vertical direction, \( \alpha = RT/p \) is the specific volume defined as a deviation from a reference state with \( R \) the gas constant, \( T \) is the temperature also defined as a deviation from a reference state, \( \omega \) is the vertical velocity (the pressure velocity), and \( \sigma = -(T/\theta)d\theta/dp \) is a stratification parameter. Note that some variables are already introduced in appendix A.

We separate the dependent variables into the vertical dependence and the remaining part:

\[
v = \hat{v}(x, y, t)\Phi(p), \quad \text{(C2a)}
\]

\[
\xi = \hat{h}(x, y, t)\Phi(p), \quad \text{(C2b)}
\]

\[
\omega = \hat{\omega}(x, y, t)W(p), \quad \text{and} \quad \text{(C2c)}
\]

\[
T = \hat{T}(x, y, t)\Theta(p), \quad \text{(C2d)}
\]

where the horizontal dependences are designated by \( (x, y) \) for short.

By substitution of the separation of the variables in Eq. (C2), Eq. (C1a) simply reduces to

\[
\frac{\partial \hat{v}}{\partial t} + f k \times \hat{v} = -\nabla \hat{h}, \quad \text{(C3)}
\]

taking out the common factor \( \Phi \). Equation (C1b) reduces to

\[
\hat{h} \left( \frac{\partial \Phi}{\partial p} \right) = -\frac{R\Theta}{p} \hat{T},
\]

which suggests that we may set

\[
\hat{h} = \hat{T} \quad \text{(C4)}
\]

so that

\[
\frac{\partial \Phi}{\partial p} = -\frac{R\Theta}{p}. \quad \text{(C5)}
\]

In similar manner, Eq. (C1c) suggests that we may set

\[
\Theta = \frac{\sigma p_0}{H} W \quad \text{(C6)}
\]

so that
\[
\frac{\partial h}{\partial t} - \frac{H}{p_0} \frac{\partial h}{\partial \omega} = 0. \tag{C7}
\]

Here, we have introduced the mean depth \( H \) and a reference pressure \( p_0 \) for retaining the dimensional consistency for both \( W \) and \( \Theta \). By substituting Eq. (C6) into Eq. (C5), we obtain

\[
W = -\frac{H}{\sigma} \left( \frac{p}{p_0 R} \right) \frac{\partial \Phi}{\partial p}.
\]  

(C8)

By further substituting Eq. (C8) into Eq. (C1d), we obtain

\[
\Phi \nabla \cdot \mathbf{v} - \frac{H}{p_0} \frac{\partial}{\partial p} \left( \frac{p}{R \sigma} \frac{\partial}{\partial p} \Phi \right) + \frac{1}{H} \Phi = 0. \tag{C9a}
\]

This equation can be separated into two parts, taking into account the dimensional consistency,

\[
\frac{\partial \omega}{\partial p} = -\nabla \cdot \mathbf{v} \quad \text{and} \quad \frac{\partial}{\partial p} \left( \frac{p}{R \sigma} \frac{\partial}{\partial p} \Phi \right) + \frac{1}{H} \Phi = 0. \tag{C9b}
\]

Substitution of Eq. (C9a) into Eq. (C7) leads to the mass continuity for the shallow-water system:

\[
\frac{\partial h}{\partial t} + H \nabla \cdot \mathbf{v} = 0. \tag{C10}
\]

Note that Eqs. (C3) and (C10) constitute the linear shallow-water system with mean depth \( H \). Conceptually, the nonlinear advection terms may be further added and consider it as an analog model for full atmospheric circulation under a prescribed vertical structure. Strictly speaking, such a shallow-water analog model is justified only if the vertical structure of the flow is reasonably homogeneous. The equivalent barotropic structure assumed in the main text is consistent with this requirement. A special case for the above is when a flow is well confined to a shallow layer. For example, the ocean currents are often strongly confined to a surface layer; thus, the nonlinear shallow-water model can extensively be used for the basic investigations of the ocean dynamics (cf. Charney and Flierl 1981). For similar reasons, the nonlinear shallow-water analog modeling has been applied to the studies of the tropical-atmospheric dynamics (e.g., Yano et al. 1995).

The vertical structure under the mean depth \( H \) is defined by Eq. (C9b) separately. Vertical structure solutions \( \Phi \) with varying mean depths are shown in, for example, Fig. 2 of Kasahara and Puri (1981) and Fig. 3 of Fulton and Schubert (1985). As seen therein, so long as the mean depth is \( H \geq 1 \) km, the vertical structure is fairly close to the pure barotropic mode (i.e., equivalent barotropic).

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