Jared Butler[1]
School of Engineering Design,
Technology, and Professional Programs,
The Pennsylvania State University,
State College, PA 16802
e-mail: butler@psu.edu

Jacob Greenwood
Compliant Mechanisms Research Group,
Department of Mechanical Engineering,
Brigham Young University,
Provo, UT 84602
e-mail: jacobgreenwood@byu.net

Larry L. Howell
Compliant Mechanisms Research Group,
Department of Mechanical Engineering,
Brigham Young University,
Provo, UT 84602
e-mail: lhowell@byu.edu

Spencer Magleby
Compliant Mechanisms Research Group,
Department of Mechanical Engineering,
Brigham Young University,
Provo, UT 84602
e-mail: magleby@byu.edu

Limits of Extramobile and Intramobile Motion of Cylindrical Developable Mechanisms

Mechanisms that can both deploy and provide motions to perform desired tasks offer a multifunctional advantage over traditional mechanisms. Developable mechanisms (DMs) are devices capable of conforming to a predetermined developable surface and deploying from that surface to achieve specific motions. This paper builds on the previously identified behaviors of extramobility and intramobility by introducing the terminology of extramobile and intramobile motions, which define the motion of developable mechanisms while interior and exterior to a developable surface. The limits of motion are identified using defined conditions. It is shown that the more difficult of these conditions to kinematically predict may be treated as a non-factor during the design of cylindrical developable mechanisms given certain assumptions. The impact of toggle positions for each case is discussed. Physical prototypes demonstrate the results. [DOI: 10.1115/1.4048833]

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1 Introduction

Mechanisms that can create customized paths, positions, and force outputs are important, and combining these behaviors with other functionalities is an area of increasing interest. In particular, mechanisms capable of both deploying and moving to perform desired tasks offer multifunctional benefits over traditional mechanisms. Examples include deployable straight-line linkages [1,2], deployable mechanisms with intentionally shaped parts [3], shape-morphing structures [4], and mechanisms that conform to or approximate predetermined shapes [5–7]. Other means of obtaining desired shapes or behaviors have been shown through the use of compliant parts [8,9] and harmonic linkages [10].

Developable mechanisms (DMs) are devices that are able to conform to a predetermined developable surface (such as a cylinder or cone [11]) and deploying from that surface to achieve specific motions. An example is shown in Fig. 1. Their ability to lie within a surface makes them compact, and if embedded into or fabricated from part of the surface (such as if a compliant mechanism were made from part of the surface), they can occupy no additional volume, becoming hyper-compact. Because of the prevalence of developable surfaces in many engineering applications, DMs provide a way to integrate multifunctionality into previously underutilized surfaces. Foundational work in this field has defined DMs [12], described behaviors unique to cylindrical [13] and conical [14] DMs, and demonstrated their usefulness in certain applications, such as minimally invasive surgical devices [15].

Because DMs are designed within the context of a developable surface, the movement of the mechanism relative to that surface becomes of particular importance. For example, a device may be made to exist on the interior of a pressurized pipe and requires all parts to remain interior to the pipe during actuation. Another possibility would be mechanisms that lie on the outside of a rocket body where penetrating the pressure vessel would lead to catastrophic failure. Past work investigated whether a given mechanism is capable of these behaviors of moving into or away from a developable surface (referred to as intramobility and extramobility [13]). These behaviors allows a DM to (1) lie on a pre-existing surface and (2) exhibit some amount of motion without penetrating the surface. Mechanisms that exhibit extramobility and intramobility can move without interfering with existing subsystems,

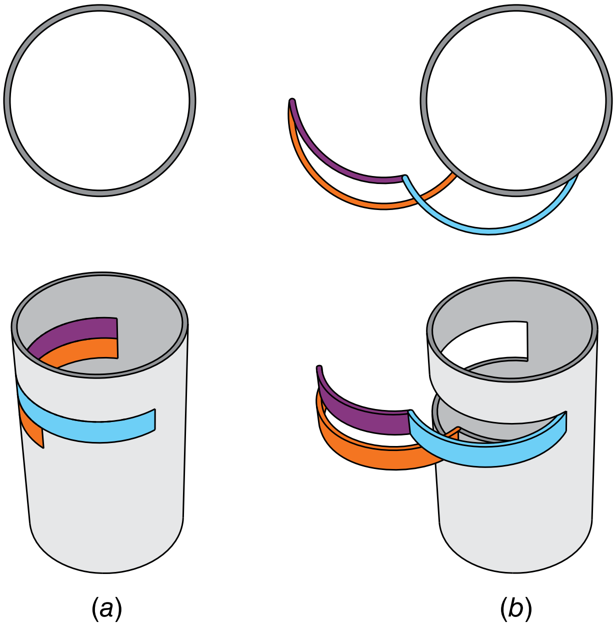
(*a*) (*b*)

Fig. 1 This four-bar linkage embedded within a cylinder illustrates a developable mechanism that (*a*) conforms to and (*b*) emerges from a cylinder

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[1] Corresponding author.

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thereby providing multifunctionality in pre-existing systems with minimal impact. However, the ability to move into or away from a developable surface does not define the possible motions of a mechanism while interior or exterior to that surface. In this paper, we advance the understanding of these mechanisms by showing how to determine the range of motion that can be achieved by a DM while interior or exterior to a developable surface.

Another challenge that exists in the design of mechanisms is the existence of toggle positions (where three pin joints are collinear) and change-points (where all joints are collinear). Early identification of these positions can aid in maintaining a mechanism's desired characteristics throughout its motion [16–18]. Change-points in DMs have not been previously investigated, meaning that the plausibility of extramobile and intramobile motion of a cylindrical DM is unclear when the linkage is a change-point mechanism.

This paper presents nomenclature and methods to determine the limits of extramobile and intramobile motions. These limits of motion are identified using three defined conditions, and it is shown that the more difficult of these conditions to model kinematically may have negligible influence on the design of cylindrical DMs given certain assumptions. It is shown that these conditions can be further reduced under certain mechanism configurations. Possibilities of change-point mechanisms existing in intramobile and extramobile DMs are investigated. A discussion is then provided on the implications of intramobile and extramobile motions.

2 Developable Mechanism Background

When modeled with zero thickness, DMs are constrained to have at least one position in which their joints are all coincident to and aligned with the ruling lines of a developable surface [12]. (This position is referred to as the conformed position.) This constraint creates unique conditions that influence the possible outcomes of mechanism synthesis. Constraining kinematic linkages to conform to predetermined developable surfaces can result in the definition of mechanism behaviors within the context of their motion relative to those surfaces, such as the ability to be extramobile, intramobile, and transmobile [13].

In a zero-thickness model, the surface to which the mechanism conforms is called the "reference" surface. When conformed, all of the mechanism's joint axes must intersect and be aligned to the ruling lines on the reference surface. It should be noted that the reference surface does not need to represent a physical surface. As such, the reference surface is merely a representation of where the joint axes must align in space. A cylindrical DM is a mechanism that has at least one position where all parts of the mechanism conform to a cylindrical reference surface.

3 Intramobile and Extramobile Motion

We define *intramobile motion* as motion where all moving parts of a mechanism remain interior to the reference surface. Similarly, *extramobile motion* is motion where all moving parts of a mechanism remain exterior to the reference surface throughout their motion.

Because DMs on cylinders are planar mechanisms, it is often convenient to model them when viewed along the cylinder centerline, as shown in Fig. 2. While cylindrical DMs physically are created using curved links, it can be advantageous in the kinematic modeling of DMs (such as determining the Grashof condition for a mechanism) to view each linkage as a straight line. Figure 2 shows a (*a*) developable mechanism and its (*b*) straight-linkage equivalent. The two mechanisms are kinematically equivalent since the distance between pivots is identical. This paper uses both methods to represent the links within a cylindrical DM. Note that for all figures in this paper, the black link is ground, the purple link is the coupler link, and the orange and blue links are links 2 and 4, respectively.

3.1 Conditions for Intramobile and Extramobile Motions.

Based on the possible motions of all links, the requirement that all parts of a mechanism must remain interior or exterior to the reference surface can be decomposed into the following conditions:

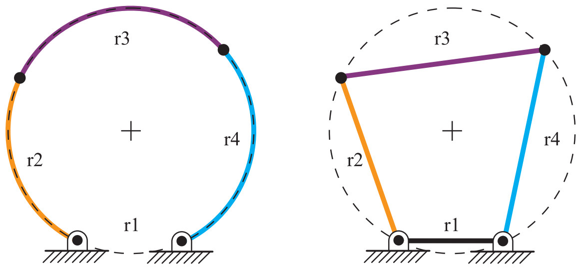
- *Condition 1:* No grounded link may rotate from the conformed position far enough to again intersect the reference surface.
- *Condition 2:* No grounded link may rotate interior to (exterior to) the reference surface for extramobile (intramobile) motion.
- *Condition 3:* No portion of the coupler may cross the reference surface.

These conditions define the limits of intramobile and extramobile motions. The motion of a cylindrical DM will remain extramobile or intramobile if none of these three conditions are violated. It is therefore useful to accurately identify the limits of each of these conditions. Predicting the motion limits of grounded links (Conditions 1 and 2) is straightforward. In contrast, predicting the location of a coupler relative to the reference surface (Condition 3) can be much more complex. However, if the first two conditions were to always occur prior to Condition 3, Condition 3 may be ignored during the design process, making its complexity a non-factor when designing for extramobile and intramobile motions. Sections 3 and 4 demonstrate that Conditions 1 and 2 will always be violated prior to Condition 3 for any four-bar mechanism exhibiting intramobile or extramobile motion given the following assumptions:

(A) All links have an arc length $\leq \pi R$.
(B) All links have the same curvature as the reference surface.
(C) All links are modeled with zero thickness.
(D) All grounded links only extend in one direction past their grounded pivot.
(E) The coupler does not extend beyond either of the moving pivots.

A grounded link is defined as any moving link pinned to ground and a coupler is a moving link attached to two grounded links.

Assumptions A and D are necessary for mechanisms to exhibit intramobility or extramobility [13]. Assumption B is a requirement for cylindrical developable mechanisms. Assumptions C and E



(*a*) (*b*)

Fig. 2 (*a*) A developable mechanism with curved links and (*b*) its straight-linkage kinematic equivalent (Color version online.)

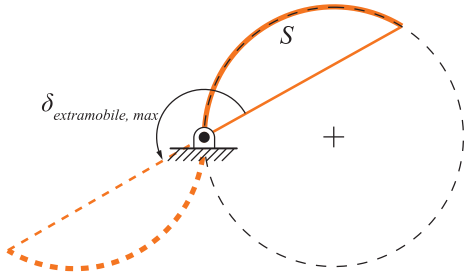


Fig. 3 The maximum amount of rotation outside the reference surface for a grounded link

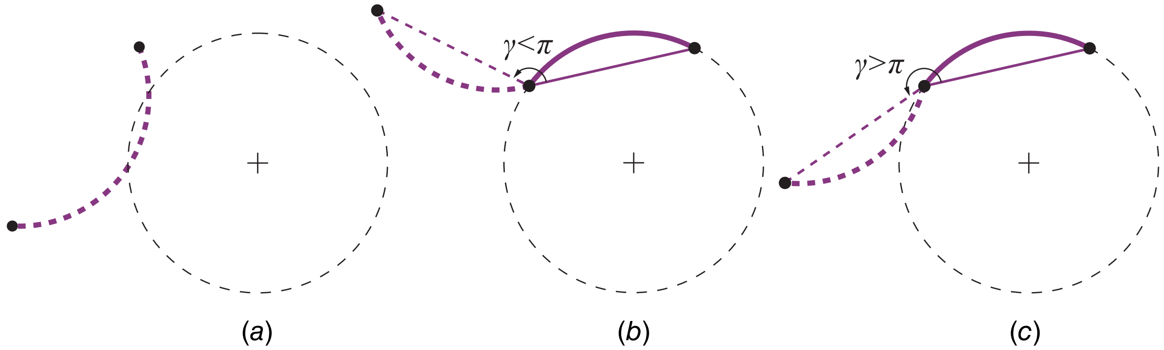


Fig. 4 (*a*) **The convex side of a coupler may intersect the reference surface before the endpoints intersect the surface. γ is used to determine if the coupler (*b*) has not or (*c*) has violated Condition 3 prior to Condition 1 or 2. Solid and dashed lines correlate to the initial and rotated positions, respectively. (Color version online.)**

build on previous work in this area [12,13] and provide a foundation for mechanisms with thickness and more complex geometries.

3.2 Conditions for Extramobile Motion.

This section will detail Conditions 1–3 for both Grashof and non-Grashof extramobile cylindrical developable mechanisms. Note that special-case Grashof mechanisms (change-point mechanisms) will be discussed in a later section.

3.2.1 Conditions 1 and 2. For grounded links (i.e., links 2 and 4 in traditional four-bar mechanisms), the maximum exterior rotation for the link (the point at which Condition 1 is violated) can be calculated as described in the equation below and shown in Fig. 3, where S is the arc length of the link.

$$\delta_{extramobile,\max} = \pi \text{ for } (0 < S \le \pi R) \tag{1}$$

To violate Condition 2, a grounded link would need to move exterior to the reference surface, then return to its initial position on the reference surface. At this point, a continuation of motion would move the link interior to the reference surface. Hence, the limits of extramobile motion for grounded links are represented by the conformed position of the link and Eq. (1).

3.2.2 Condition 3. To violate Condition 3 prior to Condition 1 or 2, the convex side of the curved coupler would need to intersect the reference surface prior to the endpoints crossing the surface, as shown in Fig. 4(*a*). Hence, there are two scenarios that must be necessary for this to happen.

First, the coupler must invert orientation (convex side of arc facing the reference surface), as shown in Fig. 4(*a*). This is only possible if the mechanism can reach both its open and crossed configurations in the same circuit (as is the case with double rockers and all non-Grashof mechanisms).

Second, the convex side of the coupler must intersect the reference surface prior to Condition 1 or 2 being violated. This can be evaluated by analyzing the rotation of the coupler, γ, when Condition 2 is violated. If $\gamma < \pi$, the coupler has not already crossed into the reference surface when Condition 2 is met, as shown in Fig. 4(*b*). If $\gamma > \pi$, the coupler has crossed into the reference surface prior to Condition 2 being met, as shown in Fig. 4(*c*).

Greenwood [13] put extramobile and intramobile mechanisms into three classes and demonstrated how only specific types of four-bar mechanisms [19,20] can be created within each class. These classes are shown in Fig. 5. (These three classes are decomposed into subclass A and B mechanisms due to symmetry, allowing our discussion to be simplified to the analysis of only subclass A mechanisms.) To demonstrate that Condition 3 is not violated prior to Conditions 1 and 2, we will look at the possible motions of the coupler in each of these three classes. Without loss of generality, we will assign θ_1 (angle of the ground link) in each class to equal 0.

Class 1. mechanisms are conformed in their open configuration (Fig. 5(*a*)). Under Class 1, and using Barker's classification for planar four-bar linkages [21], it is possible to obtain a GCCC (double crank), GCRR/GRRC (crank rocker), GRCR (double rocker), and RRR2/RRR4 (triple rocker) (excluding change-point mechanisms) [13]. Note that change-points will be discussed in a later section. Only GRCR and RRR2/RRR4 are capable of reaching both their open and crossed configurations, resulting in the convex side of the coupler facing the reference surface as the coupler moves toward the surface. The other mechanism types (GCCC and GCRR/GRRC) cannot invert the coupler and the convex side of the coupler can therefore not contact the reference surface prior to Conditions 1 and 2.

For mechanism types GRCR and RRR2/RRR4, the mechanism can deploy off the surface, toggle to its crossed configuration, and then link 2 can re-conform to the surface, as shown in Fig. 6. We will show that when link 2 comes back to the conformed position, the convex side of the coupler has not penetrated the reference surface.

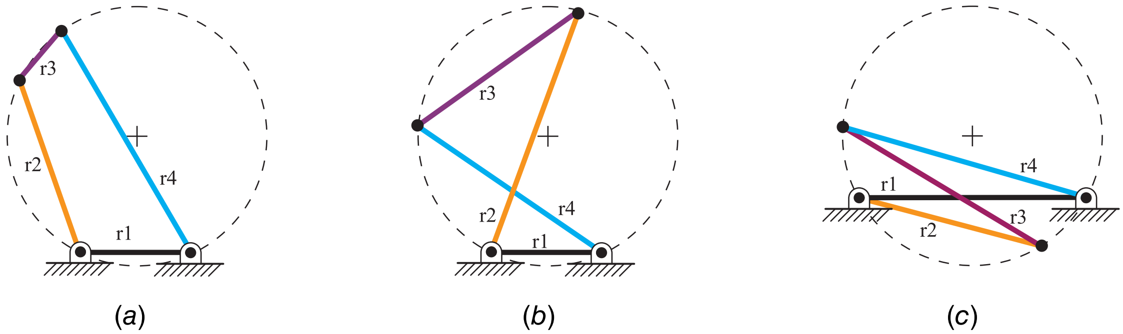


Fig. 5 There are three classes of extra/intramobile mechanisms. The black line represents the ground link: (*a*) Class 1 mechanism, (*b*) Class 2 mechanism, and (*c*) Class 3 mechanism.

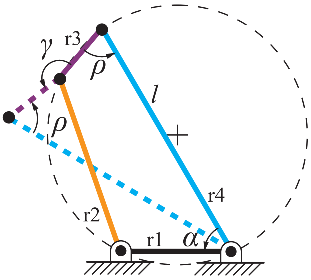


Fig. 6 Class 1 mechanism in its open (solid, conformed) and crossed (dashed) configuration. Straight lines represent curved links, as discussed previously in Fig. 2.

Under Class 1, all joints must reside on the same half of a circle while in the conformed position. This constrains the longest link l to be the link closest to the center of the circle, as shown in Fig. 6. It can be seen that the angles adjacent to l, α, and ρ must be less than $\pi/2$. Furthermore, when link 2 re-conforms to the surface (reaches the crossed configuration for the same value of θ_2 at the conformed position), a symmetric polygon is formed by links 3 and 4 in the open and crossed positions.

Because ρ must always be less than $\pi/2$, and because the mirrored polygon is symmetric, the angle opposite ρ is equivalent to ρ and must always be less than $\pi/2$. The angle γ must therefore always be less than π, preventing the coupler curve from moving past its tangent position and into the reference surface. It is then concluded that the defining limits to extramobility for Class 1, given the assumptions above, are set by Conditions 1 and 2.

Class 2. mechanisms are conformed in their crossed configuration (Fig. 5(*b*)). Under Class 2, it is possible to obtain GCCC and GRCR mechanisms (excluding change-point mechanisms). GCCC is unable to change configurations within the same circuit, which means that the convex side of the coupler cannot penetrate the surface before Condition 1 or 2 occur.

GRCR can reach both open and crossed configurations (and invert the coupler). In this case, the mechanism can deploy off the surface, toggle to its open configuration, and then link 2 can re-conform to the surface, as shown in Fig. 7. We will show that when link 2 comes back to the conformed position, the convex side of the coupler has not penetrated the reference surface.

In Class 2, $\alpha < \pi/2$ (the angle between links 1 and 4) because link 4 must not cross over the center of the circle to maintain extramobility. The angle α subtends the same arc as the angle between links 1 and 3 (β). Therefore, $\alpha = \beta$ due to the inscribed angle theorem, which states that any two angles that subtend the same arc on a circle will have the same value. Hence, $\beta < \pi/2$.

When link 2 re-conforms to the surface a symmetric polygon is formed by links 3 and 4 in their open and crossed positions. Following the same logic as Class 1, $\gamma < \pi$. It is then concluded for Class 2 that the defining limits to extramobility, given the assumptions above, are set by Conditions 1 and 2.

Class 3. mechanisms are conformed in their crossed configuration (Fig. 5(*c*)). The only possible mechanism under Class 3 is GCRR (excluding change-point mechanisms). This mechanism type cannot reach both the open and crossed circuits, meaning it cannot toggle the coupler to place the convex side adjacent to the reference surface. Therefore, for Class 3, Conditions 1 and 2 will always occur before the convex side of the coupler comes into contact with the reference surface (Condition 3).

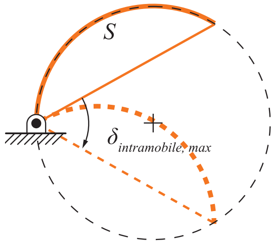


Fig. 8 The maximum amount of rotation inside the reference surface for a grounded link

3.3 Conditions for Intramobile Motion.

For grounded links (usually links 2 and 4 in traditional four-bar mechanisms), the maximum interior rotation for the link (the maximum rotation before violating Condition 1) can be calculated as described in the equation below and shown in Fig. 8, where S is the arc length of the link.

$$\delta_{intramobile,\,\max} = \pi - \frac{S}{R} \quad \text{for } (0 < S \leq \pi R) \tag{2}$$

Condition 2 can be violated if any grounded link moves away from its initial position on the reference surface then moves back to its initial position. At this point, a continuation of motion will move that link exterior to the reference surface.

To violate Condition 3, the convex side of the coupler would need to cross the reference surface. Because each link is shaped to the reference surface (see Assumption B in Sec. 3), the only way that any point on the coupler link (link 3 in traditional four-bar mechanisms) can cross the reference surface is if one or more of the endpoints has already crossed, as shown in Fig. 9. Therefore, the intramobile motion for a regular cylindrical DM is bounded by the motion of links 2 and 4 (Conditions 1 and 2).

4 Change-Point Mechanisms

Because change-point mechanisms often have unique considerations in their motion, and because many of them lie at the interface between extramobile and intramobile classes of mechanisms, they are treated separately here. Change-point mechanisms exist when

$$s + l = p + q \tag{3}$$

where s is the shortest link, l is the longest link, and p and q are the remaining two links. Because of the unique geometry that exists within a change-point mechanism, there is often crossover between where they exist in terms of the three classes of extramobile and intramobile cylindrical DMs discussed above. This

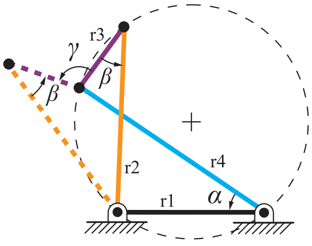


Fig. 7 Class 2 mechanism in its crossed (solid, conformed) and open (dashed) configuration

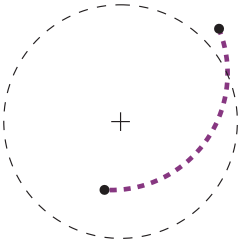


Fig. 9 The convex side of the coupler cannot cross the reference surface prior to an endpoint

Table 1 Possible change point mechanisms in each class that exhibit only extramobile and intramobile behaviors. Names follow Barker's classifications [21]

		CPCRR/ CPRRC	CPCCC	CPRCR	CP2X	CP3X
Open change-points	Class 1	✓	✓	✓	✗	✗
Crossed change-points	Class 2	✗	✗	✗	✓	✓
	Class 3	✗	✗	✗	✓	✓

section discusses where each type of change-point mechanism is found in the three classes of extramobile and intramobile mechanisms and provides logic on why these mechanisms are still unable to violate Condition 3 prior to Conditions 1 and 2. Table 1 summarizes these results and provides a reference tool to quickly identify the possible change points in each class.

The considerations described in previous sections apply for change-point mechanisms pertaining to Conditions 1 and 2 and for intramobility.

4.1 Open Change-Points. Change-point mechanisms that exist in Class 1 must, by definition, exist in an open configuration in the conformed position. As shown in Fig. 6, the longest link l in a Class 1 mechanism lies closest to the center of the circle. Hence, no other link may be of equal length to l, making it impossible to create a CP2X (two pairs of equal length links) or CP3X mechanism (all links have equal length) in a Class 1 mechanism. All other change-point mechanisms (CPCCC, CPCRR/CPRRC, and CPRCR) [21] can be created depending on the location of the shortest link relative to ground, as shown in Table 1.

4.2 Crossed Change-Points. Mechanisms in Class 2 and 3 must be in a crossed configuration when conformed. Not all change-points are capable of existing in a crossed configuration while mapped to a circle. According to Josefsson [22], the area of a crossed cyclic quadrilateral (a crossed four-bar mapped to a circle) is found by

$$K = \frac{1}{4}\sqrt{(P_1)(P_2)(P_3)(P_4)} \tag{4}$$

where

$$\begin{aligned} P_1 &= -a + b + c - d \\ P_2 &= a - b + c - d \\ P_3 &= a + b - c - d \\ P_4 &= a + b + c + d \end{aligned} \tag{5}$$

Terms a, b, c, and d represent the four link lengths of a crossed four-bar, in no particular order. Because each link must have a positive, non-zero length, $P_4 > 0$.

McCarthy and Soh showed that for a change-point mechanism, the product $P_1\ P_2\ P_3$ always equals 0 [23]. A combination of Eq. 4 and McCarthy's result suggests that a crossed change-point may exist on a circle only if the links are all co-linear in the conformed position (the circle has infinite radius). However, Hyatt et al. showed a case where a circle may have a non-infinite radius and still contain a crossed change-point mechanism at the conformed position [24]. This is only possible if at least two of the links are the same length. Therefore, the only way to obtain a change-point mechanism in Class 2 or 3 is through a CP2X or CP3X mechanism, as shown in Table 1.

4.3 Condition 3 for Extramobile Motion

Class 1. CPCCC, CPCRR/CPRRC, and CPRCR change-point mechanisms can be created depending on the location of the shortest link relative to ground. CPCCC and CPRCR mechanisms are unable to invert their coupler and therefore cannot violate Condition 3 before Condition 1 or 2 is violated, as is discussed in Sec. 3.2.2. CPCRR/CPRRC mechanisms can invert their coupler to place its convex side adjacent to the reference surface. In each case, $\gamma < \pi$, which shows that Condition 3 has not been violated before Condition 1 or 2. Therefore, for Class 1 mechanisms, Condition 1 or 2 will be reached prior to Condition 3.

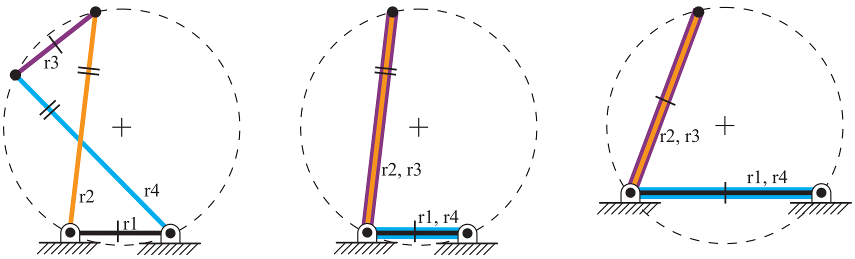
Class 2. Possible iterations of Class 2 change-point mechanisms are shown in Fig. 10. The mechanism in Fig. 10(*a*) follows the same logic as other crossed mechanisms in Class 2. If the two crossed links cross through the center of the reference surface, the toggled coupler remains tangent to the reference surface through all motion in the open configuration, meaning the coupler at no time crosses the reference surface.

To invert the coupler, the mechanisms in Figs. 10(*b*) and 10(*c*) must rotate links 2 and 3 to be co-linear with links 3 and 4 before it may reach its open configuration. Once all links are co-linear, the mechanism may only move away from the reference surface without violating Condition 1 or 2. Condition 1 is also violated prior to the convex side of the coupler contacting the reference surface.

These results lead to the conclusion that all Class 2 change-point mechanisms violate Conditions 1 or 2 prior to Condition 3 given the above assumptions.

Class 3. Possible iterations of Class 3 change-point mechanisms are shown in Fig. 11. For a Class 3 mechanism, one of the grounded links (link 2 in Fig. 5(*c*)) must be $\leq$ all other links.

Two iterations of the CP2X mechanism are possible; the first occurring when equal link lengths exist between links 1 and 4 and links 2 and 3 (Fig. 11(*a*)), and the second occurring when



(*a*) (*b*) (*c*)

Fig. 10 Class 2 change-point mechanisms in their conformed positions. Hatch marks indicate equal lengths: (*a*) CP2X where no links lie in the same position, (*b*) CP2X where links of the same length lie in the same position, and (c) CP3X where pairs of links lie in the same position and all links are the same length.

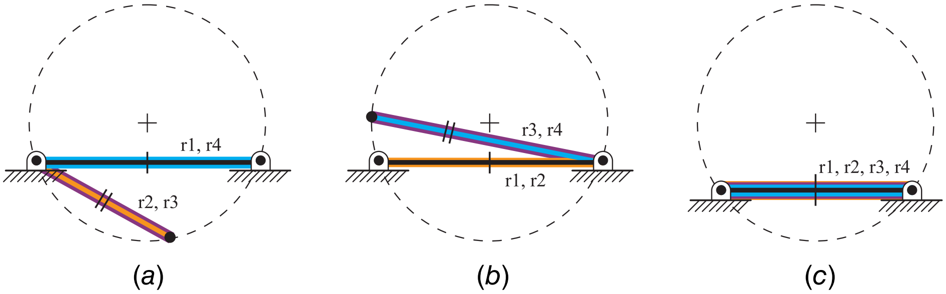


Fig. 11 Class 3 change-point mechanisms in their conformed positions. Hatch marks indicate equal lengths: (*a*) CP2X where $r_2 = r_3$ and $r_1 = r_4$, (b) CP2X where $r_1 = r_2$ and $r_3 = r_4$, and (c) CP3X where all links lie in the same position.

equal link lengths exist between links 1 and 2 and links 3 and 4 (Fig. 11(*b*)). Additionally, each pair of links that are of equivalent length must lie at the same location in the conformed position.

For the CP2X in Fig. 11(*a*), only links 2 and 3 may move from the conformed position. Since they rotate together, they may either rotate far enough away from the reference surface that they cross the reference surface simultaneously, or they may stop when collinear with the other two links. When this occurs, the mechanism moves into an open configuration, shown in Fig. 12(*a*). Because $r_2 \le r_1$, γ will never exceed π.

To invert its coupler, links 2 and 3 of CP2X shown in Fig. 11(*b*), must first rotate to the change-point position. Figure 12(*b*) demonstrates how links 2 and 3 must make a rotation greater than π radians to reach this position, violating Condition 1 prior to the mechanism being able to move back toward the reference surface.

The CP3X mechanism is obtained when all links are of the same length, as shown in Fig. 11(*c*). The convex side of the coupler cannot reach the reference surface as the coupler remains parallel with the grounded link throughout its motion.

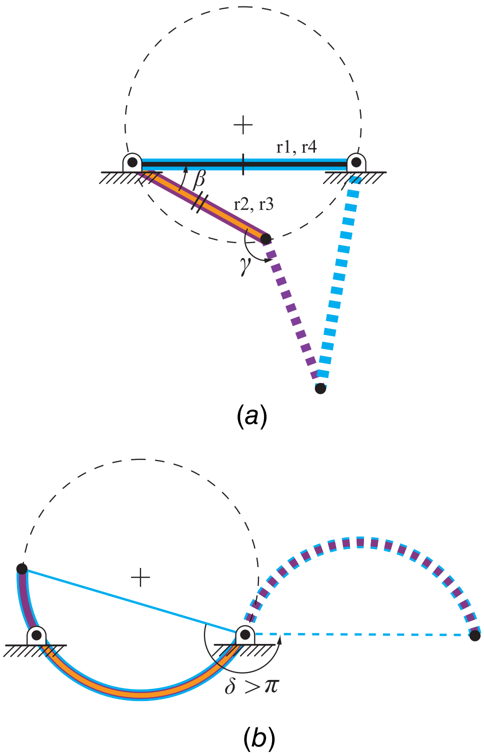


Fig. 12 Class 3 CP2X mechanisms: (*a*) a class 3 CP2X (with $r_2 = r_3$ and $r_1 = r_4$) shown in its conformed and open configurations and (*b*) a class 3 CP2X ($r_1 = r_2$ and $r_3 = r_4$) shown in its conformed position and change-point position

These results lead to the observation that, for extramobile motion, all Class 3 change-point mechanisms violate Condition 1 or 2 prior to Condition 3 given the asserted assumptions. In summary, all possible change-point mechanisms within all classes of extramobile and intramobile mechanisms will violate Conditions 1 and 2 prior to Condition 3 given the same assumptions.

5 Toggle Positions

Conditions 1 and 2 determine the limits of motion for any cylindrical developable mechanism given the stated assumptions. However, the analysis of many mechanisms may be further simplified if they have a toggle position (a position where two moving links become collinear) in their motion, as this can limit the mechanism's range of motion.

Figure 13 demonstrates a mechanism in a toggle position, where links 2 and 3 are collinear. At this position, link 4 is at an extreme of its motion and has reached $\theta_{4\min}$. Many mechanisms, including crank-rockers, double-rockers, and triple-rockers, can reach toggle positions.

Condition 1 states that no link may rotate far enough to intersect the reference surface. Figure 14 shows these limits for a single link and that links two extreme positions. If a toggle position of a mechanism is reached prior to Condition 1, such as is shown in Fig. 14, analysis of extramobile and intramobile behavior is further simplified because Condition 1 cannot be violated for that link.

When the relationship

$$|\theta_{\min,\max} - \theta_o| < \delta_{\max} \tag{6}$$

is true, then the associated link is incapable of violating Condition 1 and is only able to violate Condition 2. It should be noted that a toggle position may be considered separately for extramobile and intramobile motions. For example, a toggle position exterior to the reference surface may be reached prior to $\delta_{extramobile,\max}$ while the correlating toggle position interior to the reference surface may exceed $\delta_{intramobile,\max}$.

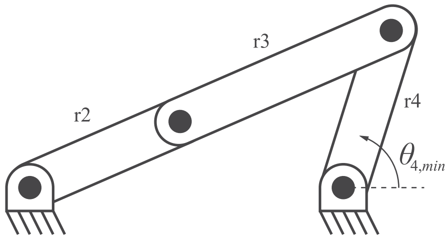


Fig. 13 A mechanism at a toggle position has reached an extreme value for one of its links

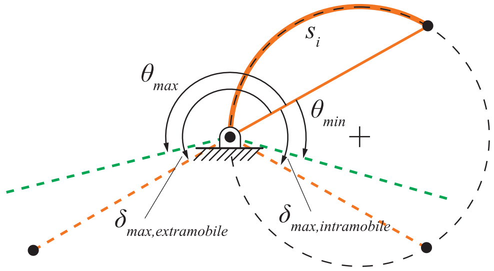


Fig. 14 It is possible for the limits of a link's motion to occur prior to δ

These toggle positions may be found through the law of cosines. Because of symmetry and the constraint that $\theta_1 = 0$, the only possible extramobile and intramobile mechanisms that may toggle are triple-rockers (RRR4), crank-rockers (GCRR), and double rockers (GRCR). In the case of a double crank, there are no toggle positions, making Condition 1 the only possible condition to violate, given the stated assumptions. Change-point mechanisms must align all links at a change-point position, making their analysis more complex than a comparison of θ and δ.

The extreme angular positions for each grounded link ($\theta_{\max}$ and $\theta_{\min}$ for subclass A mechanisms due to symmetry) can be found through the law of cosines for each case. Similar results can be shown for subclass B mechanisms (symmetrical results for each of Class 1, 2, and 3). For triple rockers (RRR4), these limits are given as follows:

$$\theta_{2\,\min,RRR4} = \arccos\left(\frac{r_1^2 + r_2^2 - (r_4 - r_3)^2}{2r_1 r_2}\right) \tag{7}$$

$$\theta_{2\,\max,RRR4} = 2\pi - \arccos\left(\frac{r_1^2 + r_2^2 - (r_4 - r_3)^2}{2r_1 r_2}\right) \tag{8}$$

$$\theta_{4\,\min,RRR4} = \pi - \arccos\left(\frac{r_1^2 + r_4^2 - (r_2 + r_3)^2}{2r_1 r_4}\right) \tag{9}$$

$$\theta_{4\,\max,RRR4} = \pi + \arccos\left(\frac{r_1^2 + r_4^2 - (r_2 + r_3)^2}{2r_1 r_4}\right) \tag{10}$$

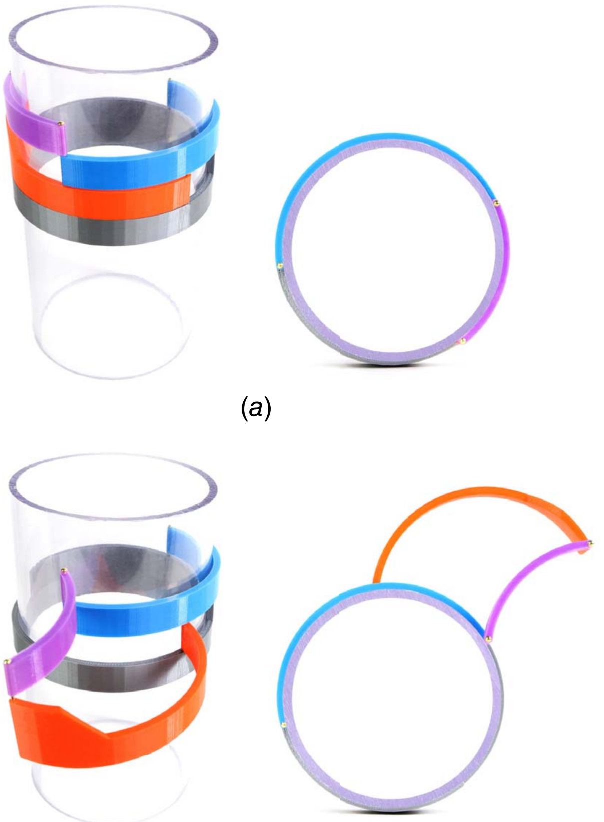
For double rockers (GRCR), the limits are

$$\theta_{2\,\min,GRCR} = \theta_{2\,\min,RRR4} \tag{11}$$

$$\theta_{2\,\max,GRCR} = \arccos\left(\frac{r_1^2 + r_2^2 - (r_3 + r_4)^2}{2r_1 r_2}\right) \tag{12}$$

$$\theta_{4\,\min,GRCR} = \theta_{4\,\min,RRR4} \tag{13}$$

$$\theta_{4\,\max,GRCR} = \pi - \arccos\left(\frac{r_1^2 + r_4^2 - (r_2 - r_3)^2}{2r_1 r_4}\right) \tag{14}$$


(*a*)

(*b*)

Fig. 15 A Class 2 extramobile mechanism demonstrating that the coupler does not cross the reference surface prior to a grounded link moving back to the conformed position: (*a*) conformed, crossed position and (*b*) limit of extramobile motion, where the coupler link has returned to the reference surface (Condition 2)


(*a*)

(*b*)

(*c*)

Fig. 16 A Class 3 intramobile mechanism moving from: (*a*) conformed crossed position, (*b*) the toggle position, and (*c*) reaching the limit of intramobile motion, where the short grounded link has rotated $\delta_{intramobile,max}$ (Condition 1)

For crank-rockers (GCRR), link 2 is fully revolute. Since there is no extreme value associated with θ_2, we will only consider θ_4.

$$\theta_{4\,\min,GCRR} = \theta_{4\,\min,RRR4} \tag{15}$$

$$\theta_{4\,\max,GCRR} = \theta_{4\,\max,GRCR} \tag{16}$$

6 Physical Demonstration

Prototypes were created to demonstrate some of the concepts discussed in this work. Two different classes were selected to demonstrate both extramobile and intramobile motions. They also demonstrate that the coupler does not cross the surface while extramobile, and that a mechanism can be made to move intramobile and reach the toggle position prior to touching the reference surface.

Figure 15 shows a sample Class 2 mechanism conformed to the exterior of a reference surface. The mechanism moves from a crossed, (*a*) conformed position into an (*b*) open position. The blue link in the image is the link that limits the extramobile motion as it moves away from the surface then back to its initial position (Condition 2). While the convex side of the coupler does invert to place it adjacent to the reference surface, it is unable to rotate sufficiently to cross the reference surface prior to the blue grounded link violating Condition 2.

Figure 16 shows a Class 3 mechanism conformed to the interior of a reference surface. This mechanism moves from the (*a*) conformed position into a (*b*) toggle position. Because this toggle is reached prior to $\delta_{intramobile,\max}$, the blue link is unable to violate Condition 1. The limit of motion then occurs when the orange link intersects the (*c*) reference surface.

7 Conclusion

The ability of a developable mechanism to have large motions exterior or interior to the reference surface provides a powerful way to create multifunctional mechanisms. The observation that Condition 3 is never violated prior to the other two conditions simplifies the design process of cylindrical DMs. As long as the assumptions above are met, a designer may create a developable mechanism and only take consideration of the grounded links to determine the limits of extramobile and intramobile motions. Additionally, under certain constraints, the design process may be further simplified if toggle positions are reached in the desired direction of motion prior to intersection with the reference surface.

This work provides a framework for the behaviors of cylindrical DMs. A next step to build on this framework would be to relax certain constraints. One direction would be to relax the constraint that the coupler must remain between the moving pivots. This would increase the functionality of the DM but require further analysis to determine the limits of mobility. Since linkages will need to physically have some appreciable thickness, another assumption to relax is that of zero-thickness. This step will help in the development of DMs that may be embedded within a material rather than only on a surface. Another logical step will be to integrate synthesis techniques with the constraints shown in this work.

An understanding of the limits of motion also enables future consideration of unique behaviors such as synthesis, bistability, and multistability in the mechanism. By integrating strain into the joints, mechanisms may be synthesized that reach desired or stable configurations while remaining exterior or interior to the reference surface.

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Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

The datasets generated and supporting the findings of this article are obtainable from the corresponding author upon reasonable request. The authors attest that all data for this study are included in the paper.

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