ASME Journal of Mechanisms and Robotics
Online journal at:
https://asmedigitalcollection.asme.org/mechanismsrobotics

Zhumadil Baigunchekov[1]
Al-Farabi Kazakh National University,
Almaty 050040, Kazakhstan
e-mail: bzh47@mail.ru

Med Amine Laribi
Department of GMSC,
Prime Institute CNRS, ENSMA,
University of Poitiers, UPR 3346.
Poitiers 86000, France
e-mail: med.amine.laribi@univ-poitiers.fr

Giuseppe Carbone
DIMEG, University of Calabria,
Via Bucci Cubo 45C,
Rende 87036, Italy
e-mail: giuseppe.carbone@unical.it

Rustem Kaiyrov
Karaganda Buketov University,
Laboratory of Applied Mechanics and Robotics,
Karaganda 100001, Kazakhstan
e-mail: kairov.rustem@mail.ru

Serik Tolenov
Al-Farabi Kazakh National University,
Almaty 050040, Kazakhstan
e-mail: tolenov.serik@mail.ru

Nurdaulet Dosmagambet
Al-Farabi Kazakh National University,
Almaty 050040, Kazakhstan
e-mail: nur_dnk@mail.ru

Structural–Parametric Synthesis of the Planar Four-Bar and Six-Bar Function Generators With Revolute Joints

This paper studies a structural–parametric synthesis of the four-bar and Stephenson II, Stephenson III A, and Stephenson III B six-bar function generators. A four-bar function generator is formed by connecting two coordinate systems with given angles of rotation using a negative closing kinematic chain (CKC) of the RR type. Six-bar function generators are formed by connecting two coordinate systems using two CKCs: a passive CKC of the RRR type and a negative CKC of the RR type. The negative CKC of the RR type imposes one geometrical constraint to the relative motion of the links, and its geometric parameters are defined by least-squares approximation. Passive CKC of the RRR type does not impose a geometrical constraint, and the geometric parameters of its links are varied to satisfy the geometrical constraint of the negative CKC. Numerical results of the four-bar and six-bar function generators parametric synthesis are presented. [DOI: 10.1115/1.4064253]

Keywords: function generator, structural–parametric synthesis, least-square approximation, mechanism design, mechanism synthesis, mechanism synthesis and analysis

1 Introduction

The first studies on the design of function-generating linkages are due to Svoboda [1,2], who designed a Watt II function generator for the generation of the logarithmic function. Kinematic synthesis (dimensional or parametric synthesis) of mechanisms, including function-generating linkages, is carried out on the basis of the kinematic geometry of finite positions of a rigid body, approximation methods (polynomials), and computers [3]. The kinematic geometry of finite positions of a rigid body, which in the case of plane motion is known as the Burmester theory [4], is used for the synthesis of function generators in the works of Hunt [5], Bottema and Roth [6], Angeles and Bai [7,8], Pira and Cunaku [9], McCarthy and Soh [10], and others. The synthesis of mechanisms by kinematic geometry is clear and simple. However, these methods are applicable for a limited number of positions. For the kinematic synthesis of mechanisms with unlimited positions of the output links, the approximation methods are used, the foundations of which were laid by Chebyshev [11]. Approximation (algebraic, optimization) methods for the kinematic synthesis of four-bar and six-bar function generators Watt II, Stephenson II, and Stephenson III were used in the works of Freudenstein [12], Hartenberg and Denavit [13], McLarnan [14], Subbian and Flugrad [15,16], Kiper et al. [17], Hwang and Chen [18], Bulatovic et al. [19], Plecnik and McCarthy [20–22], and others. In Refs. [18,19,20], the polynomial homotopic software BERTINI [23] was used. At the intersection of kinematic geometry and approximation synthesis, a new direction in the kinematic synthesis of mechanisms—approximation kinematic geometry—has been created by Sarkissyan et al. [24–26]. Approximation kinematic geometry combines the advantages of methods of kinematic geometry and approximate synthesis of mechanisms, such as simplicity and unlimited positions of the output links. Based on approximation kinematic geometry by Baigunchekov et al. [27–31], the parallel mechanism and manipulator are synthesized.

In this work, a structural–parametric synthesis of four-bar and six-bar function generators is carried out, where the structures and geometric parameters of the links of the synthesized mechanisms are determined in arbitrary given discrete values of the input and output link angles. In structural–parametric synthesis, according

[1]Corresponding author.

Contributed by the Mechanisms and Robotics Committee of ASME for publication in the JOURNAL OF MECHANISMS AND ROBOTICS. Manuscript received August 18, 2023; final manuscript received December 6, 2023; published online February 1, 2024. Assoc. Editor: Carl Nelson.

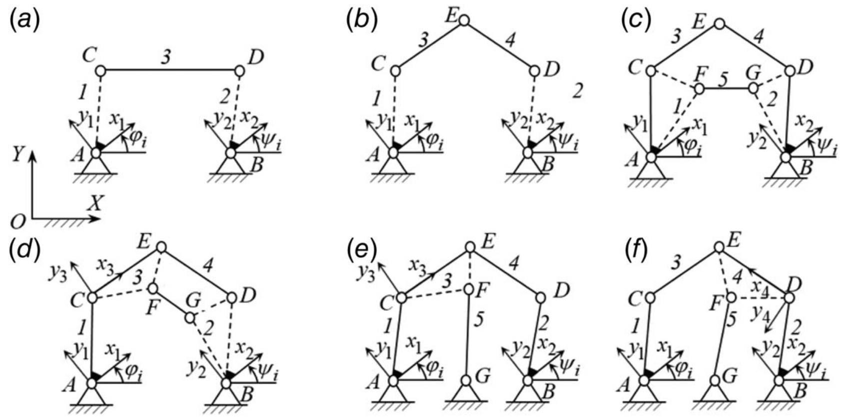

Fig. 1 Structural synthesis of planar function generators

to the given laws of motion of the input and output links, structural schemes and geometric parameters of the links of the synthesized mechanisms are simultaneously determined. At the same time, the structural–parametric synthesis of the designed mechanisms begins with the smallest number of links and becomes more complicated depending on the implementation of the specified laws of motion of the output links within the required accuracy. Depending on the complexity of the given laws of motion of the output links, it is possible to form mechanisms with complex structural schemes. Structural and kinematic analysis of complex mechanisms containing Assur groups of higher classes [32] is the subject of work [33–41]. In this paper, to analyze the positions of complex mechanisms, which is necessary to evaluate the results of parametric synthesis, a simple method of conditional generalized coordinates is proposed.

2 Structural Synthesis of the Planar Four-Bar and Six-Bar Function Generators With Revolute Joints

According to the developed principle of mechanism formation, they are formed by connecting the output object to the base using active, passive, and negative CKCs [27,28]. Active CKCs have active joints, passive CKCs have zero DOF, and negative CKCs have negative DOF. The active and passive CKCs impose geometrical constraints on the motion of the output object. The passive CKCs do not impose geometrical constraints.

The output object of the function generators is a link that performs a given rotary or translational motion relative to the base at a given motion of the input link. Let the input link and the output object perform rotational movements. We choose as the input and output links the coordinate systems Ax_1y_1 and Bx_2y_2 rotating relative to the base with rotation angles φ_i and ψ_i (Fig. 1(*a*)).

If we connect the planes of two moving coordinate systems Ax_1y_1 and Bx_2y_2 by a negative CKC *CD* of the **RR** type, then we obtain a structural scheme of a four-bar function generator *ACDB*. Connection of the planes of two moving coordinate systems Ax_1y_1 and Bx_2y_2 by binary link *CD* of the type **RR** is possible when the plane of the moving coordinate system Bx_2y_2 has a circular point (a point moving along a circle) *D* in relative motion to the coordinate system Ax_1y_1, or vice versa, i.e., when there is a circular point *C* in the plane of the coordinate system Ax_1y_1 in relative motion with respect to the coordinate system Bx_2y_2.

If none of the planes of the moving coordinate systems Ax_1y_1 and Bx_2y_2 do not have circular points in relative motion, then the planes of the two coordinate systems are connected by passive CKC *CED* of the **RRR** type. As a result, we obtain a structural scheme of the five-bar linkage *ACEDB* with two DOF (Fig. 1(*b*)). To form six-bar function generators from this five-bar linkage, we connect its non-adjacent links by binary link *FG* of the type **RR**, having one negative DOF (or a constraint that reduces the DOF of the system by one), defined by the Chebyshev formula

$$F = 3n - 2p_5 \tag{1}$$

where $n = 1$ is the number of moving links, $p_5 = 2$ is the number of kinematic pairs of the fifth class, then $F = -1$.

Consequently, the negative CKC *FG*, imposing one geometrical constraint on the five-bar linkage with two DOF, forms six-bar function generators with one DOF. Figures 1(*c*)–1(*f*) show the structural schemes of the formed six-bar function generators. If links 1 and 2 of the five-bar linkage are connected by binary link *FG*, we obtain a Stephenson I function generator (Fig. 1(*c*)). If links 3 and 2 of the five-bar linkage are connected by binary link *FG*, we obtain a Stephenson II function generator (Fig. 1(*d*)). When link 3 of the five-bar linkage is connected to the base by binary link *FG*, we obtain a Stephenson III A function generator (Fig. 1(*e*)). When link 4 of the five-bar linkage is connected to the base by binary link *FG*, we obtain a Stephenson III B function generator (Fig. 1(*f*)).

3 Parametric Synthesis of Four-Bar Function Generator

Let given the function

$$\psi_i = f(\varphi_i), \ \ i = 1, 2, \ldots, N \tag{2}$$

of the four-bar function generator (Fig. 2), where N is the number of finite positions of the moving planes Ax_1y_1 and Bx_2y_2.

It is necessary to determine the synthesis parameters (geometric parameters) of the four-bar function generator that implements function (2). The synthesis parameters are $x_C^{(1)}$, $y_C^{(1)}$, $x_D^{(2)}$, $y_D^{(2)}$ and l_{CD}, where $x_C^{(1)}$, $y_C^{(1)}$ and $x_D^{(2)}$, $y_D^{(2)}$ are the coordinates of the joints *C* and *D* in coordinate systems Ax_1y_1 and Bx_2y_2, respectively, and l_{CD} is the length of the *CD* link. Consider the movement of the coordinate system Bx_2y_2 relative to the coordinate system Ax_1y_1. In this case, point *D* moves along a circle centered at point *C* and with radius l_{CD}. Derive an equation

$$(x_{D_i}^{(1)} - x_C^{(1)})^2 + (y_{D_i}^{(1)} - x_C^{(1)})^2 - l_{CD}^2 = 0 \tag{3}$$

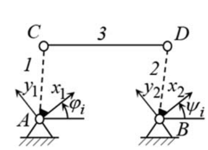

Fig. 2 Four-bar function generator

where

$$\begin{bmatrix} x_{Di}^{(1)} \\ y_{Di}^{(1)} \end{bmatrix} = \begin{bmatrix} \cos\varphi_i & \sin\varphi_i \\ -\sin\varphi_i & \cos\varphi_i \end{bmatrix} \cdot \begin{bmatrix} X_{Di} - X_A \\ Y_{Di} - Y_A \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} X_{Di} \\ Y_{Di} \end{bmatrix} = \begin{bmatrix} X_B \\ Y_B \end{bmatrix} + \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi & \cos\psi_i \end{bmatrix} \cdot \begin{bmatrix} x_D^{(2)} \\ y_D^{(2)} \end{bmatrix} \tag{5}$$

Equation (3) is an equation of geometrical constraint imposed by binary link CD of the type **RR** on the movements of two moving coordinate systems Ax_1y_1 and Bx_2y_2. The left side of Eq. (3) will be denoted by Δq_i and called the weighted difference function

$$\Delta q_i = (x_{Di}^{(1)} - x_C^{(1)})^2 + (y_{Di}^{(1)} - x_C^{(1)})^2 - l_{CD}^2 \tag{6}$$

Parametric synthesis of a four-bar function generator is to determine five geometric parameters $x_C^{(1)}$, $y_C^{(1)}$, $x_D^{(2)}$, $y_D^{(2)}$, and l_{CD} from the minimum of function (6). Substituting expressions (4) and (5) into Eq. (6), we obtain

$$\begin{aligned} \Delta q_i = 2\Big\{ & \big[-(X\_X_A)\cos\varphi_i - (Y_B - Y_A)\sin\varphi_i\big] \cdot x_C^{(1)} + \\ & + \big[(X_B - X_A)\sin\varphi_i - (Y_B - Y_A)\cos\varphi_i\big] \cdot y_C^{(1)} + \\ & + \left[\frac{1}{2}\left(x_C^{(1)^2} + y_C^{(1)^2} + x_D^{(2)^2} + y_D^{(2)^2} - l_{CD}^2\right)\right] + \\ & + \big[(X_B - X_A)\cos\psi_i + \big[(Y_B - Y_A)\sin\psi_i\big] \cdot x_D^{(2)} + \\ & + \big[-(X_B - X_A)\sin\psi_i + (Y_D - Y_A)\cos\psi_i\big] \cdot y_D^{(2)} + \\ & + \big[-\cos(\psi_i - \varphi_i) \cdot (x_C^{(1)} \cdot x_D^{(2)} + y_C^{(1)} \cdot y_D^{(2)})\big] + \\ & + \big[\sin(\psi_i - \varphi_i) \cdot (x_C^{(1)} \cdot y_D^{(2)} - y_C^{(1)} \cdot x_D^{(2)})\big] + \\ & + \frac{1}{2}\big[(X_B - X_A)^2 + (X_B - Y_A)^2\big]\Big\} \end{aligned} \tag{7}$$

If we introduce the notations

$$\begin{aligned} & p_1 = x_C^{(1)},\ p_2 = y_C^{(1)},\ p_3 = \frac{1}{2}(x_C^{(1)^2} + y_C^{(1)^2} + x_D^{(2)^2} + y_D^{(2)^2} - l_{CD}^2), \\ & p_4 = x_D^{(2)},\ p_5 = y_D^{(2)}, \\ & f_{1i} = -(X_B - X_A)\cos\varphi_i - (Y_B - Y_A)\sin\varphi_i, \\ & f_{2i} = (X_B - X_A)\sin\varphi_i - (Y_B - Y_A)\cos\varphi_i, \\ & f_3 = 1,\ f_{4i} = (X_B - X_A)\cos\psi_i + (Y_B - Y_A)\sin\psi_i, \\ & f_{5i} = -(X_B - X_A)\sin\psi_i + (Y_B - Y_A)\cos\psi_i,\ f_{6i} = -\cos(\psi_i - \varphi_i), \\ & f_{7i} = \sin(\psi_i - \varphi_i), \\ & f_{0i} = \frac{1}{2}[(X_B - X_A)^2 + (X_B - Y_A)^2] \end{aligned}$$

then Eq. (7) takes the form

$$\begin{aligned} \Delta q_i = 2[& p_1f_{1i} + p_2f_{2i} + p_3f_3 + p_4f_{4i} + p_5f_{5i} + (p_1p_4 + p_2p_5)f_{6i} + \\ & + (p_1p_5 - p_2p_4)f_{7i} - f_{0i}] \end{aligned} \tag{8}$$

Equation (8) is linear in the following two groups of synthesis parameters:

$$\mathbf{p}^{(1)} = [p_1, p_2, p_3],\ \mathbf{p}^{(2)} = [p_4, p_5, p_3] \tag{9}$$

and is represented in two linear forms

$$\begin{aligned} \Delta q_i^{(1)} = 2[& p_1(f_{1i} + p_4f_{6i} + p_5f_{7i}) + p_2(f_{2i} + p_5f_{6i} + p_4f_{7i}) + p_3f_3 + \\ & + (p_4f_{4i} + p_5f_{5i} - f_{0i})] \end{aligned} \tag{10}$$

and

$$\begin{aligned} \Delta q_i^{(2)} = 2[& p_4(f_{4i} + p_1f_{6i} - p_2f_{7i}) + p_5(f_{5i} + p_2f_{6i} + p_1f_{7i}) + p_3f_3 + \\ & + (p_1f_{1i} + p_2f_{2i} - f_{0i})] \end{aligned} \tag{11}$$

If the following notations are introduced

$$\mathbf{g}_i^{(1)} = [g_{1i}^{(1)}, g_{2i}^{(1)}, g_{3i}^{(1)}],\ \mathbf{g}_i^{(2)} = [g_{1i}^{(2)}, g_{2i}^{(2)}, g_{3i}^{(2)}]$$

where

$$\begin{aligned} & g_{1i}^{(1)} = f_{1i} + p_4f_{6i} + p_5f_{7i},\ g_{2i}^{(1)} = f_{2i} + p_5f_{6i} + p_4f_{7i},\ g_{3i}^{(1)} = f_3, \\ & g_{1i}^{(2)} = f_{4i} + p_1f_{6i} + p_2f_{7i},\ g_{2i}^{(2)} = f_{5i} + p_2f_{6i} + p_1f_{7i},\ g_{3i}^{(2)} = f_3, \\ & g_{0i}^{(1)} = p_4f_{4i} + p_5f_{5i} - f_{0i},\ g_{0i}^{(2)} = p_1f_{1i} + p_2f_{2i} - f_{0i} \end{aligned}$$

then Eqs. (10) and (11) take the form

$$\Delta q_i^{(k)} = 2(\mathbf{g}_i^{(k)^T}\mathbf{p}^{(k)} - g_{0i}^{(k)}),\ k = 1, 2 \tag{12}$$

To determine the vectors $\mathbf{p}^{(k)}$ of synthesis parameters, we minimize function (12) by least-square optimization, i.e., derive the sums

$$S^{(k)} = \sum_{i=1}^{N} (\Delta q_i^{(k)})^2 \tag{13}$$

and consider the necessary conditions for the minimum of function (13) over two groups of synthesis parameters $\mathbf{p}^{(k)}$

$$\frac{\partial S^{(1)}}{\partial p_1} = 0,\ \frac{\partial S^{(1)}}{\partial p_2} = 0,\ \frac{\partial S^{(1)}}{\partial p_3} = 0 \tag{14}$$

and

$$\frac{\partial S^{(2)}}{\partial p_4} = 0,\ \frac{\partial S^{(2)}}{\partial p_5} = 0,\ \frac{\partial S^{(2)}}{\partial p_3} = 0 \tag{15}$$

Conditions (14) and (15) lead to the following two systems of linear equations for two groups of synthesis parameters:

$$\mathbf{D}^{(1)} \cdot \mathbf{p}^{(1)^T} = \mathbf{d}^{(1)} \tag{16}$$

and

$$\mathbf{D}^{(2)} \cdot \mathbf{p}^{(2)^T} = \mathbf{d}^{(2)} \tag{17}$$

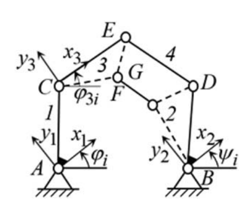

Fig. 3 Stephenson II function generator

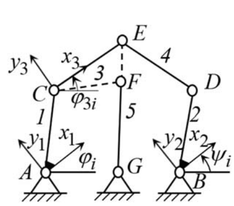

Fig. 4 Stephenson III A function generator

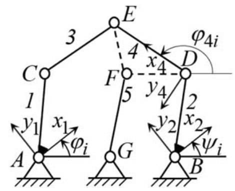

Fig. 5 Stephenson III B function generator

where

$$\mathbf{D}^{(1)}=\sum_{i=1}^{N}\begin{bmatrix} g_{1i}^{(1)^2} & g_{1i}^{(1)}\cdot g_{2i}^{(1)} & g_{1i}^{(1)} \\ g_{1i}^{(1)}\cdot g_{2i}^{(1)} & g_{2i}^{(1)^2} & g_{2i}^{(1)} \\ g_{1i}^{(1)} & g_{2i}^{(1)} & 1 \end{bmatrix},$$

$$\mathbf{d}^{(1)}=\sum_{i=1}^{N}\begin{bmatrix} g_{1i}^{(1)}\cdot g_{0i}^{(1)} \\ g_{2i}^{(1)}\cdot g_{0i}^{(1)} \\ g_{0i}^{(1)} \end{bmatrix} \quad (18)$$

$$\mathbf{D}^{(2)}=\sum_{i=1}^{N}\begin{bmatrix} g_{1i}^{(2)^2} & g_{1i}^{(2)}\cdot g_{2i}^{(22)} & g_{1i}^{(2)} \\ g_{1i}^{(2)}\cdot g_{2i}^{(2)} & g_{2i}^{(2)^2} & g_{2i}^{(2)} \\ g_{1i}^{(2)} & g_{2i}^{(2)} & 1 \end{bmatrix},$$

$$\mathbf{d}^{(1)}=\sum_{i=1}^{N}\begin{bmatrix} g_{1i}^{(2)}\cdot g_{0i}^{(2)} \\ g_{2i}^{(2)}\cdot g_{0i}^{(2)} \\ g_{0i}^{(2)} \end{bmatrix} \quad (19)$$

It is easy to show that the Hessian of matrices $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ are positive definite together with the principal minors [23], and then the solutions of the systems of linear Eqs. (16) and (17) correspond to the minimum of function (13). Therefore, for given values of the vector parameters $\mathbf{p}^{(2)}=[p_4, p_5, p_3]$, the vector parameters $\mathbf{p}^{(1)}=[p_1, p_2, p_3]$ are determined by solving the system of linear Eq. (16). Based on the obtained values of the vector parameters $\mathbf{p}^{(1)}$, the vector parameters $\mathbf{p}^{(2)}$ are determined by solving the system of linear Eq. (17). In this case, the sequence of function $S^{(k)}$ values will be decreasing and have a limit as a sequence bounded from below. This allows using the linear iteration method based on kinematic inversion to solve the least-square approximation.

4 Parametric Synthesis of Six-Bar Function Generators

Parametric synthesis of six-bar function generators (Figs. 1(*c*)–1(*f*)) consists of the parametric synthesis of the passive CKC *CED* and the negative CKC *FG*. Synthesis parameters of the passive CKC *CED* of all six-bar function generators are $x_C^{(1)}$, $y_C^{(1)}$, $x_D^{(2)}$, $y_D^{(2)}$, l_{CE}, and l_{ED}, where $x_C^{(1)}$, $y_C^{(1)}$ and $x_D^{(2)}$, $y_D^{(2)}$ are the coordinates of the joints *C* and *D* in the coordinate systems Ax_1y_1 and Bx_2y_2 of the links 1 and 2, respectively, l_{CE} and l_{ED} are the lengths of the *CE* and *ED* links. Since the passive CKC *CED* of the type **RRR** has zero DOF, it does not impose a geometrical constraint on the motion of the coordinate systems Ax_1y_1 and Bx_2y_2. Geometric parameters of the passive CKC *CED* links are varied, and the synthesis parameters of the negative CKC *FG* are approximated. For the parametric synthesis of the Stephenson I (Fig. 1(*c*)), Stephenson II (Fig. 1(*d*)), Stephenson III A (Fig. 1(*e*)), and Stephenson III B (Fig. 1(*f*)) function generators, we determine the

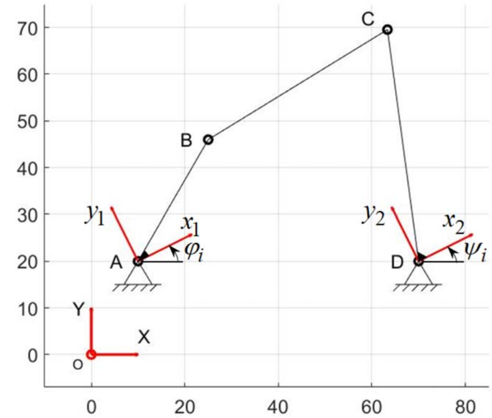

Fig. 6 Four-bar function generator with scale size

positions of the links *CE* and *ED* of the passive CKC *CED* by the equations

$$\varphi_{3i}=\varphi_{(CD)_i}+\cos^{-1}\frac{l_{CE}^2+l_{(CD)_i}^2-l_{ED}^2}{2l_{CE}\cdot l_{(CD)_i}} \quad (20)$$

$$\varphi_{4i}=\mathrm{tg}^{-1}\frac{Y_{E_i}-Y_{D_i}}{X_{E_i}-X_{D_i}} \quad (21)$$

where

$$l_{(CD)_i}=[(X_{D_i}-X_{C_i})^2+(Y_{D_i}-Y_{C_i})^2]^{\frac{1}{2}} \quad (22)$$

$$\varphi_{(CD)_i}=\mathrm{tg}^{-1}\frac{Y_{D_i}-Y_{C_i}}{X_{D_i}-X_{C_i}} \quad (23)$$

$$\begin{bmatrix} X_{C_i} \\ Y_{C_i} \end{bmatrix}=\begin{bmatrix} X_A \\ Y_A \end{bmatrix}+\begin{bmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{bmatrix}\cdot\begin{bmatrix} x_C^{(1)} \\ y_C^{(1)} \end{bmatrix} \quad (24)$$

$$\begin{bmatrix} X_{D_i} \\ Y_{D_i} \end{bmatrix}=\begin{bmatrix} X_B \\ Y_B \end{bmatrix}+\begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{bmatrix}\cdot\begin{bmatrix} x_D^{(2)} \\ y_D^{(2)} \end{bmatrix} \quad (25)$$

Synthesis parameters for the negative CKC *FG* of the Stephenson I mechanism (Fig. 1(*c*)) are $x_F^{(1)}$, $y_F^{(1)}$, $x_G^{(2)}$, $y_G^{(2)}$, and l_{FG}, which are determined similar to the parametric synthesis of the four-bar function generator (Fig. 2). Therefore, the functionality of the Stephenson I mechanism is the same as the functionality of the four-bar function generator.

Synthesis parameters for the negative CKC *FG* of the Stephenson II function generator (Fig. 3) are $x_F^{(3)}$, $y_F^{(3)}$, $x_G^{(2)}$, $y_G^{(2)}$, and l_{FG}, where $x_F^{(3)}$, $y_F^{(3)}$ and $x_G^{(2)}$, $y_G^{(2)}$ are the coordinates of the joints *F* and *G* in the coordinate systems Cx_3y_3 and Bx_2y_2 of the links 3 and 2, respectively.

For parametric synthesis of the link *FG*, we consider the movement of the coordinate system Bx_2y_2 relative to the coordinate system Cx_3y_3 and derive the equation of geometrical constraint

$$(x_{G_i}^{(3)}-x_F^{(3)})^2+(y_{G_i}^{(3)}-y_F^{(3)})^2-l_{FG}^2=0 \quad (26)$$

Table 1 The values of the angles φ_i and ψ_i of the four-bar function generator

N	1	2	3	4	5	6	7	8	9	10	11	12
φ_i	0 deg	30 deg	60 deg	90 deg	120 deg	150 deg	180 deg	210 deg	240 deg	270 deg	300 deg	330 deg
ψ_i	36 deg	38 deg	51 deg	69 deg	88 deg	102 deg	108 deg	108 deg	104 deg	98 deg	84 deg	57 deg

Table 2 The values of the synthesis parameters of the four-bar function generator

$x_C^{(1)}$	$y_C^{(1)}$	$x_D^{(2)}$	$y_D^{(2)}$	l_{CD}
25.980	15.008	25.007	43.300	45.012

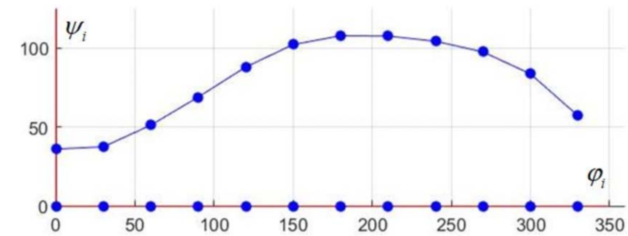

Fig. 7 Graph of the angle ψ_i change of the four-bar function generator

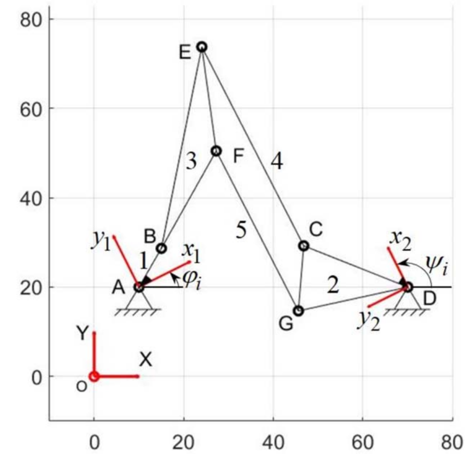

Fig. 8 Stephenson II function generator with scale size

where

$$\begin{bmatrix} X_{G_i}^{(3)} \\ Y_{G_i}^{(3)} \end{bmatrix} = \begin{bmatrix} \cos\varphi_{3i} & \sin\varphi_{3i} \\ -\sin\varphi_{3i} & \cos\varphi_{3i} \end{bmatrix} \cdot \begin{bmatrix} X_{G_i} - X_{C_i} \\ Y_{G_i} - Y_{C_i} \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} X_{G_i} \\ YG_i \end{bmatrix} = \begin{bmatrix} X_B \\ Y_B \end{bmatrix} + \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{bmatrix} \cdot \begin{bmatrix} x_G^{(2)} \\ y_G^{(2)} \end{bmatrix} \tag{28}$$

Further, the synthesis parameters of the link FG are determined similarly to the determination of synthesis parameters of the link CD of the four-bar function generator.

Synthesis parameters of the binary link FG of the Stephenson III A function generator (Fig. 4) and the Stephenson III B function generator (Fig. 5) are $x_F^{(3)}, y_F^{(3)}$ and $x_F^{(4)}, y_F^{(4)}$, respectively, and X_G, Y_G, and l_{FG} are for both function generators, where $x_F^{(3)}, y_F^{(3)}$ and $x_F^{(4)}, y_F^{(4)}$ are the coordinates of the joint F in the coordinate systems Cx_3y_3 and Dx_4y_4, respectively, X_G and Y_G are the coordinates of the joint G in the absolute coordinate system OXY.

For parametric synthesis of the link FG of the Stephenson III A and Stephenson III B function generators, we derive the following geometrical constraint equation:

$$(X_{F_i} - X_G)^2 + (Y_{F_i} - Y_G)^2 - l_{FG}^2 = 0 \tag{29}$$

where for the Stephenson III A function generator

$$\begin{bmatrix} X_{F_i} \\ Y_{F_i} \end{bmatrix} = \begin{bmatrix} X_{C_i} \\ Y_{C_i} \end{bmatrix} + \begin{bmatrix} \cos\varphi_{3i} & -\sin\varphi_{3i} \\ \sin\varphi_{3i} & \cos\varphi_{3i} \end{bmatrix} \cdot \begin{bmatrix} x_F^{(3)} \\ y_F^{(3)} \end{bmatrix} \tag{30}$$

for the Stephenson III B function generator

$$\begin{bmatrix} X_{F_i} \\ Y_{F_i} \end{bmatrix} = \begin{bmatrix} X_{D_i} \\ Y_{D_i} \end{bmatrix} + \begin{bmatrix} \cos\varphi_{4i} & -\sin\varphi_{4i} \\ \sin\varphi_{4i} & \cos\varphi_{4i} \end{bmatrix} \cdot \begin{bmatrix} x_F^{(4)} \\ y_F^{(4)} \end{bmatrix} \tag{31}$$

Further, synthesis parameters of the binary link FG are determined similar to the parametric synthesis of the four-bar function generator.

5 Numerical Results

Let us consider the parametric synthesis of the four-bar, Stephenson II, Stephenson III A, and Stephenson III B function generators. The following coordinates $X_A = 10.0$, $Y_A = 20.0$, $X_B = 70.0$, and $Y_B = 20.0$ of the four-bar Stephenson II, Stephenson III A, and Stephenson III B function generators pivots A and B are given in the absolute coordinate system OXY.

5.1 Four-Bar Function Generator. Table 1 gives the values of the angles φ_i and ψ_i of the input and output links for $N = 12$ of the four-bar function generator (Fig. 6).

Table 2 presents the obtained values of the synthesis parameters of the four-bar function generator.

Table 3 The obtained values of the angle ψ_i

N	1	2	3	4	5	6
φ_i ψ_i	0 deg 36.17 deg	30 deg 37.60 deg	60 deg 51.30 deg	90 deg 69.09 deg	120 deg 88.10 deg	150 deg 102.32 deg
N	7	8	9	10	11	12
φ_i ψ_i	180 deg 107.91 deg	2100 107.71 deg	2400 104.44 deg	2700 97.67 deg	3000 83.77 deg	3300 57.27 deg

Table 4 The values of the angles φ_i and ψ_i of the Stephenson II function generator

N	1	2	3	4	5	6	7	8	9	10	11	12
φ_i ψ_i	0 deg 104 deg	30 deg 98 deg	60 deg 93 deg	90 deg 90 deg	120 deg 90 deg	150 deg 92 deg	180 deg 98 deg	210 deg 105 deg	240 deg 112 deg	270 deg 116 deg	300 deg 116 deg	330 deg 111 deg

Table 5 The values of the synthesis parameters of the Stephenson II function generator

$x_C^{(1)}$	$y_C^{(1)}$	l_{CE}	l_{ED}	$x_D^{(2)}$	$y_D^{(2)}$	$x_F^{(3)}$	$y_F^{(3)}$	$x_G^{(2)}$	$y_G^{(2)}$	l_{GF}
8.658	5.003	45.983	50.017	12.496	21.645	23.775	−7.725	20.715	13.975	40.211

Table 6 The obtained values of the angle ψ_i

N	1	2	3	4	5	6
φ_i	0 deg	30 deg	60 deg	90 deg	120 deg	150 deg
ψ_i	104.12 deg	98.35 deg	93.18 deg	90.24 deg	89.54 deg	92.07 deg
N	7	8	9	10	11	12
φ_i	180 deg	210 deg	240 deg	270 deg	300 deg	330 deg
ψ_i	97.57 deg	105.21 deg	111.87 deg	116.16 deg	115.75 deg	110.83 deg

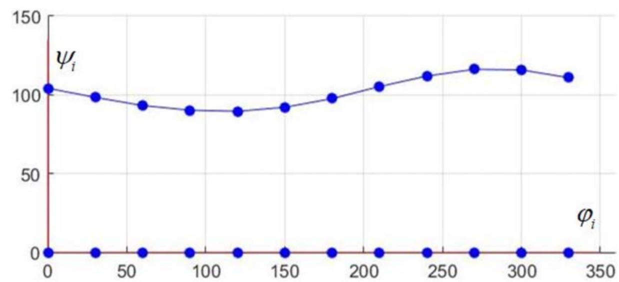

Fig. 9 Graph of the angle ψ_i change of the Stephenson II function generator

To check the parametric synthesis results of the four-bar function generator, values of the angle ψ_i corresponding to the values of the angle φ_i are determined by the equation

$$\psi_i = \varphi_{(BC)_i} - \cos^{-1}\frac{l_{BD}^2 + l_{(BC)_i}^2 - l_{CD}^2}{2l_{BD}\cdot l_{(BC)_i}} \tag{32}$$

where

$$\varphi_{(BC)_i} = \mathrm{tg}^{-1}\frac{Y_{C_i} - Y_B}{X_{C_i} - X_B} \tag{33}$$

$$l_{(BC)_i} = [(X_{C_i} - X_B)^2 + (Y_{C_i} - Y_B)^2]^{\frac{1}{2}} \tag{34}$$

$$l_{BD} = (x_D^{(2)^2} + y_D^{(2)^2})^{\frac{1}{2}} \tag{35}$$

In the expressions (33) and (34), coordinates of the joint C in the absolute coordinate system OXY are determined by the equation

$$\begin{bmatrix} X_{C_i} \\ Y_{C_i} \end{bmatrix} = \begin{bmatrix} X_A \\ Y_A \end{bmatrix} + \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{bmatrix} \cdot \begin{bmatrix} x_C^{(1)} \\ y_C^{(1)} \end{bmatrix} \tag{36}$$

Table 3 presents the obtained values of the angle ψ_i, and Fig. 7 shows a graph of its change.

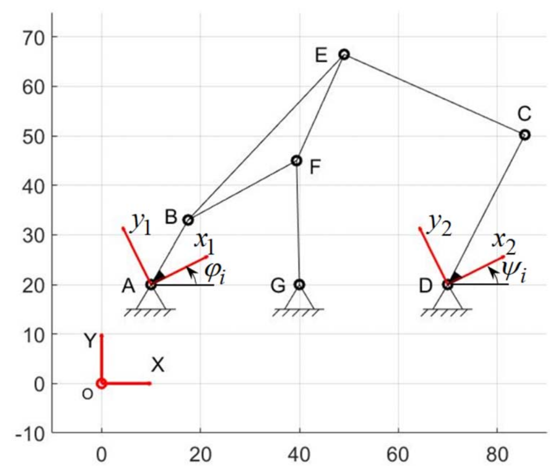

Fig. 10 Stephenson III A function generator with scale size

5.2 Stephenson II Function Generator.

Table 4 gives the values of the angles φ_i and ψ_i of the input and output links for $N = 12$ of the Stephenson II function generator (Fig. 8).

Table 5 presents the obtained values of the synthesis parameters of the Stephenson II function generator.

To check the parametric synthesis results of the Stephenson II function generator, we determine the values of the angle ψ_i corresponding to the values of the angle φ_i. To do this, the positions of the Stephenson II function generator are analyzed.

Stephenson II function generator contains an input link 1 and an Assur group of the fourth class. It has the following structural formula:

$$\mathrm{I}\ (1) \rightarrow \mathrm{IV}\ (2, 3, 4, 5) \tag{37}$$

For position analysis of the Stephenson II function generator or the mechanism of the fourth class, a simple method of conditional generalized coordinate is proposed, which is the opposite of the method of structural synthesis. If we remove a negative CKC—a binary link FG of the type of **RR** by disconnecting the elements of the joints F and G—then the considered mechanism acquires one additional DOF. If we take the link 2 as a conditional input

Table 7 The values of the angles φ_i and ψ_i of the Stephenson III A function generator

N	1	2	3	4	5	6	7	8	9	10	11	12
φ_i	0 deg	30 deg	60 deg	90 deg	120 deg	150 deg	180 deg	210 deg	240 deg	270 deg	300 deg	330 deg
ψ_i	11 deg	03 deg	07 deg	15 deg	24 deg	33 deg	44 deg	57 deg	71 deg	83 deg	77 deg	43 deg

Table 8 The values of the synthesis parameters of the Stephenson III A function generator

$x_C^{(1)}$	$y_C^{(1)}$	l_{CE}	l_{ED}	$x_D^{(2)}$	$y_D^{(2)}$	$x_F^{(3)}$	$y_F^{(3)}$	X_G	Y_G	l_{GF}
12.979	7.503	45.947	40.014	16.971	29.438	23.775	−7.725	39.983	21.087	24.897

Table 9 The obtained values of the angle ψ_i

N	1	2	3	4	5	6
φ_i ψ_i	0 deg 11.21 deg	30 deg 02.98 deg	60 deg 06.87 deg	90 deg 14.91 deg	120 deg 23.93 deg	150 deg 33.16 deg
N	7	8	9	10	11	12
φ_i ψ_i	180 deg 43.70 deg	210 deg 57.01 deg	240 deg 70.96 deg	270 deg 82.91 deg	300 deg 77.04 deg	330 deg 42.83 deg

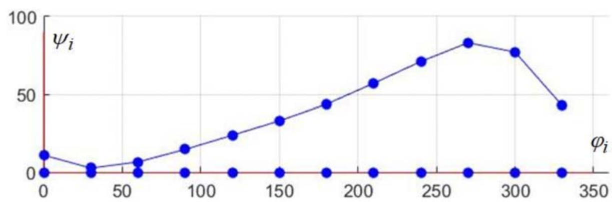

Fig. 11 Graph of the angle ψ_i change of the Stephenson III A function generator

link and the angle ψ_i as a conditional generalized coordinate, then this mechanism of the fourth class is transformed into a mechanism of the second class with the structural formula

$$\text{I (1)} \rightarrow \text{II (3, 4)} \leftarrow \text{I(2)} \tag{38}$$

For a given value of the angle φ_i and in changing the value of the conditional generalized coordinate ψ_i, the distance between the centers of the disconnected joints F and G is changed. A residual function is derived below:

$$\Delta_i = l_{FG} - l^*_{(FG)_i} \tag{39}$$

where $l^*_{(FG)_i}$ is a variable distance between the centers of the joints F and G, which is determined by the equation

$$l^*_{(FG)_i} = [(X_{G_i} - X_{F_i})^2 + (Y_{G_i} - Y_{F_i})^2]^{\frac{1}{2}} \tag{40}$$

Coordinates of the joints F and G in Eq. (40) are determined by the expressions

$$\begin{bmatrix} X_{F_i} \\ Y_{F_i} \end{bmatrix} = \begin{bmatrix} X_{C_i} \\ Y_{C_i} \end{bmatrix} + \begin{bmatrix} \cos\varphi_{3i} & -\sin\varphi_{3i} \\ \sin\varphi_{3i} & \cos\varphi_{3i} \end{bmatrix} \cdot \begin{bmatrix} x_F^{(3)} \\ y_F^{(3)} \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} X_{G_i} \\ Y_{G_i} \end{bmatrix} = \begin{bmatrix} X_B \\ Y_B \end{bmatrix} + \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{bmatrix} \cdot \begin{bmatrix} x_G^{(2)} \\ y_G^{(2)} \end{bmatrix} \tag{42}$$

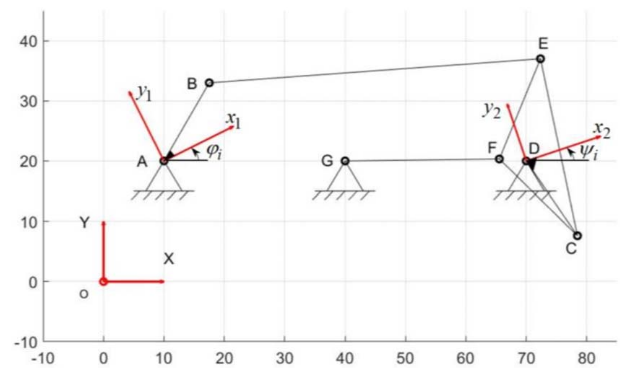

Fig. 12 Stephenson III B function generator with scale size

where

$$\begin{bmatrix} X_{C_i} \\ Y_{C_i} \end{bmatrix} = \begin{bmatrix} X_A \\ Y_A \end{bmatrix} + \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{bmatrix} \cdot \begin{bmatrix} x_C^{(1)} \\ y_C^{(1)} \end{bmatrix} \tag{43}$$

To determine the angle φ_{3i} in Eq. (41), we analyze the positions of the dyad II (3,4) and obtain

$$\varphi_{3i} = \varphi_{(CD)_i} + \cos^{-1}\frac{l_{CE}^2 + l_{(CD)_i}^2 - l_{ED}^2}{2l_{CE}\cdot l_{(CD)_i}} \tag{44}$$

where

$$\varphi_{(CD)_i} = \text{tg}^{-1}\frac{Y_{D_i} - Y_{C_i}}{X_{D_i} - X_{C_i}} \tag{45}$$

$$l_{(CD)_i} = [(X_{D_i} - X_{C_i})^2 + (Y_{D_i} - Y_{C_i})^2]^{\frac{1}{2}} \tag{46}$$

$$\begin{bmatrix} X_{D_i} \\ YD_i \end{bmatrix} = \begin{bmatrix} X_B \\ Y_B \end{bmatrix} + \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{bmatrix} \cdot \begin{bmatrix} x_D^{(2)} \\ y_D^{(2)} \end{bmatrix} \tag{47}$$

Consequently, the residual (39) is a function of the conditional generalized coordinate ψ_i. Minimizing this function by the bisection method, we obtain the values of the angle ψ_i for a given value of the

Table 10 The values of the angles φ_i and ψ_i of the Stephenson III B function generator

N	1	2	3	4	5	6	7	8	9	10	11	12
φ_i ψ_i	0 deg 223 deg	30 deg 244 deg	60 deg 266 deg	90 deg 264 deg	120 deg 245 deg	150 deg 240 deg	180 deg 245 deg	210 deg 256 deg	240 deg 268 deg	270 deg 267 deg	300 deg 245 deg	330 deg 221 deg

Table 11 The values of the synthesis parameters of the Stephenson III B function generator

$x_C^{(1)}$	$y_C^{(1)}$	l_{CE}	l_{ED}	$x_D^{(2)}$	$y_D^{(2)}$	$x_F^{(3)}$	$y_F^{(3)}$	X_G	Y_G	l_{GF}
12.978	7.493	55.017	29.962	7.507	13.004	15.029	10.134	39.995	19.917	25.545

Table 12 The obtained values of the angle ψ_i

N	1	2	3	4	5	6
φ_i	0 deg	30 deg	60 deg	90 deg	120 deg	150 deg
ψ_i	223.29 deg	244.38 deg	265.81 deg	264.01 deg	245.44 deg	240.18 deg
N	7	8	9	10	11	12
φ_i	180 deg	210 deg	240 deg	270 deg	300 deg	330 deg
ψ_i	244.96 deg	255.87 deg	267.45 deg	267.04 deg	244.83 deg	220.71 deg

angle φ_i. By changing the values of the angle φ_i, we similarly find the corresponding values of the angle ψ_i.

Table 6 presents the obtained values of the angle ψ_i, and Fig. 9 shows a graph of its change.

5.3 Stephenson III A Function Generator.

Table 7 gives values of the angles φ_i and ψ_i of the input and output links of the Stephenson III A function generator (Fig. 10).

Table 8 presents the obtained values of the synthesis parameters of the Stephenson III A function generator.

To check the parametric synthesis results of the Stephenson III A function generator, let us determine the values of the angle ψ_i corresponding to the values of the angle φ_i. To do this, we analyze the positions of the Stephenson III A function generator. This function generator has a structural formula

$$\text{I (1)} \rightarrow \text{II (3, 5)} \rightarrow \text{II (4, 2)} \tag{48}$$

i.e., it belongs to the second class mechanism. Therefore, it is necessary to consistently solve the position problems of the dyads *CFG* and *EDB*.

From the position analysis of the dyad *CFG*, we obtain

$$\varphi_{5_i} = \varphi_{(GC)_i} - \cos^{-1}\frac{l_{GF}^2 + l_{(GC)_i}^2 - l_{CF}^2}{2l_{GF}\cdot l_{(GC)_i}} \tag{49}$$

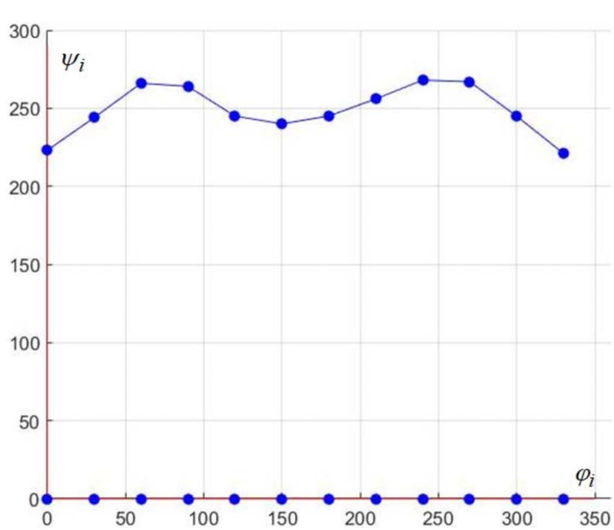

Fig. 13 Graph of the angle ψ_i change of the Stephenson III B function generator

$$\varphi_{(CF)_i} = \text{tg}^{-1}\frac{Y_{F_i} - Y_{C_i}}{X_{F_i} - X_{C_i}} \tag{50}$$

where

$$l_{(GC)_i} = [(X_{C_i} - X_G)^2 + (Y_{C_i} - Y_G)^2]^{\frac{1}{2}} \tag{51}$$

$$\varphi_{(GC)_i} = \text{tg}^{-1}\frac{Y_{C_i} - Y_G}{X_{C_i} - X_G} \tag{52}$$

Based on the obtained values of the angle $\varphi_{(CF)_i}$, we determine the value of the angle φ_{3_i}

$$\varphi_{3_i} = \varphi_{(CF)_i} + \alpha_3 \tag{53}$$

and the coordinates of the joint E

$$\begin{bmatrix} X_{E_i} \\ Y_{E_i} \end{bmatrix} = \begin{bmatrix} X_{C_i} \\ Y_{C_i} \end{bmatrix} + l_{CE}\begin{bmatrix} \cos\varphi_{3_i} \\ \sin\varphi_{3_i} \end{bmatrix} \tag{54}$$

where

$$\alpha_3 = \text{tg}^{-1}\frac{y_F^{(3)}}{x_F^{(3)}} \tag{55}$$

Then, we solve the position problem of the dyad *EDB* and determine the value of the angle ψ_i

$$\psi_i = \varphi_{(BE)_i} - \cos^{-1}\frac{l_{BD}^2 + l_{(BE)_i}^2 - l_{ED}^2}{2l_{BD}\cdot l_{(BE)_i}} - \psi^0 \tag{56}$$

where

$$l_{(BE)_i} = [(X_{E_i} - X_B)^2 + (Y_{E_i} - Y_B)^2]^{\frac{1}{2}} \tag{57}$$

$$\varphi_{(BE)_i} = \text{tg}^{-1}\frac{Y_{E_i} - Y_B}{X_{E_i} - X_B} \tag{58}$$

$$\psi^0 = \text{tg}^{-1}\frac{y_D^{(2)}}{x_D^{(2)}} \tag{59}$$

Table 9 presents the obtained values of the angle ψ_i, and Fig. 11 shows a graph of its change.

5.4 Stephenson III B Function Generator.

Table 10 gives the values of the angles φ_i and ψ_i of the input and output links of the Stephenson III B function generator (Fig. 12).

Table 11 presents the obtained values of the synthesis parameters of the Stephenson III B function generator.

Let us determine the values of the angle ψ_i corresponding to the values of the angle φ_i. To do this, we analyze the positions of the Stephenson III B function generator. This function generator has a structural formula

$$\mathrm{I}\ (1) \rightarrow \mathrm{III}\ (2, 3, 4, 5) \tag{60}$$

i.e., it belongs to the mechanism of the third class. For position analysis of this mechanism, we remove the negative CKC *FG* by disconnecting the elements of the joints *F* and *G*. Then, the considered mechanism acquires one additional DOF. If we take link 2 as a conditional input link, due to the appeared DOF, then the third class mechanism is transformed into the second class mechanism with structural formula (38).

Let us derive a residual function of the form (39), where the coordinates of the joint *F* are determined by the equation

$$\begin{bmatrix} X_{Fi} \\ Y_{Fi} \end{bmatrix} = \begin{bmatrix} X_{Di} \\ Y_{Di} \end{bmatrix} + \begin{bmatrix} \cos\varphi_{4i} & -\sin\varphi_{4i} \\ \sin\varphi_{4i} & \cos\varphi_{4i} \end{bmatrix} \cdot \begin{bmatrix} x_F^{(4)} \\ y_F^{(4)} \end{bmatrix} \tag{61}$$

To determine the angle φ_{4i} in Eq. (61), we solve a position analysis problem of the dyad *CED* and obtain

$$\varphi_{4i} = \varphi_{(DC)i} - \cos^{-1}\frac{l_{ED}^2 + l_{(DC)i}^2 - l_{CE}^2}{2l_{ED}\cdot l_{(DC)i}} \tag{62}$$

where

$$\varphi_{(DC)i} = \mathrm{tg}^{-1}\frac{Y_{Ci} - Y_{Di}}{X_{Ci} - X_{Di}} \tag{63}$$

$$l_{(DC)i} = [(X_{Ci} - X_{Di})^2 + (Y_{Ci} - Y_{Di})^2]^{\frac{1}{2}} \tag{64}$$

Consequently, the residual (39) is a function of the conditional generalized coordinate ψ_i. Minimizing this function by the bisection method, we obtain the values of the angle ψ_i. Table 12 presents the obtained values of the angle ψ_i, and Fig. 13 shows a graph of its change.

6 Conclusion

Structural synthesis of four-bar and six-bar function generators with revolute joints has been carried out. A four-bar function generator is formed by connecting two rotating coordinate systems with given rotation angles using a binary link of the type **RR**, which is a negative CKC that imposes one geometrical constraint. Six-bar function generators are formed by connecting these two rotationally moving coordinate systems using a passive CKC of the type **RRR** and by connecting non-adjacent links of the resulting five-bar linkage by binary link of the type **RR**. As a result, Stephenson I, Stephenson II, Stephenson III A, and Stephenson III B function generators have been formed. Passive CKC of the type **RRR** does not impose a geometrical constraint on the movement of two moving coordinate systems, and its geometric parameters are varied to satisfy the constraint of the negative CKC. Geometric parameters of the negative CKC of the type **RR** are determined by least-square approximation. In this case, the least-square approximation problem is reduced to a simple kinematic inversion problem based on linear iteration. Structural and parametric synthesis of the four-bar and six-bar function generators are carried out simultaneously, starting with the smallest number of links of CKCs. Numerical results of parametric synthesis of four-bar and six-bar function generators are presented.

Acknowledgment

This work was funded by the Science Committee of the Ministry of Science and High Education of Kazakhstan (Grant No. AP14872115 "Development and Research of the Novel Tripod Type Parallel Manipulators With Six Degrees of Freedom").

Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

The datasets generated and supporting the findings of this article are obtainable from the corresponding author upon reasonable request.

References

[1] Svoboda, A., 1994, "Mechanism for Use in Computing Apparatus," U.S. Patent No. 2,340,350.
[2] Svoboda, A., 1948, *Computing Mechanisms and Linkages*, McGraw-Hill, New York.
[3] McCarthy, M. J., 2011, "Kinematics, Polynomials, and Computers—A Brief History," ASME J. Mech. Rob., **3**(1), p. 010201.
[4] Burmester, L., 1888, *Lehrbuch der Kinematik*, Artur Felix Verlag, Leipzig, Germany.
[5] Hunt, K. H., 1978, *Kinematic Geometry of Mechanisms*, Oxford University Press, New York.
[6] Bottema, O., and Roth, B., 1979, *Theoretical Kinematics*, North Holland Publishing Company, Amsterdam, New York, Oxford.
[7] Angeles, J., and Bai, S., 2016, *Kinematic Synthesis, Lecture Notes*, McGill University, Montreal, PQ, Canada.
[8] Angeles, J., and Bai, S., 2005, "Some Special Cases of the Burmester Problem for Four and Five Poses," Proceedings of IDETC/CIE 2005, Long Beach, CA, Sept. 24–26, pp. 307–314, PAper No. DETC 2005-84871.
[9] Piza, B., and Cunaku, I., 2017, "Synthesis of Watt and Stephenson Six Bar Mechanisms Using Burmester Theory," Int. J. Curr. Technol. Eng., **7**(1), p. 5.
[10] McCarthy, J. M., and Soh, G. S., 2010, *Geometric Design of Linkages*, 2nd ed., Springer-Verlag, Berlin.
[11] Chebyshev, P. L., 1897, "Sur Les Parallelogrammes Composes de Trois Elements Quelcongues," Memoires de l'Academic des Sciences de Saint-Petersbourg, **36**(Suppl. 3).
[12] Freudenstein, F., 1954, "An Analytical Approach to the Design of Four-Link Mechanisms," Trans. ASME, **76**, pp. 483–492.
[13] Hartenberg, R. S., and Denavit, J., 1964, *Kinematic Synthesis of Linkages*, McGraw-Hill, New York, NY.
[14] McLarnan, C. W., 1963, "Synthesis of Six-Link Mechanisms by Numerical Analysis," ASME J. Eng. Ind., **85**(1), pp. 5–10.
[15] Subbian, T., and Flugrad, D. R., 1993, "Five Position Triad Synthesis With Applications to Four- and Six-Bar Mechanisms," ASME J. Mech. Des., **115**(2), pp. 262–268.
[16] Subbian, T., and Flugrad, D. R., Jr., 1994, "Six and Seven Position Triad Synthesis Using Continuation Methods," ASME J. Mech. Des., **116**(2), pp. 660–665.
[17] Kiper, G., Dede, M. İ. C., Maaroof, O. W., and Özkahya, M., 2017, "Function Generation With Two-Loop Mechanisms Using Decomposition and Correction Method," Mech. Mach. Theory, **110**, pp. 16–26.
[18] Huang, W. M., and Chen, Y. J., 2010, "Defect-Free Synthesis of Stephenson II Function Generators," ASME J. Mech. Rob., **2**(4), p. 041012.
[19] Bulatovic', R. R., Dozdevic', S. R., and Dordevic, V. S., 2013, "Cuckoo Search Algorithm: A Metaheuristic Approach to Solving the Problem of Optimum Synthesis of a Six-Bar Double Dwell Linkage," Mech. Mach. Theory, **61**, pp. 1–13.
[20] Plecnik, M., and McCarthy, J. M., 2014, "Numerical Synthesis of Six-Bar Linkages for Mechanical Computation," ASME J. Mech. Rob., **6**(2), p. 0310012.
[21] Plecnik, M. M., and McCarthy, J. M., 2016, "Kinematic Synthesis of Stephenson III Six-Bar Function Generators," Mech. Mach. Theory, **97**, pp. 112–126.
[22] Plecnik, M., and McCarthy, J. M., 2013, "Synthesis a Stephenson II Function Generator for Eight Precision Positions," Proceedings of the IDETC/CIE 2013, Portland, OR, Aug. 4–7, p. 10.
[23] Bates, D. J., Hauenstein, J. D., Sommese, A. J., and Wampler, C. W., 2013, *Numerically Solving Polynomial Systems With Bertini*, SIAM Press, Philadelphia, PA.
[24] Sarkissyan, Y. L., Gupta, K. C., and Roth, B., 1973, "Kinematic Geometry Associated With the Least Square Approximation of a Given Motion," J. Eng. Ind., **95**(2), pp. 503–510.
[25] Sarkissyan, Y. L., 1982, *Approximation Synthesis of Mechanisms (in Russian)*, Nauka, Moscow.
[26] Sarkissyan, Y. L., Stepanyan, K. G., and Verlinski, S. V., 2015, "Rigid Body Points Approximating Concentric Circles in Given Sets of Its Planar Displacements," Proceedings of the 14th IFToMM World Congress, Taipei, Taiwan, Oct. 25–30, Vol. 1, pp. 57–61.
[27] Baigunchekov, Z., Laribi, M. A., Carbone, G., Mustafa, A., Amanov, B., and Zholdassov, Y., 2021, "Structural-Parametric Synthesis of the RoboMech Class Parallel Mechanism With Two Sliders," Appl. Sci., **11**(21), pp. 9831; 18.
[28] Baigunchekov, Z., Laribi, M. A., Mustafa, A., and Kassinov, A., 2021, "Kinematic Synthesis and Analysis of the RoboMech Class Parallel Manipulator With Two Grippers," Robotics, **10**(3), pp. 99, 16.

[29] Baigunchekov, Z., Mustafa, A., Sobh, T., Patel, S., and Utenov, M., 2020, "A RoboMech Class Parallel Manipulator With Three DOF," East.-Eur. J. Enterp. Technol., **3**(7–105), pp. 44–56.

[30] Baigunchekov, Z., Izmambetov, M., Zhumasheva, Z., Baigunchekov, T., and Mustafa, A., 2019, "Parallel Manipulator of a Class RoboMech for Generation of Horizontal Trajectories Family," Mech. Mach. Sci., **73**, pp. 1395–1402.

[31] Baigunchekov, Z., Ibrayev, S., Izmambetov, M., Naurushev, B., and Mustafa, A., 2019, "Synthesis of Cartesian Manipulator of a Class RoboMech," Mech. Mach. Sci., **66**, pp. 69–76.

[32] Assur, L., 1914, "Research of a Planar Linkage with Lower Pairs on the Basis of Their Structure and Classification," Proceedings of the Saint-Petersburg Polytechnic Institute, Saint-Petersburg, pp. 20–21.

[33] Peng, H., and Huafeng, D., 2020, "Structural Synthesis of Assur Groups With up to 12 Links and Creation of Their Classified Databases," Mech. Mach. Theory, **145**, p. 103668.

[34] Yang, W., Ding, H., and Kecskeméthy, A., 2022, "Structural Synthesis Towards Intelligent Design of Planar Mechanisms: Current Status and Future Research Trend," Mech. Mach. Theory, **171**, pp. 104715.

[35] Morlin, F. V., Carboni, A. P., and Martins, D., 2023, "Synthesis of Assur Groups via Group and Matroid Theory," Mech. Mach. Theory, **184**, pp. 105279.

[36] Mlynarski, T., 1996, "Position Analysis of Planar Linkages Using the Method of Modification of Kinematic Unit," Mech. Mach. Theory, **31**(6), pp. 831–838.

[37] Mitsi, S., 1999, "Position Analysis in Polynomial Form of Planar Mechanisms With a Closed Chain of the Assur Group of Class 4," Mech. Mach. Theory, **34**(8), pp. 1195–1209.

[38] Mitsi, S., Bouzakis, K. D., and Mansour, G., 2004, "Position Analysis in Polynomial Form of Planar Mechanism With an Assur Group of Class 4 Including One Prismatic Joint," Mech. Mach Theory, **39**(3), pp. 237–245.

[39] Han, L., Liao, Q., and Liang, C., 2000, "Closed-Form Displacement Analysis for a Nine-Link Barranov Truss or an Eight-Link Assur Group," Mech. Mach. Theory, **35**(3), pp. 379–390.

[40] Zhang, Q., Zou, H. J., and Guo, W. Z., 2006, "Position Analysis of Higher-Class Assur Groups by Virtual Variable Searching and Its Application in a Multifunction Domestic Sewing Machine," Int. J. Adv. Manuf. Technol., **28**(5–6), pp. 602–609.

[41] Sun, Y. W., Ge, W., Zheng, G., and Dong, D., 2016, "Solving the Kinematics of the Planar Mechanism Using Data Structures of Assur Groups," ASME J. Mech. Rob., **8**(6), p. 061002.