Geometric Construction-Based Realization of Planar Elastic Behaviors With Parallel and Serial Manipulators

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This paper addresses the passive realization of any selected planar elastic behavior with a parallel or a serial manipulator. Sets of necessary and sufficient conditions for a mechanism to passively realize an elastic behavior are presented. These conditions completely decouple the requirements on component elastic properties from the requirements on mechanism kinematics. The restrictions on the set of elastic behaviors that can be realized with a mechanism are described in terms of acceptable locations of realizable elastic behavior centers. Parallel–serial mechanism pairs that realize identical elastic behaviors (dual elastic mechanisms) are described. New construction-based synthesis procedures for planar elastic behaviors are developed. Using these procedures, one can select the geometry of each elastic component from a restricted space of kinematically allowable candidates. With each selection, the space is further restricted until the desired elastic behavior is achieved. [DOI: 10.1115/1.4037019]

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1 Introduction

Compliant behavior in manipulation is an important topic in robotics research and industrial application. General compliant behavior can be modeled as a body suspended by an elastic parallel or serial mechanism. For small displacements from equilibrium, elastic behavior can be described by a symmetric positive semidefinite (PSD) matrix, the stiffness matrix **K**, which maps the applied force to displacement, or its inverse, the compliance matrix **C**, which maps the displacement to force.

A desired compliance can be obtained using mechanisms containing multiple passive elastic components, with each providing compliant constraint along or about a single axis. In robotic applications, a desired elastic behavior may be achieved using an elastic mechanism mounted on the manipulator end effector or the desired behavior may be designed into the robot manipulator itself. In the design of the behavior using either approach, the geometric construction of the mechanism/manipulator is an important consideration.

In some manipulation tasks, a time-varying compliant behavior is needed. For this purpose, variable stiffness actuators (VSAs) [1] that allow joint compliance to be changed in real time are used. Although the use of VSA's significantly enlarges the space of realizable compliant behaviors, varying the joint stiffness values alone, however, may not be adequate to achieve a desired behavior. Identification of the mechanism geometry required to realize a given compliance (as well as the joint compliances) is the primary motivation for this work.

This work is also motivated by the desire for a better understanding of compliant behavior achieved with a parallel or serial mechanism. In the planar case, since the elastic components in a mechanism are easy to illustrate, the physical significance of realization conditions can be readily understood in terms of the mechanism geometry.

1.1 Related Work. Screw theory [2] has been widely used in elastic behavior analysis [3–6], while Lie groups [7] have also been used.

In previous work in the realization of *spatial* compliances, the bounds of elastic behaviors achieved with *simple mechanisms* (i.e., parallel and serial mechanisms without helical joints) were identified [8,9]. Synthesis procedures to achieve a simple-mechanism realizable stiffness or compliance matrix were developed [8,9] and later refined [10,11]. The synthesis of an *arbitrary* spatial stiffness matrix with a parallel system with both screw and simple springs was presented in Ref. [12] and the process further refined in Ref. [13]. Stiffness matrix decompositions for the purpose of realization with screw and simple springs revealed inherent elastic behavior properties [14,15].

Each of these approaches to spatial elastic behavior realization involved a decomposition of the stiffness matrix without regard to mechanism geometry. More recent work has included some geometric considerations in the realization of spatial elastic behaviors [16–19].

In recent work [20] on *planar* elastic mechanism realization, a procedure to synthesize an arbitrary planar stiffness was developed for a restricted class of mechanism. As part of the procedure, the geometric parameters of a symmetric four-spring parallel mechanism were selected.

Most recently, the realization of a specified point planar elastic behavior (compliance in Euclidian space $E(2)$) using 3R serial mechanisms with specified link lengths has been addressed. In Ref. [21], optimization was used to identify the combination of mechanism configuration and joint stiffnesses that achieve an approximation of the desired elastic behavior. In Refs. [22] and [23], the synthesis of isotropic compliance in $E(2)$ and $E(3)$ with serial mechanisms has been addressed. In Ref. [24], conditions on mechanism geometry to achieve all compliances in $E(2)$ were identified and synthesis procedures for the realization of an arbitrary 2×2 compliance were presented. The results obtained for 3R mechanisms [24] were then extended to general serial mechanisms having three (revolute and/or prismatic) joints [25].

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1.2 Overview. This paper addresses the passive realization of an arbitrary planar (3×3) elastic behavior with either a parallel or a serial mechanism. Unlike most previous synthesis procedures that involved mathematically decomposing the stiffness matrix in one step without regard to mechanism geometry, the synthesis procedures presented here are completely geometry based (no matrix decomposition needed). This allows one to select the geometry of each elastic component from a restricted space of kinematically allowable candidates. Then, with each selection, the space is further restricted until the desired elastic behavior is achieved.

The paper is outlined as follows: Section 2 presents the theoretical background for planar compliance realization with a parallel or serial mechanism. Necessary and sufficient conditions for an elastic behavior to be realized with a mechanism are obtained. In Sec. 3, the physical implications of the realization conditions are presented. The restrictions on the set of elastic behaviors that can be realized (described in terms of the locations of realizable elastic behavior centers) with a mechanism are identified and the concept of dual elastic mechanisms is introduced. In Sec. 4, geometric construction-based syntheses of a planar compliant behavior using either a parallel or a serial mechanism are presented. In Sec. 5, a numerical example is provided to illustrate the synthesis procedures. A brief summary is presented in Sec. 6.

2 Planar Compliance Realization Conditions

In this section, the technical background for compliance realization with a parallel or serial mechanism is presented. Necessary and sufficient conditions to realize a planar elastic behavior are derived for both parallel and serial mechanisms.

2.1 Technical Background. It is known that any rank-m 6×6 PSD matrix $\mathbf{K}$ can be decomposed into a sum of m rank-1 PSD matrices, i.e.,

$$\mathbf{K} = k_1\mathbf{w}_1\mathbf{w}_1^T + k_2\mathbf{w}_2\mathbf{w}_2^T + \cdots + k_m\mathbf{w}_m\mathbf{w}_m^T \tag{1}$$

where $k_i > 0$ is a constant and $\mathbf{w}_i \in \mathbb{R}^6$ is a unit wrench defined as the *spring wrench* [8]. Each rank-1 PSD stiffness $\mathbf{K}_i = k_i\mathbf{w}_i\mathbf{w}_i^T$ can be uniquely realized with a simple spring or a screw spring [12] having a line of action along $\mathbf{w}_i$ and spring constant k_i. The decomposition (1) can be written as

$$\mathbf{K} = \mathbf{W}\mathbf{K}_J\mathbf{W} \tag{2}$$

where $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m]$ is the wrench matrix and $\mathbf{K}_J = \mathrm{diag}(k_1, k_2, \ldots, k_m)$ is the joint-space stiffness matrix.

If stiffness is decomposed into the form of Eq. (1), the elastic behavior can be realized with a set of springs connected in parallel in which each spring provides a single axis of compliant constraint. In general, the rank-1 decomposition of $\mathbf{K}$ is not unique. There are infinitely many sets of springs that realize a given elastic behavior.

By duality [26], a decomposition of a compliance matrix $\mathbf{C}$ (the inverse of stiffness matrix $\mathbf{K}$) yields a set of compliant joint twists associated with a serial mechanism. Using a similar process, a compliance matrix $\mathbf{C}$ can be realized with a serial mechanism in which each joint twist provides a rank-1 PSD component.

For the planar case, an elastic behavior is characterized by a 3×3 PSD stiffness matrix $\mathbf{K}$ or its inverse, the compliance matrix $\mathbf{C}$. The spring wrenches in a parallel mechanism and the joint twists in a serial mechanism are 3-vectors. To realize an arbitrary elastic behavior, only simple mechanisms (zero or infinite pitch spring wrenches or joint twists) are needed. For a parallel mechanism, only line springs and torsional springs are needed. For a serial mechanism, only revolute and prismatic joints are needed.

The planar spring wrench for a line spring and for a torsional spring can be expressed in Plücker ray coordinates as

$$\mathbf{w}_l = \begin{bmatrix} \mathbf{n} \\ d \end{bmatrix}, \quad \mathbf{w}_t = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3}$$

where $\mathbf{n}$ is a unit 2-vector indicating the direction of the spring axis and $d = (\mathbf{r} \times \mathbf{n}) \cdot \mathbf{k}$ is a scalar indicating the distance of the spring axis from the coordinate frame used to describe the stiffness $\mathbf{K}$, $\mathbf{r}$ is the perpendicular position vector from the coordinate frame to the spring axis, and $\mathbf{k}$ is the unit vector perpendicular to the plane of the mechanism.

When the value of a wrench $\mathbf{w}_l$ is given, the perpendicular position $\mathbf{r}$ to the wrench axis can be calculated using

$$\mathbf{r} = -d\Omega\mathbf{n} \tag{4}$$

where Ω is the 2×2 anti-symmetric matrix associated with a cross product

$$\Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

For a torsional spring, since the spring wrench is a free vector, its location is arbitrary.

The planar joint twist for a revolute joint and for a prismatic joint can be expressed in Plücker axis coordinates as

$$\mathbf{t}_r = \begin{bmatrix} \mathbf{v} \\ 1 \end{bmatrix}, \quad \mathbf{t}_p = \begin{bmatrix} \mathbf{n} \\ 0 \end{bmatrix} \tag{6}$$

where $\mathbf{v} = \mathbf{r} \times \mathbf{k}$ and $\mathbf{r}$ is a 2-vector indicating the location of the revolute joint relative to the coordinate frame used to describe the compliance $\mathbf{C}$, and where $\mathbf{n}$ is a unit 2-vector indicating the direction of the prismatic joint axis.

Given the value of a twist $\mathbf{t}_r$, a unique point, the instantaneous center of rotation for the twist motion is calculated using

$$\mathbf{r} = \Omega\mathbf{v} \tag{7}$$

For a twist associated with a prismatic joint, since $\mathbf{t}_p$ is a free vector in twist space, the location of the joint is arbitrary in the mechanism chain.

A wrench $\mathbf{w}$ and twist $\mathbf{t}$ are called reciprocal [2] if $\mathbf{w}$ performs no work along $\mathbf{t}$. If wrench $\mathbf{w}$ and twist $\mathbf{t}$ are expressed in Plücker ray and axis coordinates (as in Eqs. (3) and (6)), respectively, then $\mathbf{w}$ and $\mathbf{t}$ are reciprocal if and only if

$$\mathbf{w}^T\mathbf{t} = \mathbf{t}^T\mathbf{w} = 0 \tag{8}$$

2.2 Realization Conditions. The space of stiffness matrices that can be realized with a given mechanism by adjusting the spring constant of each spring in the mechanism is determined by the mechanism kinematics.

Consider a parallel mechanism having three spring wrenches $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$. Below, we prove that a necessary and sufficient condition for a stiffness matrix $\mathbf{K}$ to be realized with the mechanism is

$$\mathbf{w}_i \times \mathbf{K}(\mathbf{w}_j \times \mathbf{w}_k) = 0, \quad \{i, j, k\} = \{1, 2, 3\} \tag{9}$$

To prove the condition is necessary, we suppose that $\mathbf{K}$ is realized with the mechanism. Then, by Eq. (2), $\mathbf{K}$ can be expressed as

$$\mathbf{K} = \mathbf{W}\mathbf{K}_J\mathbf{W}^T \tag{10}$$

where $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]$ is the wrench matrix and $\mathbf{K}_J = \mathrm{diag}(k_1, k_2, k_3)$ (with $k_i \geq 0$) is the joint stiffness matrix. For the 3×3 matrix $\mathbf{W}$, its inverse can be expressed as

$$\mathbf{W}^{-1} = \frac{1}{\lambda}[\mathbf{w}_2 \times \mathbf{w}_3, \mathbf{w}_3 \times \mathbf{w}_1, \mathbf{w}_1 \times \mathbf{w}_2]^{\mathrm{T}} \tag{11}$$

where λ is the triple product of $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$

$$\lambda = (\mathbf{w}_1 \times \mathbf{w}_2) \cdot \mathbf{w}_3 \tag{12}$$

Multiplying Eq. (10) by $\mathbf{W}^{-T}$ from the right yields

$$\mathbf{K}[\mathbf{w}_2 \times \mathbf{w}_3, \mathbf{w}_3 \times \mathbf{w}_1, \mathbf{w}_1 \times \mathbf{w}_2] = \lambda[k_1\mathbf{w}_1, k_2\mathbf{w}_2, k_3\mathbf{w}_3]$$

Thus, for $\forall\{i,j,k\} = \{1,2,3\}$

$$\mathbf{K}(\mathbf{w}_j \times \mathbf{w}_k) = \lambda k_i \mathbf{w}_i \tag{13}$$

Thus, $\mathbf{w}_i \times \mathbf{K}(\mathbf{w}_j \times \mathbf{w}_k) = 0$, which proves that condition (9) is necessary.

In the evaluation of the condition sufficiency, consider that condition (9) is satisfied, then, there exist scalars α_i such that

$$\mathbf{K}[\mathbf{w}_2 \times \mathbf{w}_3, \mathbf{w}_3 \times \mathbf{w}_1, \mathbf{w}_1 \times \mathbf{w}_2] = [\alpha_1\mathbf{w}_1, \alpha_2\mathbf{w}_2, \alpha_3\mathbf{w}_3]$$

Using Eq. (11)

$$\mathbf{W}^{-1}\mathbf{K}\mathbf{W}^{-T} = \mathrm{diag}\left(\frac{\alpha_1}{\lambda}, \frac{\alpha_2}{\lambda}, \frac{\alpha_3}{\lambda}\right) = \mathbf{K}_J$$

Since $\mathbf{K}$ is PSD, $(\alpha_i/\lambda) \geq 0$. Thus

$$\mathbf{K} = \mathbf{W}\mathbf{K}_J\mathbf{W}^T$$

which proves that the stiffness $\mathbf{K}$ is realized with the mechanism. The three joint stiffness constants can be calculated using

$$k_i = \frac{\alpha_i}{\lambda}$$

where λ is the scalar defined in Eq. (12) and

$$\alpha_1 = \frac{\mathbf{w}_1^T\mathbf{K}(\mathbf{w}_2 \times \mathbf{w}_3)}{\mathbf{w}_1^T\mathbf{w}_1}$$

$$\alpha_2 = \frac{\mathbf{w}_2^T\mathbf{K}(\mathbf{w}_3 \times \mathbf{w}_1)}{\mathbf{w}_2^T\mathbf{w}_2}$$

$$\alpha_3 = \frac{\mathbf{w}_3^T\mathbf{K}(\mathbf{w}_1 \times \mathbf{w}_2)}{\mathbf{w}_3^T\mathbf{w}_3}$$

By duality, condition (9) applies to a serial mechanism having joint twists $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ and a compliance matrix $\mathbf{C}$. In Eq. (9), one can simply replace the stiffness matrix $\mathbf{K}$ with the compliance matrix $\mathbf{C}$ and replace the spring wrenches $\mathbf{w}_i$ with the joint twists $\mathbf{t}_i$ to obtain the condition for a serial mechanism. The joint compliances can also be obtained accordingly.

In summary, we have:

Proposition 1. *Consider a parallel mechanism having spring wrenches* $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ *and a serial mechanism having joint twists* $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$. *Then, if the elastic constants (k_i or c_i) are selectable,*

(a) *A stiffness matrix* $\mathbf{K}$ *can be realized with the parallel mechanism if and only if*

$$\mathbf{w}_i \times \mathbf{K}(\mathbf{w}_j \times \mathbf{w}_k) = 0, \quad \{i,j,k\} = \{1,2,3\} \tag{14}$$

(b) *A compliance matrix* $\mathbf{C}$ *can be realized with the serial mechanism if and only if*

$$\mathbf{t}_i \times \mathbf{C}(\mathbf{t}_j \times \mathbf{t}_k) = 0, \quad \{i,j,k\} = \{1,2,3\} \tag{15}$$

The realization conditions (14) and (15) are mathematical requirements for a parallel and a serial mechanism, respectively, to achieve an arbitrary given compliance behavior. In these conditions, each spring wrench $\mathbf{w}_i$ or joint twist $\mathbf{t}_i$ is treated as a vector in $\mathbb{R}^3$ and the cross product is an operation between these 3-vectors. The physical significance of these conditions is provided in Sec. 2.3.

Below, for any full-rank compliance behavior, an equivalent set of conditions is derived from Eqs. (14) and (15). These conditions do not use the cross product operation and have clear physical significance.

For a full-rank $\mathbf{K}$, $\alpha_i \neq 0$. Multiplying Eq. (13) from the left by $\mathbf{C} = \mathbf{K}^{-1}$ yields

$$(\mathbf{w}_j \times \mathbf{w}_k) = \alpha_i\mathbf{C}\mathbf{w}_i$$

For any $i \neq j$

$$\alpha_i\mathbf{w}_j^T\mathbf{C}\mathbf{w}_i = \mathbf{w}_j^T(\mathbf{w}_j \times \mathbf{w}_k) = 0 \tag{16}$$

Since $\alpha_i \neq 0$, Eq. (16) can be expressed as

$$\mathbf{w}_i^T\mathbf{C}\mathbf{w}_j = 0, \quad \forall i \neq j$$

To determine the spring constants k_i, consider

$$\mathbf{K} = k_1\mathbf{w}_1\mathbf{w}_1^T + k_2\mathbf{w}_2\mathbf{w}_2^T + k_3\mathbf{w}_3\mathbf{w}_3^T \tag{17}$$

Multiplying $\mathbf{K}$ in Eq. (17) from the right by $\mathbf{C}\mathbf{w}_i$ yields

$$\mathbf{K}\mathbf{C}\mathbf{w}_i = k_i\mathbf{w}_i(\mathbf{w}_i^T\mathbf{C}\mathbf{w}_i) \Rightarrow \mathbf{w}_i = k_i(\mathbf{w}_i^T\mathbf{C}\mathbf{w}_i)\mathbf{w}_i$$

Thus

$$k_i = \frac{1}{\mathbf{w}_i^T\mathbf{C}\mathbf{w}_i}$$

The result obtained for stiffness matrix $\mathbf{K}$ for a parallel mechanism applies to its dual involving the compliance matrix $\mathbf{C}$ for a serial mechanism. Thus, we have:

Proposition 2. *Consider a parallel mechanism having spring wrenches* $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ *and a serial mechanism having joint twists* $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$. *Then, if the elastic constants (k_i or c_i) are selectable*

(a) *A full-rank elastic behavior* $\mathbf{K}$ ($\mathbf{C}$) *can be realized with a parallel mechanism if and only if*

$$\mathbf{w}_i^T\mathbf{C}\mathbf{w}_j = 0, \quad \forall i \neq j \tag{18}$$

The spring constant associated with $\mathbf{w}_i$ *is determined using*

$$k_i = \frac{1}{\mathbf{w}_i^T\mathbf{C}\mathbf{w}_i}, \quad i = 1,2,3 \tag{19}$$

(b) *A full-rank elastic behavior* $\mathbf{C}$ ($\mathbf{K}$) *can be realized with a serial mechanism if and only if*

$$\mathbf{t}_i^T\mathbf{K}\mathbf{t}_j = 0, \quad \forall i \neq j \tag{20}$$

The joint compliance associated with $\mathbf{t}_i$ *is determined using*

$$c_i = \frac{1}{\mathbf{t}_i^T\mathbf{K}\mathbf{t}_i}, \quad i = 1,2,3 \tag{21}$$

Note that the conditions in Eq. (14) or Eq. (18) for parallel mechanism and the conditions in Eq. (15) or Eq. (20) for serial mechanisms can be used to determine whether a given elastic behavior can be realized based on the mechanism kinematics alone. If these conditions are satisfied, the realization of the specified behavior is ensured if the non-negative spring coefficients in Eq. (19) or joint compliances in Eq. (21) can be physically attained.

Also, note that realization condition (14) for a parallel mechanism applies for all stiffness matrices (including those nonfull-rank elastic behaviors) while condition (18) applies only to full-rank elastic behaviors. Since full-rank elastic behaviors are of most interest, in the rest of this paper, only full-rank stiffness and compliance matrices are considered and the conditions presented in Proposition 2 are used.

2.3 Physical Significance of Realization Conditions. Since the cross product operation is normally not used on screws, the physical significance of conditions (14) and (15) is not evident. However, if the three vectors in Eq. (14) associated with the cross product are interpreted as planar twists (i.e., $\mathbf{t} = \mathbf{w}_i \times \mathbf{w}_j$), the physical meaning of the realization conditions can be obtained.

For a parallel mechanism having three spring wrenches $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$, consider a twist $\mathbf{t}$ reciprocal to the two spring wrenches $\mathbf{w}_i$ and $\mathbf{w}_j$. By Eq. (8), the planar twist can be expressed as

$$\mathbf{t} = \gamma(\mathbf{w}_i \times \mathbf{w}_j)$$

where γ is a scalar and the twist $\mathbf{t}$ is located at the intersection of the two spring wrench axes. Realization condition (14) requires

$$\mathbf{K}\mathbf{t} = \alpha \mathbf{w}_k$$

where α is a scalar. Thus, if a parallel mechanism realizes the stiffness, a twist located at the intersection of any two spring axes ($\mathbf{w}_i$ and $\mathbf{w}_j$) yields a wrench along the axis of the third spring $\mathbf{w}_k$.

Similarly, for a serial mechanism having three joint twists $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$, realization condition (15) implies that a wrench passing through any two joints ($\mathbf{t}_i$ and $\mathbf{t}_j$) results in a twist motion about the third joint $\mathbf{t}_k$.

The physical significance of conditions (18) and (20) is evident. For an elastic behavior realized with a parallel mechanism, the twist resulting from a force along one spring wrench must be reciprocal to the other two spring wrenches. For an elastic behavior realized with a serial mechanism, the wrench resulting from a motion along one joint twist must be reciprocal to the other two joint twists. The realization conditions for parallel and serial mechanisms provide the relationship between the mechanism geometry and the elastic behavior to be realized. If the conditions are not satisfied, then a specified planar elastic behavior cannot be obtained no matter how the joint stiffnesses vary.

3 Planar Elastic Behaviors and Mechanisms

The implications of the realization conditions can be understood in the geometry of the mechanism. First, additional physical interpretations of conditions (18) and (20) are presented. Next, the bounds on the realizable space of elastic behaviors for a given mechanism are interpreted in terms of the locus of elastic behavior centers using these conditions. Then, conditions on the parallel and serial mechanisms that can achieve the same subspace of elastic behaviors are identified. The two mechanisms are defined to be dual elastic mechanisms. The geometric properties of a pair of dual elastic mechanisms are presented.

3.1 Implications of the Realization Conditions. Consider a parallel mechanism consisting of three line springs. Spring wrench behavior is independent of the location along the spring axis and, if none of the springs is parallel to another, the three spring axes form a triangle *ABC* as shown in Fig. 1.

If a force $\mathbf{f}_1$ is applied to the elastically constrained body along spring wrench $\mathbf{w}_1$, the force can be expressed as $\mathbf{f}_1 = \alpha \mathbf{w}_1$ where α is a scalar. The twist motion $\mathbf{t}_1$ resulting from $\mathbf{f}_1$ is

$$\mathbf{t}_1 = \mathbf{C}\mathbf{f}_1 = \alpha \mathbf{C}\mathbf{w}_1$$

By condition (18), twist $\mathbf{t}_1$ is reciprocal to the other two spring wrenches $\mathbf{w}_2$ and $\mathbf{w}_3$. Thus, the instantaneous center associated with $\mathbf{t}_1$ must be located at the intersection of these two spring axes, point *C*. The resulting motion is a rotation about vertex *C* as illustrated in Fig. 2(*a*). Therefore, if an applied force is along one spring axis, the resulting motion is a rotation about the opposite vertex of the triangle formed by the three spring axes.

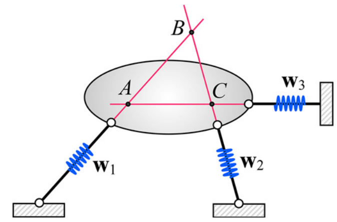

Fig. 1 Planar parallel mechanism with three line springs. The three spring axes typically form a triangle *ABC*.

Also, it can be proved that if an applied force passes through a vertex of the triangle, the resulting twist corresponds to an instantaneous center located on the spring axis opposite to the vertex. To prove this, consider a force $\mathbf{f}$ passing through vertex *A* as shown in Fig. 2(*b*). Then, $\mathbf{f}$ can be expressed as a linear combination of $\mathbf{w}_1$ and $\mathbf{w}_2$

$$\mathbf{f} = \alpha \mathbf{w}_1 + \beta \mathbf{w}_2$$

where α and β are arbitrary scalars. Condition (18) requires

$$\mathbf{w}_3^T \mathbf{C}\mathbf{f} = \mathbf{w}_3^T \mathbf{C}(\alpha \mathbf{w}_1 + \beta \mathbf{w}_2) = 0$$

which means that the twist resulting from $\mathbf{f}$ acting on the body must be reciprocal to $\mathbf{w}_3$. Thus, the twist instantaneous center must be on spring axis $\mathbf{w}_3$.

In summary, for a planar parallel mechanism having three line springs,

(i) A force along one spring axis results in a twist having an instantaneous center of rotation located at the opposite vertex of the triangle formed by the three spring axes.
(ii) A force passing through a vertex of the triangle results in a twist center on the line along the opposite side of the triangle.

By duality and condition (20), for a planar serial mechanism having three revolute compliant joints:

(i) A rotation about one joint results in a wrench passing through the other two joints;
(ii) A rotation about an arbitrary point on the line passing through two joints results in a wrench passing through the third joint.

3.2 Center of Planar Elastic Behavior. For any planar elastic behavior, there is a unique point at which the behavior can be

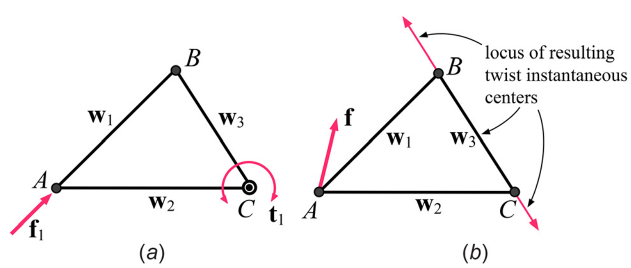

Fig. 2 Force and resulting motion of a three-spring parallel mechanism: (*a*) a force along one spring axis results in a rotation about the opposite vertex of the triangle and (*b*) a force passing through one vertex results in a twist with an instantaneous center located on the line along the opposite side of the triangle

described by a diagonal stiffness (compliance) matrix. This point is defined as the center of stiffness (compliance). For a given *planar* elastic behavior, the centers of stiffness and compliance are coincident. Any force passing through the center results in a pure translation (i.e., a twist with infinite pitch or with instantaneous center at infinity), and any twist at the center results in a pure couple (i.e., a wrench with zero pitch). It can be seen that, if a force $\mathbf{f}$ results in a pure translation, the line of action of the force must pass through the center of stiffness.

If, in a given coordinate frame, a 3×3 stiffness $\mathbf{K}$ and compliance $\mathbf{C}$ are expressed in a partitioned form as

$$\mathbf{K} = \begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & k_{33} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{E} & \mathbf{g} \\ \mathbf{g}^T & c_{33} \end{bmatrix} \tag{22}$$

where $\mathbf{A}$ and $\mathbf{E}$ are 2×2 symmetric matrices, and $\mathbf{b}$ and $\mathbf{g}$ are 2-vectors, the locations of stiffness center and compliance center can be determined using [20]

$$\mathbf{r}_k = -\Omega \mathbf{A}^{-1}\mathbf{b} \text{ and } \mathbf{r}_c = \Omega \mathbf{g}/c_{33} \tag{23}$$

where $\Omega \in \mathbb{R}^{2\times 2}$ is the anti-symmetric matrix defined in Eq. (5). For the same planar elastic behavior ($\mathbf{C} = \mathbf{K}^{-1}$), $\mathbf{r}_k = \mathbf{r}_c$.

Conditions (18) and (20) constrain the possible location of the elastic centers associated with a parallel or serial mechanism. Below, we show that, for any stiffness behavior realized with a parallel mechanism of three line springs, the center of the behavior must be inside the triangle formed by the three spring axes.

Consider a force $\mathbf{f}$ that passes through vertex A between $\mathbf{w}_1$ and $\mathbf{w}_2$ which can be expressed as

$$\mathbf{f}(q) = (1-q)\mathbf{w}_1 + q\mathbf{w}_2, \quad q \in [0, 1] \tag{24}$$

As q varies from $0 \rightarrow 1$, the direction of $\mathbf{f}$ varies from $\mathbf{w}_1$ to $\mathbf{w}_2$, and as previously shown, since $\mathbf{f}$ passes through vertex A, the instantaneous center $T(q)$ of the resulting twist must vary from point C to point B along the $\mathbf{w}_3$ axis. We show that $T(q)$ cannot move from $C \rightarrow B$ along finite line CB as q varies from $0 \rightarrow 1$.

Suppose that for all $q \in [0, 1]$ the center of the twist is finite. Then the location of the twist instantaneous center, $T(q)$ (which moves along $\mathbf{w}_3$), is a continuous function of q. Note that $T(0) = C$ and $T(1) = B$. With the finite path from $C \rightarrow B$ along the axis of $\mathbf{w}_3$, there is a point $\hat{q} \in (0, 1)$ such that $\mathbf{f}(\hat{q})$ passes through the instantaneous center $T(\hat{q})$ (Fig. 3(*a*)), which means $\mathbf{f}(\hat{q})$ is reciprocal to the motion caused by itself. Thus, at $q = \hat{q}$

$$\mathbf{f}^T\mathbf{C}\mathbf{f} = 0$$

This conflicts the fact that $\mathbf{C}$ is positive definite and $\mathbf{f} \neq 0$. Therefore, the finite path from $C \rightarrow B$ is not valid. The path of $T(q)$ as q increases from 0 to 1 must be opposite to that illustrated in Fig. 3(*a*), and there must be $q \in (0, 1)$ such that $\mathbf{f}(q)$ in Eq. (24) results in a twist at infinity, which is a pure translation. Thus, the line of action of $\mathbf{f}(q)$ must pass through the center of stiffness.

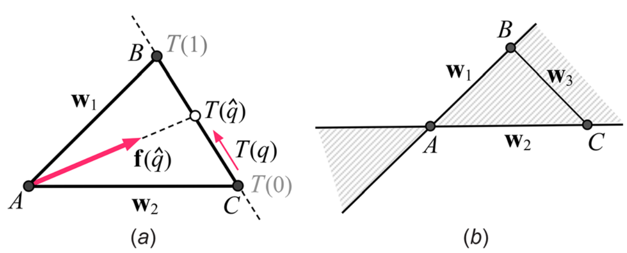

Fig. 3 Location of center of elastic behavior: (*a*) if *T*(*q*) moves from $C \rightarrow B$ along the finite segment *CB* as *q* varies $0 \rightarrow 1$, there must be a $\hat{q}$ such that $f(\hat{q})$ passes points *A* and $T(\hat{q})$ and (*b*) the center must be in the shaded area

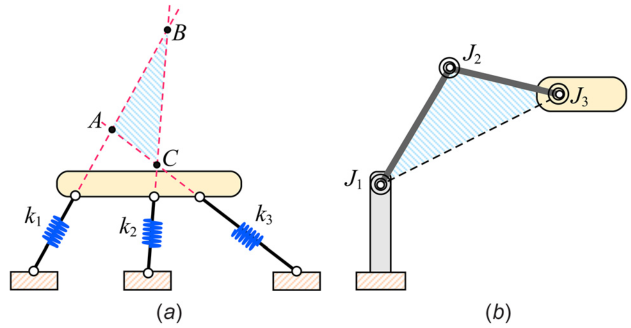

Fig. 4 Location of stiffness center associated with a parallel and serial mechanism: (*a*) for a parallel mechanism, the center must lie within the triangle formed by the three spring axes and (*b*) for a serial mechanism, the center must lie within the triangle formed by the three joints

Since the force $\mathbf{f}(q)$ is a positive combination of $\mathbf{w}_1$ and $\mathbf{w}_2$, the line of action of $\mathbf{f}(q)$ is within the area bounded by spring axes $\mathbf{w}_1$ and $\mathbf{w}_2$ (the shaded area of Fig. 3(*b*)). Therefore, the center of stiffness must be in this area.

Applying the same reasoning to vertices B and C, it is proved that the location of the stiffness center is within the triangle ABC formed by the three spring axes (Fig. 4(*a*)).

By duality, for a serial mechanism having three revolute joints, no matter how the joint compliances are selected, the center of compliance must be within the triangle formed by the locations of the three joints $J_1J_2J_3$ as shown in Fig. 4(*b*).

Since the locus of stiffness centers is determined for a given mechanism with fixed geometry, it is easy to assess whether a specified elastic behavior can be attained by evaluating the location of the behavior center. If the center is not in the region bounded by the mechanism geometry, then the behavior cannot be realized with the mechanism regardless of the value of each joint stiffness/compliance. Also, the location of the center can be used to: (1) help determine the placement of the elastic components in the design of a new mechanism or (2) determine the location within the manipulator workspace that a specified compliance can be achieved in an existing mechanism.

3.3 Dual Mechanisms in Parallel and Serial Constructions.

Suppose a parallel mechanism has three line spring wrenches $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$. Consider the following three 3-vectors $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ defined by

$$\mathbf{s}_1 = \mathbf{w}_2 \times \mathbf{w}_3, \quad \mathbf{s}_2 = \mathbf{w}_3 \times \mathbf{w}_1, \quad \mathbf{s}_3 = \mathbf{w}_1 \times \mathbf{w}_2 \tag{25}$$

If the three 3-vectors $\mathbf{s}_i$ are viewed as planar twists in Plücker axis coordinates, and $\mathbf{t}_i$ is the unit twist associated with $\mathbf{s}_i$, then the three unit twists $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ are uniquely determined by the three wrenches $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and each $\mathbf{t}_i$ is reciprocal to two wrenches $(\mathbf{w}_j, \mathbf{w}_k)$.

Consider the serial mechanism composed of three joints J_1, J_2, and J_3 having joint twists $\mathbf{t}_1$, $\mathbf{t}_2$, and $\mathbf{t}_3$, respectively. Since twist $\mathbf{t}_1$ is reciprocal to wrenches $\mathbf{w}_2$ and $\mathbf{w}_3$, J_1 must be at the intersection of the two wrenches. Similarly, joint J_2 must be at the intersection of wrenches $\mathbf{w}_1$ and $\mathbf{w}_3$, and joint J_3 must be at the intersection of wrenches $\mathbf{w}_1$ and $\mathbf{w}_2$. Therefore, as shown in Fig. 5, the triangle formed by the three line spring axes in the parallel mechanism is coincident with the triangle formed by the three revolute joints in the serial mechanism. We define such a pair of parallel and serial mechanisms as *dual elastic* mechanisms.

Given a pair of dual elastic mechanisms with spring wrenches $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ and joint twists $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$, it can be proved that for an elastic behavior described in stiffness matrix $\mathbf{K}$ or compliance matrix $\mathbf{C} = \mathbf{K}^{-1}$

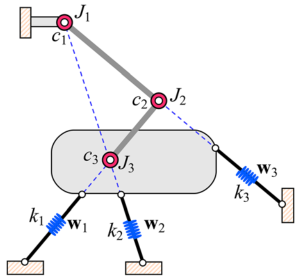

Fig. 5 Dual elastic mechanisms in parallel and serial construction. The triangle formed by the three spring axes in the parallel mechanism is coincident with the triangle formed by the three joints in the serial mechanism.

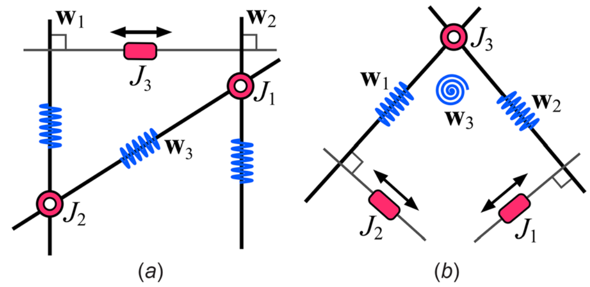

Fig. 6 Dual elastic mechanisms in nongeneric cases. (*a*) A parallel mechanism with two parallel springs. The dual elastic serial mechanism has two revolute joints each located at the intersection of nonparallel springs and one prismatic joint perpendicular to the two parallel springs. (*b*) A parallel mechanism with one torsional spring. The dual elastic serial mechanism has two prismatic joints perpendicular to the two line springs and one revolute joint located at the intersection point of the two line springs.

$$\mathbf{t}_i^T\mathbf{K}\mathbf{t}_j = 0 \Longleftrightarrow \mathbf{w}_i^T\mathbf{C}\mathbf{w}_j = 0, \quad \forall i \neq j \tag{26}$$

Thus, an arbitrary elastic behavior can be realized with one mechanism if and only if it can be realized with its dual elastic mechanism. The realizable spaces of elastic behaviors for the two mechanisms are exactly the same. Also, it can be proved that, if k_i is the spring constant associated with spring wrench $\mathbf{w}_i$ in the parallel mechanism and c_i is the joint compliance associated with joint twist $\mathbf{t}_i$, then k_i and c_i satisfy

$$k_i c_i = \frac{1}{\left(\mathbf{w}_i^T\mathbf{t}_i\right)^2}, \quad i = 1, 2, 3 \tag{27}$$

Note that, since $\mathbf{t}_1$ is reciprocal to two wrenches $\mathbf{w}_2$ and $\mathbf{w}_3$, $\mathbf{t}_1$ cannot be reciprocal to wrench $\mathbf{w}_1$, unless the three wrenches are linearly dependent or $\mathbf{t}_1 = 0$. Thus, $\mathbf{w}_1^T\mathbf{t}_1 \neq 0$. Similarly, $\mathbf{w}_i^T\mathbf{t}_i \neq 0$ for $i = 2, 3$.

It can be seen that, for the generic case, a three-spring parallel mechanism and a three-joint serial mechanism are a pair of dual elastic mechanisms if and only if the two triangles formed by the three springs and formed by the three joints are coincident. For some (nongeneric) cases, the triangle for a parallel or serial mechanism does not exist. The dual mechanisms have different geometry. Below, two cases are considered:

(a) Two springs are parallel in a parallel mechanism. Suppose a parallel mechanism has three springs $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ with $\mathbf{w}_1 \parallel \mathbf{w}_2$. The dual elastic serial mechanism has two revolute joints and one prismatic joint. The two revolute joints are located at the two intersection points of springs $\mathbf{w}_1$ and $\mathbf{w}_3$ and $\mathbf{w}_2$ and $\mathbf{w}_3$. The prismatic joint is perpendicular to the two parallel spring axes. The geometry of the two mechanism's wrench and twist axes is illustrated in Fig. 6(*a*).
(b) A parallel mechanism has one torsional spring. Suppose $\mathbf{w}_3$ is the torsional spring in a parallel mechanism. The dual elastic serial mechanism has two prismatic joints J_1 and J_2 and one revolute joint J_3. The directions of the two prismatic joints are perpendicular to the two line springs $\mathbf{w}_1$ and $\mathbf{w}_2$, respectively, and the revolute joint is located at the intersection of the two line springs $\mathbf{w}_1$ and $\mathbf{w}_2$. The geometry of the two mechanism's wrench and twist axes is illustrated in Fig. 6(*b*).

4 Elastic Behavior Synthesis Procedures

In this section, procedures for the realization of planar elastic behavior using geometric construction-based methods are presented. First, a synthesis procedure for a parallel mechanism having three springs is provided. Next, a synthesis procedure for a serial mechanism having three joints is presented. These two types of mechanisms are the most general in that all full-rank planar stiffness/compliance matrices can be realized with these two types. Then, synthesis procedures for a parallel mechanism having a torsional spring and for a serial mechanism having prismatic joints are discussed.

4.1 Parallel Elastic Mechanism. Suppose a stiffness $\mathbf{K}$ described in a body-based frame is to be realized. The following synthesis procedure identifies a set of spring axes and their corresponding spring constants that realize the given $\mathbf{K}$. The location of the stiffness center of the behavior, C_k, can be calculated using Eq. (23). The geometry associated with the sequence of operations in the synthesis procedure is illustrated in Fig. 7.

(1) Select the first spring $\mathbf{w}_1$: The spring axis can be chosen arbitrarily relative to the stiffness center C_k.
(2) Calculate the twist $\mathbf{t}_1$ resulting from wrench $\mathbf{w}_1$

$$\mathbf{t}_1 = \mathbf{C}\mathbf{w}_1$$

The location of the instantaneous center of rotation associated with $\mathbf{t}_1$, T_1, is calculated using Eq. (7).
(3) Select the second spring $\mathbf{w}_2$: Due to the reciprocal condition (18), all candidate wrenches are from the pencil of lines passing through point T_1. Choose a direction for a wrench passing through T_1, then $\mathbf{w}_2$ is determined.
(4) Calculate the twist $\mathbf{t}_2$ resulting from $\mathbf{w}_2$

$$\mathbf{t}_2 = \mathbf{C}\mathbf{w}_2$$

The location of the instantaneous center associated with $\mathbf{t}_2$, T_2, is determined using Eq. (7). Since $\mathbf{t}_2$ satisfies the reciprocal condition (18), T_2 must be on the axis of the first spring $\mathbf{w}_1$.
(5) Identify the third spring $\mathbf{w}_3$: The axis of $\mathbf{w}_3$ is uniquely determined by the line passing through points T_1 and T_2.

With the final step, all three spring wrenches are determined. The stiffness coefficient for each spring can be calculated using Eq. (19).

Note that in the generic case, the three wrenches $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ generated from the procedure are associated with line springs and form a triangle. If, in the process of realizing a given elastic behavior, one or more spring axes are selected to pass through the center of stiffness, then the three spring axes will not form a triangle. Consider the following two cases:

(a) If the first spring is selected to pass through the stiffness center C_k in step 1, the twist $\mathbf{t}_1 = \mathbf{C}\mathbf{w}_1$ is a pure translation. Due to the reciprocal condition (18), both the second and third spring axes, $\mathbf{w}_2$ and $\mathbf{w}_3$, must be perpendicular to the

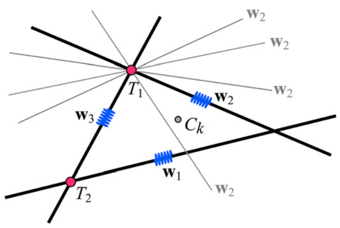

Fig. 7 Realization of a planar stiffness with a parallel mechanism. The first spring axis $\mathbf{w}_1$ can be arbitrarily selected. The second spring can be selected from the pencil of lines passing through point T_1. The third spring axis is determined by the line passing through the instantaneous centers of twists $\mathbf{t}_1$ and $\mathbf{t}_2$, T_1 and T_2.

direction of translation $\mathbf{t}_1$. The location of the second spring axis $\mathbf{w}_2$ can be arbitrarily selected. The location of $\mathbf{w}_3$ can be determined by passing through the instantaneous center of the twist $\mathbf{t}_2 = \mathbf{C}\mathbf{w}_2$ (as shown in Fig. 8(*a*)). If both the first and second springs are chosen to pass through the stiffness center, then the third spring must be a torsional spring.

(b) If in step 3, the second spring wrench $\mathbf{w}_2$ (passing through point T_1, the instantaneous center of $\mathbf{t}_1 = \mathbf{C}\mathbf{w}_1$) is chosen to pass through the stiffness center C_k (illustrated in Fig. 8(*b*)), then the twist $\mathbf{t}_2 = \mathbf{C}\mathbf{w}_2$ is a pure translation (twist instantaneous center at infinity). The third spring wrench $\mathbf{w}_3$ must also pass through T_1 and be parallel to $\mathbf{w}_1$. Thus, the behavior is realized with three line springs $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ as shown in Fig. 8(*b*).

4.2 Serial Elastic Mechanism. Similar to the parallel mechanism case, the synthesis procedure identifies the set of joint locations (configuration of the mechanism) and corresponding joint compliance constants that realize the given $\mathbf{C}$. The location of the compliance center, C_c, can be calculated using Eq. (23) and $\mathbf{K} = \mathbf{C}^{-1}$. The geometry associated with the sequence of operations in the synthesis procedure is illustrated in Fig. 9.

(1) Select the first joint location for $\mathbf{t}_1$: The location of the joint, J_1, can be arbitrarily chosen relative to the center C_c.

(2) Calculate the wrench $\mathbf{w}_1$ resulting from the twist

$$\mathbf{w}_1 = \mathbf{K}\mathbf{t}_1$$

The perpendicular position to the line of action of $\mathbf{w}_1$ is determined using Eq. (4).

(3) Select the second joint location for $\mathbf{t}_2$: Due to the reciprocal condition (20), all candidate joints are located on the line of action of wrench $\mathbf{w}_1$. Choose a joint location on the line along $\mathbf{w}_1$, then the joint twist $\mathbf{t}_2$ is determined.

(4) Calculate the wrench $\mathbf{w}_2$ resulting from $\mathbf{t}_2$

$$\mathbf{w}_2 = \mathbf{K}\mathbf{t}_2$$

Since $\mathbf{w}_2$ satisfies the reciprocal condition, the line of action of $\mathbf{w}_2$ passes through the first joint J_1.

(5) Identify the third joint twist $\mathbf{t}_3$: the joint is uniquely determined by the intersection of the two lines along wrenches $\mathbf{w}_1$ and $\mathbf{w}_2$.

With the final step, all three joint locations are determined. The joint compliance coefficient for each elastic joint can be calculated using Eq. (21).

Note that in the generic case, the three twists $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ generated from the procedure are joint twists of revolute joints. If in the process of realizing a given elastic behavior, the location of one joint is selected to be at the center of compliance, then one or two prismatic joints must be used. Consider the following two cases.

(a) If the location of the first joint is selected at the compliance center C_c in step 1 (illustrated in Fig. 10(*a*)), the wrench

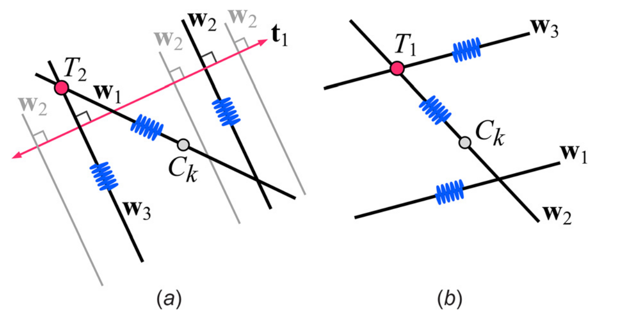

Fig. 8 Nongeneric parallel mechanism stiffness realization cases. (*a*) If the first spring axis is selected to pass through C_k, then the spring axes $\mathbf{w}_2$ and $\mathbf{w}_3$ must be perpendicular to the translation $\mathbf{t}_1$ resulting from $\mathbf{w}_1$. The location of $\mathbf{w}_2$ can be selected arbitrarily and the spring axis $\mathbf{w}_3$ passes through the instantaneous center of twist $\mathbf{t}_2 = \mathbf{C}\mathbf{w}_2$, T_2. (*b*) If the second spring axis $\mathbf{w}_2$ passes through the stiffness center C_k, the third spring axis $\mathbf{w}_3$ must be parallel to spring axis $\mathbf{w}_1$ and pass through T_1.

$\mathbf{w}_1 = \mathbf{K}\mathbf{t}_1$ is a pure couple. Due to the reciprocal condition (20), both the second and third joint twists, $\mathbf{t}_2$ and $\mathbf{t}_3$, must be pure translation and the corresponding joints must be prismatic. The direction of the prismatic joint axis for J_2, $\mathbf{n}_2$, can be selected arbitrarily. Given the selection of $\mathbf{n}_2$, the direction of the prismatic joint axis for J_3 must be perpendicular to the wrench axis determined by

$$\mathbf{w}_2 = \mathbf{K}\mathbf{t}_{p2}$$

where $\mathbf{t}_{p2} = [\mathbf{n}_2^T, 0]^{\mathrm{T}}$ is the joint twist of J_2.

(b) If in step 3, the location of the second joint J_2 is selected such that line J_1J_2 passes through the center of compliance C_c (illustrated in Fig. 10(*b*)), then the wrench $\mathbf{w}_2 = \mathbf{K}\mathbf{t}_2$ is parallel to $\mathbf{w}_1$. The third joint must be prismatic and perpendicular to $\mathbf{w}_1$. Thus, the compliance is realized with a serial mechanism having two revolute joints and one prismatic joint. Since translation is a free vector, the location of the prismatic joint on the serial chain is arbitrary.

4.3 Discussion. In the generic case, the synthesis procedure presented in Sec. 4.1 yields three line springs in a parallel mechanism, and the synthesis procedure presented in Sec. 4.2 yields three revolute joints in a serial mechanism. In the case that a torsional spring is desired in a parallel mechanism or a prismatic joint is desired in a serial mechanism, the synthesis procedures can be modified.

A three-spring parallel mechanism can have, at most, one torsional spring to realize a full-rank stiffness matrix. Since the spring wrench associated with a torsional spring has the form of $\mathbf{w}_t$ in Eq. (3), it must be selected as the first spring in the synthesis

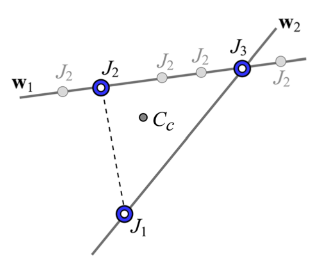

Fig. 9 Realization of a planar compliance with a serial mechanism. The location of the first joint J_1 can be arbitrarily selected. The second joint can be selected from any point on the line along wrench $\mathbf{w}_1$. The third joint is determined by the intersection of the two lines along wrenches $\mathbf{w}_1$ and $\mathbf{w}_2$.

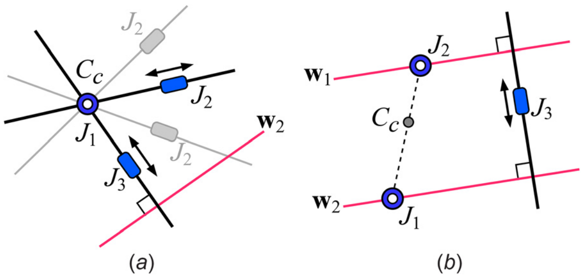

Fig. 10 Nongeneric serial mechanism compliance realization cases. (*a*) If the first joint is selected to pass through the center of compliance C_c, the other two joints must be prismatic. The direction of the second prismatic axis can be arbitrarily chosen. The direction of the third prismatic axis is perpendicular to the line of action of the wrench $\mathbf{w}_2 = \mathbf{K}\mathbf{t}_2$. (*b*) If the second joint J_2 is on the line passing through J_1 and the compliance center C_c, the resulting wrench $\mathbf{w}_2 = \mathbf{K}\mathbf{t}_2$ must be parallel to $\mathbf{w}_1$. The third joint is prismatic and is perpendicular to $\mathbf{w}_1$.

procedure presented in Sec. 4.1 (or the last spring if the other two springs pass through the center of stiffness).

A three-joint serial mechanism can have, at most, two prismatic joints to realize a full-rank compliance matrix. The twist associated with a prismatic joint has the form of $\mathbf{t}_p$ in Eq. (6). Since the third joint is uniquely determined by the first two joints, the revolute joint should be assigned last in the synthesis procedure.

5 Examples

Examples are provided to demonstrate the geometry-based synthesis procedures for the realization of a specified elastic behavior. First, the realization of a stiffness matrix with a parallel mechanism is demonstrated. Then, the realization of the same elastic behavior with a serial mechanism is presented.

The elastic behavior to be realized in a known coordinate frame is given by

$$\mathbf{K} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 6 & 5 \\ 1 & 5 & 9 \end{bmatrix}, \quad \mathbf{C} = \frac{1}{25}\begin{bmatrix} 29 & 23 & -16 \\ 23 & 26 & -17 \\ -16 & -17 & 14 \end{bmatrix}$$

The location of the center of stiffness/compliance for this behavior is calculated using Eq. (23) to be at $((17/14), -(16/14))$. Since the center must be inside the triangle formed by the spring components (or location of the elastic joints), this point is used as a reference in selecting each component.

5.1 Parallel Mechanism Realization.

The geometry associated with the sequence of operations in the synthesis procedure is illustrated in Fig. 11.

The first spring can be arbitrarily chosen. Here, a horizontal spring passing through the origin is selected. The spring wrench is

$$\mathbf{w}_1 = [1, 0, 0]^{\mathrm{T}}$$

The twist associated with $\mathbf{w}_1$ is calculated to be

$$\mathbf{t}_1 = \mathbf{C}\mathbf{w}_1 = \frac{16}{25}\left[-\frac{29}{16}, -\frac{23}{16}, 1\right]^{\mathrm{T}}$$

Using Eq. (7), the location the instantaneous center T_1 associated with twist $\mathbf{t}_1$ is found to be $(1.4375, -1.8125)$.

Given the selection of the first spring, the second spring wrench is any one that passes through point T_1. Here, the slope of the line is chosen to be 4. The unit wrench passing through T_1 with this slope is

$$\mathbf{w}_2 = [0.2425, 0.9701, 1.8342]^{\mathrm{T}}$$

The twist associated with the second spring wrench is

$$\mathbf{t}_2 = \mathbf{C}\mathbf{w}_2 = 0.2122[0, -0.0716, 1]^{\mathrm{T}}$$

Using Eq. (7), the location of the instantaneous center T_2 associated with $\mathbf{t}_2$ is calculated as (0.0716, 0).

The third spring wrench is along the line passing through T_1 and T_2. The spring wrench is

$$\mathbf{w}_3 = [0.6019, -0.7986, -0.0570]^{\mathrm{T}}$$

The three spring constants calculated using Eq. (19) are

$$k_1 = 0.8621, \quad k_2 = 2.6699, \quad k_3 = 5.4680$$

The process is verified by summing the stiffness components using Eq. (1) yielding

$$\mathbf{K} = \sum_{i=1}^{3} k_i \mathbf{w}_i \mathbf{w}_i^T = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 6 & 5 \\ 1 & 5 & 9 \end{bmatrix}$$

Note that the synthesis procedure identifies the line of action and stiffness constant for each spring. In the construction of a parallel mechanism, each spring can be anywhere along its line of action.

5.2 Serial Mechanism Realization.

For the parallel mechanism obtained in Sec. 5.1, the dual elastic serial mechanism is readily determined using the results of Sec. 3.3. The three joints of the serial mechanism are located at the three vertices T_1, T_2, and T_3 of the triangle formed by the three spring axes (shown in Fig. 11). The joint compliances calculated using Eq. (27) are

$$c_1 = 0.3531, \quad c_2 = 0.1202, \quad c_3 = 0.0867$$

An alternative design using the synthesis procedure of Sec. 4.2 is derived below. The geometry associated with the sequence of operations in the synthesis procedure is illustrated in Fig. 12.

The first joint location can be chosen arbitrarily. Here, the location of J_1 is selected at (1, 0). The unit joint twist associated with J_1 is calculated using Eq. (6)

$$\mathbf{t}_1 = [0, -1, 1]^{\mathrm{T}}$$

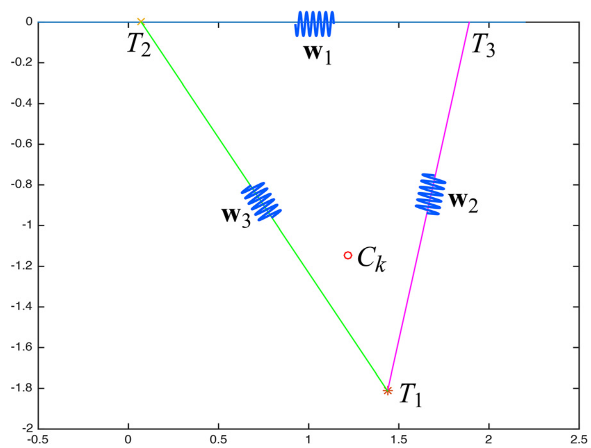

Fig. 11 Synthesis of planar stiffness with a parallel mechanism. The line of action for each spring is identified based on its geometry. A parallel mechanism can be constructed with three springs along the three spring wrenches $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$.

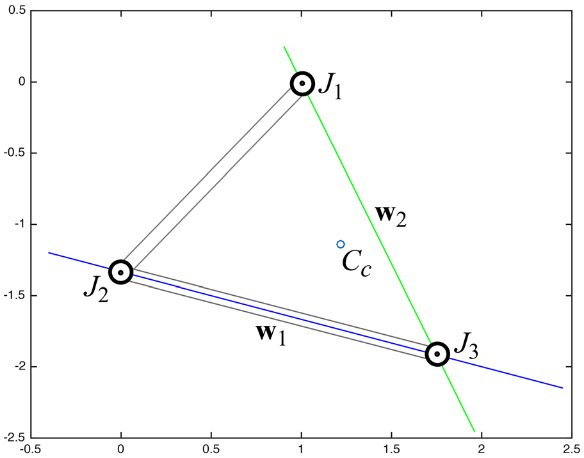

Fig. 12 Synthesis of planar compliance with a serial mechanism. The location for each joint is identified based on its geometry. A serial mechanism can be constructed with three joints located at J_1, J_2, and J_3.

The wrench $\mathbf{w}_1$ associated with $\mathbf{t}_1$ is calculated to be

$$\mathbf{w}_1 = \mathbf{K}\mathbf{t}_1 = [3, -1, 4]^{\mathrm{T}}$$

Using Eq. (4), the equation of the line of action of $\mathbf{w}_1$ is determined to be

$$y = -\frac{1}{3}x - \frac{4}{3}$$

Given the selection of the first joint, the second joint J_2 is located at any point on the line of action of $\mathbf{w}_1$. Here, the location of J_2 is selected to be $\mathbf{r}_2 = [0, -(4/3)]^{\mathrm{T}}$. Then, joint twist associated with the second joint, J_2, is calculated using Eq. (6) to be

$$\mathbf{t}_2 = \left[-\frac{4}{3}, 0, 1\right]^{\mathrm{T}}$$

The wrench associated with $\mathbf{t}_2$ is calculated to be

$$\mathbf{w}_2 = \mathbf{K}\mathbf{t}_2 = \left[-3, \frac{23}{3}, \frac{23}{3}\right]^{\mathrm{T}}$$

The line of action of $\mathbf{w}_2$ is

$$y = -\frac{23}{9}x + \frac{23}{9}$$

The intersection of the two action lines $\mathbf{w}_1$ and $\mathbf{w}_3$ is $((7/4), (23/12))$, which is the location of J_3. The unit twist associated with this location is

$$\mathbf{t}_3 = \left[-\frac{23}{12}, -\frac{7}{4}, 1\right]^{\mathrm{T}}$$

The three joint compliances are calculated using Eq. (21)

$$c_1 = 0.2, \quad c_2 = 0.0857, \quad c_3 = 0.2743$$

The process is verified by summing the compliance components, similar to Eq. (1) yielding

$$\mathbf{C} = \sum_{i=1}^{3} c_i \mathbf{t}_i \mathbf{t}_i^T = \frac{1}{25}\begin{bmatrix} 29 & 23 & -16 \\ 23 & 26 & -17 \\ -16 & -17 & 14 \end{bmatrix}$$

Note that the synthesis procedure identifies the location and joint compliance coefficient for each elastic joint. In the construction of a serial mechanism, the connection order of these joints does not influence the elastic behavior achieved with the mechanism.

6 Summary

In this paper, the realization of an arbitrary planar elastic behavior using parallel and serial mechanisms is addressed. A set of necessary and sufficient conditions for a mechanism to realize a given planar compliance is presented and the physical interpretations of the realization conditions are provided. The methods presented in this paper allow one to synthesize any compliant behavior by selecting each elastic component in a parallel or serial mechanism based on its geometry without decomposition of the compliance/stiffness matrix. Each selected component restricts the space of allowable candidates in subsequent selection. Since the conditions on the mechanism geometry and joint compliances are decoupled, the methods identified can be used for mechanisms having VSAs to realize a desired compliant behavior by changing the mechanism configuration and joint stiffnesses. In application, one can use the method to judiciously select a better mechanism geometry for a specified compliance from the infinite, but restricted, set of options available. This method makes those restrictions to the mechanism geometry explicit.

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