Dynamics of Synoptic Eddy and Low-Frequency Flow Interaction. Part III: Baroclinic Model Results

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ABSTRACT

In this three-part study, a linear closure has been developed for the synoptic eddy and low-frequency flow (SELF) interaction and demonstrated that internal dynamics plays an important role in generating the leading low-frequency modes in the extratropical circulation anomalies during cold seasons.

In Part III, a new linearized primitive equation system is first derived for time-mean flow anomalies. The dynamical operator of the system includes a traditional part depending on the observed climatological mean state and an additional part from the SELF feedback closure utilizing the observed climatological properties of synoptic eddy activity. The latter part relates nonlocally all the anomalous eddy-forcing terms in equations of momentum, temperature, and surface pressure to the time-mean flow anomalies. Using the observational data, the closure was validated with reasonable success, and it was found that terms of the SELF feedback in the momentum and pressure equations tend to reinforce the low-frequency modes, whereas those in the thermodynamic equation tend to damp the temperature anomalies to make the leading modes equivalent barotropic. Through singular vector analysis of the linear dynamical operator, it is highlighted that the leading modes of the system resemble the observed patterns of the Arctic Oscillation, Antarctic Oscillation, and Pacific–North American pattern, in which the SELF feedback plays an essential role, consistent with the finding of the barotropic model study in Part II.

1. Introduction

In the extratropics, atmospheric variability is dominated by abundant transient synoptic eddies. Embedded in such a turbulent circulation, there are prominent recurring patterns such as the Arctic Oscillation (AO) (Thompson and Wallace 1998, 2000) or North Atlantic Oscillation (NAO) (Wallace and Gutzler 1981; Hurrell 1995; Wallace 2000), Antarctic Oscillation (AAO) (Gong and Wang 1999; Thompson and Wallace 2000), and Pacific–North American (PNA) pattern (Wallace and Gutzler 1981), which are also known as teleconnection patterns dominating in the monthly or seasonal mean anomaly fields. The AO and AAO are often referred together to as annular modes (Thompson and Wallace 2000).

Dynamical origins of these patterns of the low-frequency variability have been the central subject in many studies (e.g., Qin and Robinson 1992; Branstator 1992; Robinson 1991, 1994; Kimoto et al. 2001; Lorenz and Hartmann 2001; Feldstein 2002; Koo and Ghil 2002; Koo et al. 2002; Kravtsov et al. 2003, 2005; Watanabe and Jin 2004; Kondrashov et al. 2004), and are still in debates (Shindell et al. 1999; Baldwin and Dunkerton 1999; DeWeaver and Nigam 2000; Ambaum and Hoskins 2002). Given that earlier observational studies suggest that the transient eddy forcing associated with changes in storm tracks is the major source of the persistent low-frequency anomalies (Egger and Schilling 1983; Hoskins et al. 1983; Lau 1988, 1991;
Metz 1987; Karoly 1990; Cai and van den Dool 1991), a variety of stochastic dynamical models have been developed to parameterize the transient eddy forcing (e.g., Penland and Ghil 1993; Penland and Matrosova 1994; Newman et al. 1997; Whitaker and Sardeshmukh 1998; Zhang and Held 1999; Majda et al. 1999, 2003; DelSole 2001; Winkler et al. 2001). Whether the transient eddies serve merely as a stochastic forcing or a positive feedback to the mean flow was later questioned (Feldstein and Lee 1998). However, the increasing evidence (Hartmann and Lo 1998; Limpasuvan and Hartmann 2000; Pan and Jin 2005) indicates that transient synoptic eddy momentum fluxes associated with the low-frequency zonal flow anomaly tend to reinforce the zonal wind variation: a two-way positive feedback. It is becoming clear that the internal dynamical processes, particularly the synoptic eddy and low-frequency flow (SELF) feedback, are indispensable for the low-frequency variability (Pan 2003).

In Parts I and II of this three-part study (Jin et al. 2006a,b, hereafter Part I and Part II), we have proposed an alternative dynamical approach for describing the SELF feedback, and derived a linear SELF closure in a barotropic model (see also section 2b). The anomalous time-mean eddy forcing is linearly related to time-mean flow anomaly through the SELF closure. Based on this closure, we established a linear barotropic framework with respect to a stochastic basic flow that includes not only basic mean flow but also climatological properties of the synoptic eddy statistics. The barotropic model study reveals that the scale-selective SELF feedback is of essential importance in the origin of low-frequency modes such as the AO, AAO, and PNA, and it also can amplify the atmospheric response to remote forcing.

In this Part III, we extended the previous barotropic model study to a primitive equation framework. Section 2 describes the linear primitive equation model [referred to as the linear baroclinic model (LBM)] and presents the derivation of the general form of linear dynamic system with SELF feedback, based on observed climatological mean flow and climatological properties of the three-dimensional synoptic eddy statistics. Section 3 presents the construction of the three-dimensional stochastic flow. Section 4 gives validation of the linear SELF feedback closure using observational data. Sections 5 and 6 elucidate the role of the SELF feedback in the formation of low-frequency modes and regulating the atmospheric response to remote forcing by analyzing the leading modes of the linear dynamic operator and forced solutions to idealized forcing, respectively. The conclusions are given in section 7, whereas the details of the model equations are provided in the appendix.

2. A primitive equation framework with SELF feedback

a. The primitive equation model

In this study, we develop the primitive equation framework from the LBM used in earlier studies (Watanabe and Kimoto 2000, 2001; Watanabe and Jin 2004), and the detailed equations are given in the appendix. The model variables consist of vorticity (ξ), divergence (D), temperature (T), and logarithm of surface pressure (π). We choose a version with five vertical levels (σ = 0.8987, 0.6983, 0.4439, 0.2220, 0.06224), and a T21 spectral resolution. Furthermore, we only consider zonal wavenumbers 0–15 to reduce the computational burden even though the total degrees of freedom are 7056, still quite large. Three dissipation terms are included: a biharmonic horizontal diffusion with the damping time scale of one day for the smallest wave, a weak vertical diffusion (damping time scale of 1000 days), and the Newtonian damping and Rayleigh friction. The latter is represented by a linear drag that has a one-day damping at the lowest layer while the damping rate is fifteen days elsewhere. This linear damping is adequate to suppress the unstable growth of the linear baroclinic waves (Hall and Sardeshmukh 1998).

b. The closure for SELF feedback

The concepts and approaches described in Parts I and II for deriving the SELF feedback closure in barotropic model can be naturally extended into a general primitive equation framework. For simplicity, we let \( \mathbf{x} \) represent state vector of the variables in the baroclinic model, and separate it into three parts:

\[
\mathbf{x} = \mathbf{x}^c + \mathbf{x}^a + \mathbf{x}',
\]

where the superscripts \( a \) and \( c \) stand for anomaly and climatology, respectively. The prime indicates high-frequency variability, and the overbar represents time mean such as monthly mean. Thus \( \mathbf{x}' \) is the climatological mean state obtained through long-term average of the observations; \( \mathbf{x}^a \) denotes the time-mean flow anomaly, which we will loosely refer to as the low-frequency anomaly; and \( \mathbf{x}^c \) represents the high-frequency transient eddy field.

We write symbolically the linearized equation for the low-frequency variability \( \mathbf{x}^a \) as follows:

\[
\frac{\partial}{\partial t} \mathbf{x}^a + \mathbf{L}(\mathbf{x}')\mathbf{x}^a = \mathbf{A}(\mathbf{x})\mathbf{x}^a + \mathbf{Q}^a,
\]
where the detailed equations are given in the appendix. Here, \( \mathbf{L} \) is a linear operator depending on climatological basic state, \( \overline{Q^a} \) denotes time-mean anomalies in the external forcing. This equation extends the simple barotropic model of Part I to a primitive equation model. The anomalous transient eddy forcing term \( \overline{A(x)\overline{Q^a}} \) is defined as the difference between the time-mean (such as monthly mean) eddy forcing \( \overline{A(x)\overline{Q}} \) and its long-term climatological mean \( \overline{A(x)\overline{Q}} \). Without \( \overline{A(x)\overline{Q}} \), Eq. (2) reduces to the conventional LBM (e.g., Hoskins and Karoly 1981).

Similar to Part I, we assume that the anomalous time-mean eddy forcing can be linearly related to time-mean flow anomaly through a SELF feedback closure. In the primitive equation model, this closure can be written as

\[
\overline{A(x)\overline{Q^a}} = -\mathbf{L}_f \overline{Q^a},
\]

where \( \mathbf{L}_f \) is the closure operator for the SELF feedback. In the next section, we will derive an explicit, nonlocal form of the closure operator \( \mathbf{L}_f \) in the primitive equation model.

c. The derivation of the closure

We consider a hypothetical ensemble of the quasi-stationary stochastic state, \( \mathbf{X} \), in which the observed atmospheric circulation \( \mathbf{x} \) is regarded as one particular realization. Let angle brackets represent the ensemble mean. Then \( \mathbf{X} \) can be decomposed as

\[
\mathbf{X} = \langle \mathbf{X} \rangle + \mathbf{X}',
\]

where \( \mathbf{X}' \) represents the stochastic eddy field in which \( \mathbf{x}' \) is one particular realization as well. With an ergodic assumption (see section 3 of Part I), the ensemble-mean quantities can be substituted by the time means of one realization as

\[
\langle \mathbf{X} \rangle \approx \overline{\mathbf{x}} = \overline{\mathbf{x}}^c + \overline{\mathbf{x}}^a.
\]

As in Part I, the stochastic eddy field \( \mathbf{X}' \) is separated into two parts: a major but stochastic stationary component \( \mathbf{X}'_s \) that characterizes the climatological properties of the observed storm track and a minor but quasi-stationary stochastic component \( \mathbf{X}'_i \) that represents the synoptic eddies modulated by the low-frequency anomaly. Thus \( \mathbf{X}' \) can be expressed as

\[
\mathbf{X}' = \mathbf{X}'_s + \mathbf{X}'_i.
\]

Based on the ergodic assumption, the time mean of eddy flux term, its climatology, and low-frequency anomaly can be approximated by the corresponding ensemble means based on \( \mathbf{X}' \); namely,

\[
\overline{A(x)\overline{Q^a}} = \langle A(X'_s)\overline{Q^a} \rangle,
\]

\[
\overline{A(x)\overline{Q^a}} = \langle A(X'_s)\overline{Q^a} \rangle + \langle A(X'_i)\overline{Q^a} \rangle = \langle L_f(X'_i)\overline{Q^a} \rangle,
\]

where \( \mathbf{L}_f \) is a linear operator depending on \( \mathbf{X}'_i \). In Eq. (7b) we have neglected the nonlinear term \( \langle A(X'_i)\overline{Q^a} \rangle \) by assuming that it is relatively small. Using Eq. (7b), Eq. (2) becomes

\[
\frac{\partial}{\partial t} \overline{Q^a} + \mathbf{L}(\overline{Q^a})\overline{Q^a} = \mathbf{L}_f(X'_i)\overline{Q^a} + \overline{Q^a},
\]

To obtain the closure for the term representing the ensemble-mean anomalies of synoptic eddy forcing in Eq. (8), we use the following linearized equation for \( \mathbf{X}'_i \) (refer to the appendix for the detailed derivation),

\[
\frac{\partial}{\partial t} \mathbf{X}'_i + \mathbf{L}(\overline{Q^a})\mathbf{X}'_i = \mathbf{L}_f(X'_i)\overline{Q^a} + \mathbf{Q}'_i,
\]

where all nonlinear terms are neglected. This equation is an extension of the barotropic version of Eq. (8) in Part I. For simplicity, we also omit the high-order temporal terms in the transient flow equation. Equations (8) and (9) now form a coupled dynamic system that describes the interaction between the low-frequency anomaly \( \mathbf{Q}^a \) and anomalous synoptic eddies \( \mathbf{Q}'_i \).

For a given \( \mathbf{X}^c \), Eq. (9) is a linear model of anomalous storm track activities. Our approach for modeling anomalous storm tracks using Eq. (9) avoids the time integration for the basic stochastic eddy–flow ensemble \( \mathbf{X}'_i \). Instead, the basic stochastic eddy state is prescribed as a part of the climatological state, which enables us to explicitly solve Eq. (9) to obtain a closure for Eq. (8).

Recall that \( \mathbf{X}'_i \) represents typical, synoptic traveling disturbances, and the stochastic high-frequency state may be expressed as

\[
\mathbf{X}'_i(\lambda, \varphi, \sigma, t) = \sum_{n=1}^{N_0} \alpha_n^2 \mathbf{r}_n(t) \mathbf{X}_n(\lambda, \varphi, \sigma) e^{-i\omega_n t} + \text{c.c.},
\]

where c.c. stands for complex conjugate; \( \lambda, \varphi, \) and \( \sigma \) represent longitude, latitude and vertical level, respectively. Each \( \mathbf{X}_n(\lambda, \varphi, \sigma) e^{-i\omega_n t} \) is a pair of complex empirical orthogonal function (CEOF) patterns associated with the angle phase speed \( \omega_n \) and variance \( \alpha_n^2 \), all obtained from the observation. As in Part I, each \( \mathbf{r}_n(t) \) is a normalized and independent red noise process [\( d\mathbf{r}_n/dt = -\xi_d \tau_n + w_n(t) \); \( w_n(t) \) is a white noise] with the decorrelation time scale \( \tau_n \), which is estimated from the \( e \)-folding time of the autocorrelations of each CEOF time series. In Eq. (10), \( N_0 \) is the number of CEOF modes used. The details of the CEOF will be given in section 3.
With the definition of $X_i$ given in Eq. (10), the solution for the anomalous high-frequency ensemble has the following form:

$$X_i(\lambda, \varphi, \sigma, t) = \sum_{n=1}^{N} X_n(\lambda, \varphi, \sigma, t)e^{-i\omega_n t} + \text{c.c.} \quad (11)$$

Then using Eqs. (10) and (11) and defining

$$\tilde{X}_n = \langle \tilde{\xi}_0(t) X_n \rangle,$$  

where the tilde means complex conjugate, we have

$$\langle \text{L}_f(\tilde{X}_n)^\dagger \tilde{X}_n \rangle = \sum_{n=1}^{N} \alpha_n^2 \langle \text{L}_f(\tilde{X}_n)^\dagger \tilde{X}_n \rangle + \text{c.c.}. \quad (13)$$

Multiplying $\tilde{\xi}_0(t)$ to both sides of Eq. (9) and taking the ensemble average, we obtain

$$\{\tau_n^{-1} - i\omega_n + L(\overline{X})\} \tilde{X}_n = \text{L}_f(\tilde{\xi}_0)\overline{X}_n. \quad (14)$$

Here we used the relation

$$-\frac{d}{dt} \frac{\delta X_i}{\delta t} + \frac{\delta \tilde{X}_0}{\delta t} + \frac{\tilde{\xi}_0(t)}{\tau_n} \sim \frac{\tilde{\xi}_0(t) X_n}{\tau_n}, \quad (15)$$

where $d\xi_n/dt = -\xi_n/\tau_n + w_n(t)$, the definition of $\xi_n(t)$, is used. We assumed $d\tilde{X}_n/dt = 0$ following the quasi-equilibrium assumption (see appendix A of Part I).

Combining Eqs. (8), (13), and (14), we obtain an explicit form of the closed system for the low-frequency anomalies:

$$\frac{\partial}{\partial t} \overline{X'} + L(\overline{X'})\overline{X'} = -L(\overline{X'}) + Q', \quad (16a)$$

$$\text{L}_f \overline{X'} = -\sum_{n=1}^{N} \alpha_n^2 \text{L}_f(\tilde{X}_n)^\dagger(\tau_n^{-1} - i\omega_n)$$

$$+ L(\overline{X'})^{-1} \text{L}_f(\tilde{X}_n)\overline{X'} + \text{c.c.}, \quad (16b)$$

where the operator $\text{L}_f$ represents a linear, nonlocal closure for the SELF interaction. In Eq. (16), the dynamical impact of the climatological stationary waves on the low-frequency variability is included in the conventional linear operator $L$. The impact of climatological properties (such as its typical spatial pattern, variance, typical lifetime, and phase speed) of the synoptic eddies on the low-frequency variability is included in the SELF feedback closure operator $L_f$. The one and only difference between Eq. (16) and the traditional LBM is the SELF feedback closure. With the SELF feedback, Eq. (16) becomes a new baroclinic framework for the low-frequency variability. Since this is an anomaly modeling approach, the climatological properties of the storm track is externally given as a part of the basic state, but the small change in the storm track associated with the low-frequency anomaly is dynamically determined in the system.

The system (16) is symbolically identical to the barotropic counterpart described in Part I, which in fact mounts to a conceptual difference between the traditional and this extended linear framework, as schematically illustrated by Fig. 4 of Part I. In a traditional LBM framework, linear dynamics of the low-frequency variability is controlled by the climatological mean flow and other sources for low-frequency variability, including eddy forcing, are all considered as external forcing. In the extended LBM framework (16), the climatological properties of synoptic eddies also become important elements in generating low-frequency modes of variability.

3. Properties of a three-dimensional stochastic synoptic eddy field

a. Reconstruction of three-dimensional synoptic eddy fields

To ensure that the stochastic basic state as expressed by Eq. (10) captures the basic nature of observed synoptic eddies, we take an approach similar to Part I (see also Pan 2003), but extend it to three-dimensional eddy fields. We adopt a CEOF decomposition method (Barnett 1983; Horel 1984) to derive the basic patterns and associated parameters that characterize the climatological storm track. The primary dataset is the National Centers for Environmental Prediction–National Center for Atmospheric Research (NCEP–NCAR) reanalysis (Kalnay et al. 1996). It includes daily data of zonal wind ($u$), meridional wind ($v$), temperature ($T$), and surface pressure ($p_s$) during the cold season (November–April for the Northern Hemisphere and May–October for the Southern Hemisphere) between the years of 1979 and 1995. A bandpass filter retaining 2–8 days is applied to these fields in order to extract transient baroclinic eddies. Each variable of the transient fields is weighted by its global-mean variance, horizontal area, and vertical mass distribution before performing the CEOF analysis.

As in the previous section, let $x'$ represent the filtered fields ($u', v', T', p_s'$), then the CEOF provides the following expansion,

$$x'(\lambda, \varphi, \sigma, t) = \sum_{n=1}^{N} X_n(\lambda, \varphi, \sigma, t)Y_n(t) + \text{c.c.}$$

$$= \sum_{n=1}^{N} X_n(\lambda, \varphi, \sigma, t)\omega_n e^{-i\phi_n(t)} + \text{c.c.}, \quad (17)$$

$$\phi_n(t) = \arctan\left[ \frac{\text{Im} Y_n(t)}{\text{Re} Y_n(t)} \right] = -\omega_n t + \phi_n(t), \quad (18)$$

$$\alpha_n^2 = \frac{\text{Var} \left[ X_n(\lambda, \varphi, \sigma, t) \right]}{\text{Var} \left[ Y_n(t) \right]}, \quad (19)$$
where $X_n(\lambda, \varphi, \sigma)$ and $Y_n(t)$ are the $n$th complex EOF pattern and time coefficient, respectively. From the real and imaginary parts of $Y_n(t)$, we obtain the angle phase $\phi_n(t)$, which is further decomposed into a linearly fitted phase propagation $-\omega_n t$ and a small residual $\phi'_n(t)$. The absolute of the time coefficient, $Y_n(t)Y'_n(t)$, is used to compute the variance $\alpha^2_n$ and $\varepsilon$-folding decay time ($\tau_n$), the latter being estimated from the autocorrelation function. Through the above analysis, high-frequency fields are approximately described in terms of variance $\alpha^2_n$, primary frequency $\omega_n$, spatial structure $X_n(\lambda, \varphi, \sigma)$, and the decorrelation time $\tau_n$. Using the set of $(X_n, \alpha_n, \omega_n, \tau_n, n = 1, N_e)$ to capture the climatological properties of the observed synoptic eddy field, the stochastic synoptic eddy field $X'_n$ in Eq. (10) is reconstructed. It should be noted that the constructed field of $X'_n$ is different from in two senses: the CEOFs are truncated at $N_e$ in constructing $X'_n$ besides the phase, amplitude, and decay time are fixed at constant for each CEOF. Consequently, the constructed $X'_n$ excludes an irregular, or anomalous, part of the high-frequency states, which will be internally solved in the SELF closure.

Figure 1a presents the spatial structure for the real part of the first CEOF as represented by the meridional wind $v'$ during the cold season of the Northern Hemisphere (NH). There are approximately 4–5 negative/positive centers in the $v'$ field, located mainly over the Pacific and the Atlantic regions. The first CEOF accounts for 10% of the total variance. The pattern for the imaginary part is similar to the real part but is shifted eastward around 15° in longitude and the phase difference is 90°. The results for other variables, such as $u'$, $T'$, and $p'_z$, are similar [not shown here, for details see Pan (2003)]. The wavelike structure in the velocity field tilts slightly to the west in the vertical cross section (Fig. 1b) and the $T'$ field tilts slightly eastward (not shown).

The relative position between $p'_z$, $u'$, $v'$, and $T'$ at the lowest level ($\sigma = 0.8987$) is shown in Fig. 1c. Warm temperature anomalies are located on the southeast side of the low pressure center, where the southerly anomaly dominates. This structure of synoptic eddies is consistent with those of typical cyclones and anticyclones. The second CEOF, which accounts about 8% of the total variance, shares many common features with the first CEOF. Similar results are also obtained for the Southern Hemisphere (SH; not shown).

b. Representation of the climatological eddy forcing

We now examine whether the constructed stochastic eddy states capture the climatological eddy forcing fields. On one hand, we calculate the climatological eddy forcing term $\bar{A}(\bar{x})\bar{x}^f$ directly using the observed data of the high-frequency fields. On the other hand, we can use the reconstructed stochastic synoptic field $X'_n$ to evaluate the ensemble mean of the climatological eddy forcing $\langle A(X'_n)X'_n \rangle$. Then we will compare $\langle A(X'_n)X'_n \rangle$ with $\bar{A}(\bar{x})\bar{x}^f$ to verify that $X'_n$ captures the climatological properties of synoptic eddies. Using the orthogonality of the CEOF time coefficients in Eq. (17), we have

$$\bar{A}(\bar{x})\bar{x}^f = \sum_{n=1}^{N_e} \bar{A}(X'_n)\bar{X}^f_n + \text{c.c.}$$

Owing to the assumption of independence of each complex red noise processes in Eq. (10), $\langle A(X'_n)X'_n \rangle$ can be expressed as

$$\langle A(X'_n)X'_n \rangle = \sum_{n=1}^{N_e} \bar{A}(X'_n)\bar{X}^f_n + \text{c.c.},$$

which includes the eddy-induced forcing term for the vorticity, divergence, temperature and surface pressure.
fields. The details of these terms are given in Eqs. (A7), (A10), (A11), and (A12) in the appendix.

Thus the difference between $\Delta(x)\Delta^\tau$ and $\langle A(X)X\rangle$ is mainly due to the truncation error, namely $N_c < N$. By setting $N_c = 20$, the majority of mean eddy variance and eddy fluxes is well captured. The vorticity eddy forcing terms are expressed in terms of the streamfunction tendency by operating an inverse Laplacian factor. For the NH, the climatological eddy vorticity forcing (Figs. 2a and 2b) is positive in the polar region at the lower level and negative at the upper level. The eddy thermal forcing has a north–south dipole structure with positive value in the north and negative value in the south, corresponding to a northward eddy heat transport (Figs. 2c and 2d). The results are consistent with previous studies (e.g., Lau and Holopainen 1984). Similar results are also obtained for surface pressure eddy forcing (Figs. 2e and 2f). All main centers of these eddy forcing fields (Figs. 2a–f) are located over the Pacific and Atlantic storm track regions, respectively. The results for the SH are very similar except that the eddy forcing terms are more zonally symmetric. Thus with a relative small number of $N_c$, the stochastic representation of the synoptic eddies captures well the climatological eddy forcing fields.

4. The validation of the SELF feedback closure

As expressed in Eq. (3), we can relate the anomalous eddy forcing directly to low-frequency flow anomalies by the SELF feedback operator $L_F$, which is derived through a series of assumptions and approximations and depends only on the climatological mean state and climatological measures of the synoptic eddies $(X_c, \alpha_c, \omega_h, \tau_f; N = 1, N_c)$. It uses neither any information about observed anomalous eddy forcing $A(x)\Delta^\tau$ nor information about observed low-frequency variability of $x\Delta$, so the feedback operator $L_F$ has no direct built-in answer about the observed anomalous eddy activity and low-frequency anomalies. Therefore, we can validate this dynamically derived SELF feedback operator by applying $L_F$ onto the observed $x\Delta$ to calculate parameterized eddy forcing as $L_Fx\Delta$ and then comparing it with the observed eddy-forcing anomalies.

As in Part I, tests are conducted with the focus on the leading low-frequency modes and the associated eddy forcing patterns. We first regressed the anomalous transient eddy forcing $A(x)\Delta^\tau$, calculated directly from the filtered data of $x$, onto the monthly means of indices for low-frequency variability such as the AO and AAO during the cold season. The definitions for the AO and AAO indices are the same as those defined by Thompson and Wallace (1998) and Gong and Wang (1999), respectively. A similar regression is made to the monthly mean anomaly field $x\Delta$, which represents the three-dimensional structure of the AO and AAO from the observation. Then we calculate the parameterized eddy-forcing term $-L_F\Delta$ by applying the SELF feedback operator on the regressed low-frequency patterns.

The observed and parameterized transient eddy forcing fields for streamfunction, temperature, and the logarithm of surface pressure associated with the AO resemble each other reasonably well (Figs. 3a–c). Both parameterized and observed streamfunction forcing (Fig. 3a) have negative and positive values at low and middle latitudes, respectively, with the magnitude stronger in the upper troposphere. The forcing at low
levels of the polar region is slightly positive. The regressed streamfunction anomaly field associated with the AO (Figs. 4a and 4b) correlates positively with the tendency fields induced by synoptic eddies. Given that the observed eddy-forcing pattern can be reproduced as the response to the AO in the dynamical closure, this strongly suggests a positive SELF feedback in the vorticity equation. The temperature forcing field is positive at high-latitude regions, negative at middle-latitude regions, and slightly positive at low latitudes (Fig. 3b). The regressed temperature anomaly field associated with the AO (Fig. 4c) correlates negatively with the temperature tendency field induced by synoptic eddies. Thus, the transient eddy forcing has a damping effect on the meridional temperature gradient associated with the AO. This indicates that the eddy forcing in the temperature equation tends to reduce the vertical shear of zonal flow. As the result, eddy forcing tends to make
the mode to become equivalent barotropic in its vertical structure. The eddy forcing for the surface pressure field is positive at midlatitudes and negative at high latitudes, which also correlates positively with the regression pattern for surface pressure anomaly corresponding to the AO (Fig. 4d). This also mounts to the positive eddy feedback for the AO. Overall, the SELF feedback operator captures the different roles of the eddy feedback for different components of the AO mode.

The results for the AAO (Figs. 5a–c) are essentially similar. The observed eddy forcing is well captured by the SELF feedback closure. The synoptic eddy-induced streamfunction tendency also shows a clear zonal mean component, with the positive tendency near the polar region and a negative tendency in the middle latitude, serving as a positive eddy feedback to reinforce the streamfunction field of the AAO (Fig. 5a). The eddy forcing in temperature equation again tends to reduce the north–south gradient (Fig. 5b). Compared with the NH, the transient eddy forcing for the SH is clearly more zonally distributed, owing to the zonally uniform distributions of the mean storm track. Similar validation and analysis are also conducted for the PNA, which also indicate the SELF feedback closure captures reasonably well the impact of the eddy feedback on this dominating low-frequency flow pattern (not shown).

5. The leading modes in the linear dynamical system

In this section, we further calculate the singular vectors of the linear operator, \( L(\mathbf{x}) + L_\mathbf{f} \), by means of the singular value decomposition (SVD; Navarra 1993; Kimoto et al. 2001; Watanabe and Jin 2004), to examine the leading low-frequency modes of the baroclinic model. We will examine separately the role of the SELF feedback as well as the stationary waves of the basic state in the leading dynamical modes.

The leading singular mode of the linear matrix for the NH cold season (Fig. 6), defined as a mode having the smallest singular value hence corresponding to the near-neutral mode, is found to be of similar patterns to the observed AO (Fig. 4). The patterns of streamfunction anomalies associated with the leading mode at different vertical levels (Figs. 6a and 6b) generally well resemble those observed (Figs. 4a and 4b). In particular, the dipole structure is locally pronounced over the Atlantic sector, as it is observed. This leading mode has an equivalent barotropic structure in the vertical (not shown), which is also consistent with observations (Thompson and Wallace 2000). Both the simulated
(Fig. 6c) and observed (Fig. 4c) temperature anomalies decrease with height. The surface pressure anomaly pattern of the leading singular mode (Fig. 6d) also closely resembles the observed pattern of the AO (Fig. 4d), with a negative anomaly over the polar region, a strong positive center over the North Atlantic, and weak one at Pacific regions.

The second leading mode under the NH cold season background condition has a wavelike pattern (not shown), which does not appear to be closely related to the observed leading low-frequency modes. However, third singular mode is reminiscent of the PNA pattern, having a wave-train type pattern located mainly over the Pacific sector with four action centers over the North Pacific and North America (Fig. 7). A great part of the variability of the PNA is directly related to remote forcing from Tropics, nevertheless, it is also known that even in the absence of external forcing, PNA-like pattern with significant amount of low-frequency variance is still simulated in atmospheric models (Straus and Shukla 2002). That the PNA-like pattern appears as one of the leading modes of the baroclinic model supports the notion that internal dynamics, particularly the SELF feedback, plays a significant role in the formation of the PNA pattern.

The leading singular mode of the linear matrix for the SH cold season is also similar to the observed AAO (Figs. 8 and 9). The main features of AAO are well reproduced by the leading mode of linear system with the SELF feedback. For instance, both the leading mode and observed AAO have equivalent barotropic structures in the streamfunction field (Figs. 8a,b and 9a,b). The observed and simulated patterns of temperature fields have negative anomalies in the polar region and positive anomalies in the extrapolar region (Figs. 8c and 9c). The surface pressure anomaly of the leading singular mode is also similar to that of the observed AAO (Figs. 8d and 9d).

To isolate the effect of the SELF feedback, we set it to zero and recalculate the singular vectors of the conventional linear operator \( L(x_c) \). The leading mode of \( L(x_c) \) from traditional LBM under the SH cold season...
background conditions bears no resemblance with the AAO (not shown), which is also consistent with the barotropic study in Part II. The leading mode of under the NH winter climate state does have dipolelike zonal mean flow anomalies and bears some similarity to the AO pattern, which is consistent with the other studies (Limpasuvan and Hartmann 2000; Kimoto et al. 2001; Watanabe and Jin 2004) who found that stationary waves in the nonzonal basic flow may contribute to the formation of AO. Yet, the spatial pattern of this leading mode (Fig. 10) without SELF feedback resembles the observed AO mode to a significantly less degree than Figs. 6a and 6b, suggesting the dominating impact of the SELF feedback on the formation of the AO mode.

To further demonstrate the effect of the SELF feedback, we deliberately suppress the zonally asymmetric part of the climatological mean state (i.e., stationary waves) in the linear operator, but still retain the full SELF feedback operator, which depends on the entire full basic flow and basic eddy properties. In this case, we found that the AO- and AAO-like modes still appear as the leading singular modes of the system (Figs. 11–12) of the dynamical system. A PNA-like mode also exists in the system as the third mode (not shown). The strong regional characteristics of the AO in the Atlantic region, for instance, are thus almost exclusively related to the storm track captured in \( \mathbf{L}_f \). In other words, these localized features of AO and AAO are generated by the SELF feedback in the storm track region (see also Part II; Pan 2003).

In summary, our analysis indicates that the SELF feedback is of essential importance in forming the AO-, AAO-, and PNA-like leading modes. The baroclinic model results show that the main conclusions drawn in the barotropic study are valid. This is because SELF feedback in the baroclinic framework tends to naturally select the leading modes with the equivalent barotropic structure. One indication that SELF feedback favors the equivalent barotropic structure is the fact that eddy terms tend to damp temperature anomalies of the leading mode, as discussed in section 4. However, more in-depth study is needed to understand this effect of the SELF feedback.

6. Role of SELF feedback in teleconnections

Some of the teleconnection patterns in the extratropics are known to be excited by tropical forcing (Hosk-
ins and Karoly 1981; Jin and Hoskins 1995). However, the eddy forcing also plays an important role in the middle-latitude response to such remote forcing (e.g., Held et al. 1989; Ting and Peng 1995; Peng et al. 2005). With the proposed SELF closure in the primitive equation model, we revisit the role of SELF feedback in regulating the atmospheric teleconnection.

A forced experiment is conducted with Eq. (16) by putting a diabatic heating over the central Pacific ($0^\circ$, $140^\circ$W). The vertical heating distribution in the tropical region follows the observed result by Yanai and Tomita (1998), which has a top-heavy profile. The remote response in midlatitude (Fig. 13a) is modest without the eddy feedback ($L = 0$), but significantly enhanced when the eddy feedback is included (Fig. 13b), which is consistent with previous numerical studies (e.g., Held et al. 1989). The difference between the responses with and without the SELF feedback, as shown in Fig. 13c, has some projections onto the PNA- and NAO-like patterns. This result is consistent with our findings in the barotropic model (see Part I). Namely, the external forcing excites Rossby wave trains that tend to perturb the synoptic eddy activity in the stormy basic flow in an organized manner to intensify the remote response. Therefore, the SELF feedback is an important part of the relay for the atmospheric teleconnections.

7. Summary and concluding remarks

In this three-part study, we proposed a linear dynamical framework with respect to a stochastic basic flow and derived a linear SELF closure to investigate the dynamics of the low-frequency variability of the atmospheric circulation. The approach relies on separating the quadratic, eddy–eddy interaction term into a climatological component and an anomalous one; the anomalous component presumably varies on a slow time scale and slaved by the large-scale low-frequency flow. The key idea is to prescribe the climatological mean flow and climatological properties of the synoptic eddy flow and then to provide a linear closure between
the anomalous synoptic eddy activity and the anomalous low-frequency flow.

We laid out in Part I, with a barotropic framework, the basic steps for establishing a linear dynamic closure for the SELF feedback. After this linear closure is implemented and tested in Part I using a barotropic model, we analyzed in Part II leading singular modes of this barotropic model with SELF feedback and their relevance to the observed low-frequency modes, and further provided a theoretical analysis of the SELF feedback.

In Part III, we demonstrate that the barotropic model study of Parts I and II can be naturally extended to the primitive equation model. We first derived the general forms of the linear dynamic equations with the SELF feedback closure, which depends on observed climatological mean state and observed climatological properties of synoptic eddy activity. We used the CEOF decomposition to the observed bandpass-filtered data to attain the climatological properties of the synoptic eddies, such as the spatial structure, variance, e-folding time scale, and propagation speeds.

We validated the nonlocal dynamical SELF feedback closure by using the observational data. The observed anomalous three-dimensional synoptic eddy-forcing fields associated with the AO and AAO are reasonably reproduced by the SELF feedback closure. The observed positive feedbacks associated with the momentum and surface pressure eddy forcing and negative feedback associated with eddy thermal forcing are successfully captured by the SELF feedback closure. The feedback associated with eddy–heat forcing tends to mainly weaken the baroclinic structure of the low-frequency modes, which may be responsible for their equivalent barotropic structures.

We further performed SVD analysis using the linear baroclinic model to investigate the role of the SELF feedback in forming the leading low-frequency modes. Our analyses show that both AO- and AAO-like modes are the leading internal modes of the atmospheric low-frequency dynamic system in each hemisphere under cold season background conditions, whereas the PNA-like pattern emerges as the third SVD mode in the NH. With the full SELF feedback in the model, the AO-, AAO-, and PNA-like modes remain as the leading modes of the system even when the zonal asymmetric part of the climatological mean state is removed. Thus, the SELF feedback is of essential importance in the formation of the leading low-frequency modes of the extratropical circulations. Moreover, all the leading modes are of equivalent barotropic structure, and thus the main findings on the dynamical origin of the low-frequency modes from this baroclinic model study and the barotropic model study in the Parts I and II are consistent. Both barotropic and baroclinic model results showed that the SELF feedback significantly amplifies the middle-latitude atmospheric responses to the tropical forcing.

Our finding that the linear dynamic system with SELF feedback closure can produce basic tropospheric features of the AO and AAO suggests that the fundamental patterns of these low-frequency modes are largely controlled by internal dynamics within the troposphere. However, it has been noted that the annular modes (or AO and AAO) are influenced by interacting with the stratosphere (Baldwin and Dunkerton 1999, 2001; Shindell et al. 1999; Ambaum and Hoskins 2002).

Owing to the limited vertical resolution, our current model framework is inadequate to address this issue; however, with higher resolutions, we will further investigate this problem. The new linear framework developed in this study may also be used to investigate the possible role of the extratropical ocean–atmosphere interaction in the dynamics of the extratropical low-frequency variability. Moreover, nonlinear corrections to the SELF feedback closure, other nonlinearities of the slow dynamics, feedbacks between dynamics and physical processes, may also be incorporated into this framework for more comprehensive studies of the low-frequency variability.

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APPENDIX

Model Equations

The primitive equation model employs \( \sigma = p/p_s \) coordinate and has four variables \((\pi, \xi, D, T)\). The continuity equation in terms of the logarithm of surface pressure \((\pi = \ln p_s)\) is

\[
\frac{\partial \pi}{\partial t} + V \cdot \nabla \pi + \nabla \cdot V + \frac{\partial \sigma}{\partial \sigma} = 0, \quad (A1)
\]

where \( \sigma \) is vertical velocity, \( V = (u, v) \) the zonal and meridional velocity.
Each variable is separated into three terms as in Eq. (1). The continuity equation can be rewritten as

\[
\frac{\partial (\pi' + \bar{\pi}^e + \bar{\pi}^a)}{\partial t} = -(\mathbf{V}' + \mathbf{V}^a + \mathbf{V}^c) \cdot \nabla (\pi' + \bar{\pi}^e + \bar{\pi}^a) - \nabla \cdot (\mathbf{V}' + \mathbf{V}^a + \mathbf{V}^c) - \frac{\partial (\sigma' + \bar{\sigma}^e + \bar{\sigma}^a)}{\partial \sigma}.
\]  

(A2)

By assuming that the basic flow term is much larger than the anomaly term for each variable, we can get the climatological equation after performing the long-term average,

\[
\mathbf{V}' \cdot \nabla \bar{\pi}^e + \nabla \cdot \mathbf{V}' = -\nabla \cdot \nabla \bar{\pi}^e.
\]  

(A3)

Taking a time average such as monthly (or seasonal) average and removing the climatology part, we obtain the linearized anomaly equations for the low-frequency variability:

\[
\frac{\partial \bar{\pi}^a}{\partial t} + \bar{\mathbf{V}}^a \cdot \nabla \bar{\pi}^e + \bar{\mathbf{V}}^e \cdot \nabla \bar{\pi}^a + \nabla \cdot \bar{\mathbf{V}}^a + \frac{\partial \bar{\sigma}^a}{\partial \sigma} = -\nabla \cdot \nabla \bar{\pi}^a.
\]  

(A4)

Based on the discussion in section 2, we use roman and bold italic variables below to distinguish the stochastic eddy state such as \( \mathbf{V}' \) from its one particular realization \( \mathbf{V} \). Let

\[
\mathbf{V}' = \mathbf{V}' + \mathbf{V}'_a, \quad \pi' = \pi' + \pi'_a
\]  

(A5)

and consider the ergotic approximation, we have

\[
\mathbf{V}' \cdot \nabla \bar{\pi}^e \approx \langle \mathbf{V}' \cdot \nabla \pi'_a \rangle, \quad \mathbf{V}' \cdot \nabla \bar{\pi}^a \approx \langle \mathbf{V}' \cdot \nabla \pi'_a \rangle + \langle \mathbf{V}' \cdot \nabla \pi'_a \rangle.
\]  

(A6)

where \( \mathbf{V}'_a, \pi'_a, \pi'_a \) and \( \pi'_a \) are the basic and anomalous stochastic states. Substituting Eq. (A6) into Eqs. (A3) and (A4), we have

\[
\bar{\mathbf{V}}^e \cdot \nabla \bar{\pi}^e + \nabla \cdot \bar{\mathbf{V}}^e + \frac{\partial \bar{\sigma}^e}{\partial \sigma} = -\langle \mathbf{V}' \cdot \nabla \pi'_a \rangle, 
\]  

(A7)

\[
\frac{\partial \bar{\pi}^a}{\partial t} + \bar{\mathbf{V}}^a \cdot \nabla \bar{\pi}^e + \bar{\mathbf{V}}^e \cdot \nabla \bar{\pi}^a + \nabla \cdot \bar{\mathbf{V}}^a + \frac{\partial \bar{\sigma}^a}{\partial \sigma} = -\langle \mathbf{V}' \cdot \nabla \pi'_a \rangle - \langle \mathbf{V}'_a \cdot \nabla \pi'_a \rangle.
\]  

(A8)

Using Eqs. (A1), (A7), and (A8), we get an equation for \( \pi' \). Since \( \mathbf{V}'_a, \pi'_a \), and \( \sigma' \) are all one particular realization of \( \mathbf{V}', \pi', \sigma' \), and can be expanded according to Eq. (A5), we obtain the linearized equation for \( \pi'_a \) as follows:

\[
\frac{\partial \bar{\pi}^a}{\partial t} + \bar{\mathbf{V}}^a \cdot \nabla \bar{\pi}^e + \bar{\mathbf{V}}^e \cdot \nabla \bar{\pi}^a + \nabla \cdot \bar{\mathbf{V}}^a + \frac{\partial \bar{\sigma}^a}{\partial \sigma} = -\langle \mathbf{V}' \cdot \nabla \pi'_a \rangle.
\]  

(A9)

Here higher-order and nonlinear terms are neglected.

Similarly, we can get the corresponding equations for vorticity, divergence, and temperature. The climatological mean equations are

\[
\frac{1}{a \cos \varphi} \frac{\partial \bar{A}^{we}}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial (\bar{A}^{we} \cos \varphi) + \gamma (\bar{\xi}^c)}{\partial \varphi} = \frac{1}{a \cos \varphi} \frac{\partial (\bar{A}_a^{we})}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\bar{A}_a^{we} \cos \varphi),
\]  

(A10)

\[
\frac{1}{a \cos \varphi} \frac{\partial \bar{A}^{we}}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\bar{A}^{we} \cos \varphi) + \nabla^2 \left( \bar{\mathbf{V}}^e + R[T'] \bar{\pi}^e + \frac{\bar{\pi}^{e2} + \bar{V}^{e2}}{2} + \gamma (\bar{T}^e) \right) + 
\]  

\[
= \frac{1}{a \cos \varphi} \frac{\partial (\bar{A}_a^{we})}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\bar{A}_a^{we} \cos \varphi) - \nabla^2 \left( \frac{\langle \bar{u}'_e^2 \rangle + \langle \bar{v}'_e^2 \rangle}{2} \right),
\]  

(A11)

\[
\frac{1}{a \cos \varphi} \frac{\partial \bar{\pi}^a}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\bar{\mathbf{V}}^a \cos \varphi) - \bar{\pi}^a \bar{D}^e + \bar{\sigma} \frac{\partial \bar{\pi}^e}{\partial \sigma} - \kappa \bar{T}' \left( \frac{\partial \bar{\pi}^e}{\partial t} + \bar{\mathbf{V}}^e \cdot \nabla \bar{\pi}^e + \frac{\bar{\sigma}}{\sigma} \right) + \gamma (\bar{T}^e)
\]  

\[
= \frac{1}{a \cos \varphi} \frac{\partial (\bar{u}'_e \bar{\pi}'_c \cos \varphi)}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left( \langle \bar{u}'_e \bar{\xi}'_c \cos \varphi \rangle + \langle \bar{v}'_e \bar{\xi}'_c \cos \varphi \rangle - \langle \bar{\sigma}'_c \bar{\xi}'_c \cos \varphi \rangle - \kappa \langle \bar{u}'_e \bar{u}'_c \rangle \right) + \kappa \langle \bar{v}'_e \bar{v}'_c \rangle \bar{\mathbf{V}}^e \cdot \nabla \bar{\pi}'_c + \bar{\mathbf{V}}^e \cdot \nabla \bar{\pi}'_c + \frac{\sigma'}{\sigma},
\]  

(A12)
where \( a \) represents the radius of the earth, \( \gamma \) is linear damping rate, \( \Phi \) the geopotential height, \( R \) the atmospheric gas constant, \( C_p \) the specific heat at constant pressure, \( \kappa = R/C_p \), and

\[
\mathcal{A}^{vc} = (\xi^c + f \pi^c) - \frac{\partial \pi^c}{\partial \sigma} - \frac{R \bar{T}^c \partial \bar{T}^c}{a \cos \phi \partial \phi},
\]

\[
\langle \mathcal{A}_{uc} \rangle = \langle \xi, u \rangle - \left\langle \sigma_c \frac{\partial u_c}{\partial \sigma} \right\rangle + \left\langle \frac{R \bar{T}^c_c}{a \cos \phi \partial \phi} (\partial \bar{T}^c_c \partial \bar{T}^c_c) \right\rangle.
\]

The equations for low-frequency anomalies are

\[
\frac{\partial \tilde{\xi}^a}{\partial t} - \frac{1}{a \cos \phi} \frac{\partial \tilde{\mathcal{A}}^{ua}}{\partial \lambda} + \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \tilde{\mathcal{A}}^{ua} \cos \phi \right) + \gamma(\tilde{\xi}^a) = \frac{1}{a \cos \phi} \frac{\partial (\tilde{\mathcal{A}}^{ua})}{\partial \lambda} - \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} (\langle \tilde{\mathcal{A}}^{ua} \rangle \cos \phi)
\]

\[
\frac{\partial \tilde{D}^a}{\partial t} - \frac{1}{a \cos \phi} \frac{\partial \tilde{\mathcal{A}}^{ua}}{\partial \lambda} - \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \tilde{\mathcal{A}}^{ua} \cos \phi \right) + \nabla^2 (\tilde{\mathcal{A}}^{ua} \cos \phi) - \tilde{T}^a - \tilde{\mathcal{A}}^{ua} \cos \phi - \gamma(\tilde{D}^a)
\]

\[
\frac{\partial \tilde{T}^a}{\partial t} + \frac{1}{a \cos \phi} \frac{\partial \tilde{\mathcal{A}}^{ua}}{\partial \lambda} + \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left( \tilde{\mathcal{A}}^{ua} \cos \phi \right) - \kappa \tilde{T}^a \left( \frac{\partial \pi^a}{\partial t} + \nabla \cdot \nabla \pi^a + \frac{\bar{\sigma}^c}{\sigma} \right) - \kappa \tilde{T}^a \left( \frac{\partial \pi^a}{\partial t} + \nabla \cdot \nabla \pi^a + \frac{\bar{\sigma}^c}{\sigma} \right) + \gamma(\tilde{T}^a)
\]

\[
\frac{\partial \tilde{T}^a}{\partial t} - \frac{1}{a \cos \phi} \frac{\partial (\tilde{T}^a \tilde{T}^a)}{\partial \lambda} - \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} \left[ \langle \tilde{T}^a \tilde{T}^a \rangle \cos \phi + \langle \tilde{T}^a \tilde{D}^a \rangle + \langle \tilde{T}^a \tilde{D}^a \rangle - \left\langle \sigma_c \tilde{T}^a \tilde{T}^a \right\rangle \right]
\]

\[
\frac{\partial \tilde{T}^a}{\partial t} + \left\langle \kappa \tilde{T}^a \left( \frac{\partial \pi^a}{\partial t} + \nabla \cdot \nabla \pi^a + \frac{\bar{\sigma}^c}{\sigma} \right) \right\rangle + \langle \kappa \tilde{T}^a (V_a \cdot \nabla \pi^a + V_a \cdot \nabla \pi^a) \rangle.
\]

where

\[
\mathcal{A}^{ua} = (\xi^a + f \pi^a) - \frac{\partial \pi^a}{\partial \sigma} - \frac{R \bar{T}^a \partial \bar{T}^a}{a \cos \phi \partial \phi} - \frac{R \bar{T}^a \partial \bar{T}^a}{a \cos \phi \partial \phi}.
\]

\[
\mathcal{A}^{ua} = (\xi^a + f \pi^a) - \frac{\partial \pi^a}{\partial \sigma} - \frac{R \bar{T}^a \partial \bar{T}^a}{a \cos \phi \partial \phi}.
\]

\[
\langle \mathcal{A}_{ua} \rangle = \langle \xi, u \rangle - \left\langle \sigma_c \frac{\partial u_c}{\partial \sigma} \right\rangle + \left\langle \frac{R \bar{T}^c_c}{a \cos \phi \partial \phi} (\partial \bar{T}^c_c \partial \bar{T}^c_c) \right\rangle.
\]

\[
\langle \mathcal{A}{\prime}_{ua} \rangle = \langle \xi, u \rangle - \left\langle \sigma_c \frac{\partial u_c}{\partial \sigma} \right\rangle - \left\langle \frac{R \bar{T}^c_c}{a \cos \phi \partial \phi} (\partial \bar{T}^c_c \partial \bar{T}^c_c) \right\rangle.
\]
The equations for the ensemble eddy anomaly (or high-frequency anomaly) can be expressed as

$$\frac{\partial \xi_{\mu}^{a}}{\partial t} = \frac{1}{\cos \phi} \frac{\partial A'_{\mu}}{\partial \lambda} + \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left( A'_{\mu} \cos \phi \right) = \frac{1}{\cos \phi} \frac{\partial A'_{\mu}}{\partial \lambda} - \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left( A'_{\mu} \cos \phi \right) - \gamma(\xi_{\mu}^{a}),$$  \hspace{1cm} (A25)

$$\frac{\partial T_{a}'}{\partial t} = \frac{1}{\cos \phi} \frac{\partial u_{a} T_{a}'}{\partial \lambda} + \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left( u_{a} T_{a}' \cos \phi + v_{a} A_{a}' \cos \phi \right) - \frac{T_{a}' d_{a} \nabla T_{a}'}{\cos \phi} + \frac{\partial T_{a}'}{\partial \sigma} + \frac{\partial}{\partial \sigma} \left( \frac{\partial T_{a}'}{\partial \sigma} \right) + \frac{\partial}{\partial \sigma} \left( \frac{\partial T_{a}'}{\partial \sigma} \right),$$  \hspace{1cm} (A26)

where

$$\frac{\partial}{\partial t} X_{a}^{d} + L(X^{d})X_{a}^{d} = L_{E}(X_{a}^{d})\mathbf{x}^{a},$$  \hspace{1cm} (A32)

$$\frac{\partial}{\partial t} \mathbf{x}^{a} + L(X^{d})\mathbf{x}^{a} = L_{E}(X_{a}^{d})\mathbf{x}_{a}^{d},$$  \hspace{1cm} (A33)

using \( \mathbf{x} \) to represent the variable group \((\xi, D, T, \pi)\) and \( \mathbf{X} \) to represent the corresponding stochastic state, we can symbolically express the high-frequency and low-frequency anomaly equations as Eqs. (8) and (9):

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