Parameterizing Surface and Internal Tide Scattering and Breaking on Supercritical Topography: The One- and Two-Ridge Cases

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ABSTRACT

A parameterization is presented for turbulence dissipation due to internal tides generated at and impinging upon topography steep enough to be “supercritical” with respect to the tide. The parameterization requires knowledge of the topography, stratification, and the remote forcing—either barotropic or baroclinic. Internal modes that are arrested at the crest of the topography are assumed to dissipate, and faster modes assumed to propagate away. The energy flux into each mode is predicted using a knife-edge topography that allows linear numerical solutions. The parameterization is tested using high-resolution two-dimensional numerical models of barotropic and internal tides impinging on an isolated ridge, and for the generation problem on a two-ridge system. The recipe is seen to work well compared to numerical simulations of isolated ridges, so long as the ridge has a slope steeper than twice the critical steepness. For less steeply sloped ridges, near-critical generation becomes more dominant. For the two-ridge case, the recipe works well when compared to numerical model runs with very thin ridges. However, as the ridges are widened, even by a small amount, the recipe does poorly in an unspecified manner because the linear response at high modes becomes compromised as it interacts with the slopes.

1. Introduction

The fate of tidal energy in the deep ocean is still not fully understood, despite the expectation that it is important in driving the vertical mixing of heat into the abyss and ultimately driving the global overturning circulation (Munk and Wunsch 1998). Approximately 1 TW of energy is believed to be lost from the barotropic tide in the deep ocean, compared to 2 TW on shallow shelves. However, the fate of that 1 TW of energy is still unknown.

Experiments at sites where surface tides are converted to internal tides (Polzin et al. 1997; Rudnick et al. 2003) indicate modest local turbulence. At topography that is predominantly subcritical to the internal tides, like some midocean spreading centers, it is estimated that perhaps 30% of the energy lost from the surface tide goes into local dissipation (St. Laurent and Nash 2004), though more recent theoretical estimates make that fraction more variable depending on the local forcing (Polzin 2009) and the Coriolis frequency (Nikurashin and Legg 2011). These theories probably need more testing. However, the fate of the rest of the energy, predominantly in low modes, is not well understood. Similarly, for abrupt topography typical of midocean ridges, like Hawaii, the fraction of energy dissipated locally is expected to be
modest, both from available observations (Klymak et al. 2006) and from a theory similar to the one in this paper (Klymak et al. 2010b). Except for the strongest forcing or shallower topography, the local dissipation is expected to be less than 10% of the energy removed from the surface tide; the rest again radiates away as low modes.

The local fraction of dissipation, while small compared to the total barotropic energy converted to internal wave energy, is still spectacular. Observations near Hawaii show overturns exceeding 200-m height at the ridge crest (Levine and Boyd 2006; Aucan et al. 2006; Klymak et al. 2008). These near-bottom “breakers” were shown to be dominated by trapped lee waves that are generated near the ridge crest during each tidal cycle and then propagate past the moorings as the tide changes (Legg and Klymak 2008; Klymak et al. 2010b). This lee wave mechanism seems to dominate dissipation at supercritical ridges, though the observational tests of this are still being carried out (Alford et al. 2011). This motivated us to create a parameterization that predicts the energy in these lee waves. In steady state, the lee-wave size can be predicted by comparing the speed of the horizontally propagating internal modes to the speed of the flow at the crest of the obstacle (Klymak and Legg 2010), with the lee wave representing a critical mode such that \( c_c \approx U_m \), where \( c_c \) is the deep-water speed of the critical mode and \( U_m \) is the speed of the barotropic flow at the ridge crest. Noting that such waves set up quickly if the topography is supercritical (Klymak et al. 2010a), we used the arrested wave criteria in an oscillating flow to determine the critical vertical modes at the obstacle crest.

Dissipation in these waves was determined by considering the energy put into the modes higher than the critical mode (i.e., the trapped slow modes) as determined from a knife-edge model (St. Laurent et al. 2003; Llewellyn Smith and Young 2003). The resulting recipe was tested versus numerical simulations (Klymak et al. 2010b, hereafter KLP10) with quite good effectiveness. However, the fraction of energy dissipated locally from such ridges remained quite low, with much of the energy radiating away.

Given that much of the energy escapes supercritical topography, the question arises as to where it goes. One possibility is scattering of the radiated low modes at remote topography. Here we consider the dissipation that can occur when incoming internal tides impinge on a steep isolated obstacle, as well as the dissipation occurring during barotropic tidal flow over a double ridge system. In doing so, we extend the KLP10 recipe for dissipation during barotropic tidal flow over a single steep ridge to include multiple ridges and incoming baroclinic tides. The extended recipe is applied to several test cases, for which it is evaluated by comparison with explicit numerical simulations. The incoming tide problem may have applications to observations south of Hawaii on the tides impacting the Line Islands Ridge and similar systems. The two-ridge problem was specifically aimed at Luzon Strait on the east side of the South China Sea. Recent efforts have indicated that double ridge systems may have stronger dissipation under barotropic tidal forcing than single ridges, with estimates ranging from 10% in a two-dimensional system (Buijsman et al. 2012) to 40% from a coarse three-dimensional model (Alford et al. 2011). Testing this configuration seems warranted, though preliminary efforts indicate that the two-dimensional assumption in the recipe presented here may be too simple (Buijsman et al. 2012).

We start with a discussion of the numerical model that we use to test our recipe with (section 2) and then describe the phenomenology we are trying to model (section 3). The recipe itself is described (section 4) and tested on a one-ridge topography and a two-ridge topography, using the two dimensional numerical model as the “truth.” For both setups the recipe uses a knife-edge model to generate internal tides (described in the appendices), so the effect of varying width is tested (section 5) before summarizing and discussing further caveats (section 6).

### 2. The numerical model

As in KLP10, the proposed dissipation recipe will be tested against a two-dimensional simulation using the Massachusetts Institute of Technology general circulation model (MITgcm) (Marshall et al. 1997). This model has been used for other two-dimensional wave breaking problems (Legg and Adcroft 2003; Legg and Huijts 2006; Legg and Klymak 2008; Klymak et al. 2010a). Forcing was applied via velocity and density nudging at boundaries more than two mode-1 horizontal wavelengths from the topography. The model was run using the hydrostatic approximation for numerical efficiency; tests with the nonhydrostatic terms did not reveal substantially different responses for this particular scenario (Klymak and Legg 2010). The models all use Gaussian topography, defined as \( h(x) = h_1 \exp(-x^2/\sigma^2) \). For the idealized runs below, we use a constant initial stratification of \( N_0 = 5.2 \times 10^{-3} \text{ s}^{-1} \) in a total water depth of \( H = 2000 \text{ m} \), and a Coriolis frequency of \( f = 10^{-4} \text{ s}^{-1} \). As noted in KLP10, the Coriolis frequency enters into the generation–dissipation problem with more energy generated and dissipated for lower \( f \), but that the barotropic recipe does very well across the range of \( f \), so we did not vary \( f \) here.

The dissipation scheme employed in this model is described in Klymak and Legg (2010) and consists of...
applying a high vertical viscosity and diffusivity whenever there are density overturns owing to breaking waves. The diffusivity is scaled by the size of the density overturns so that the energy loss $\epsilon$ is consistent with the Ozmidov scale $L_O$.

$$\epsilon = L_O^2 N^3,$$  \hspace{1cm} (1)

where $N$ is the stratification after overturns have been removed by density sorting. From this we obtain a turbulent viscosity and diffusivity of $K = 0.2 \epsilon / N^2$ or $K = 10^{-5} \text{ m}^2 \text{ s}^{-1}$, whichever is larger. The limitations of this scheme are that it does not account for shear-driven mixing, and it does not work well if the breaking internal waves are small compared to the vertical grid size. This scheme does a better job than a local Mellor–Yamada 2.0 scheme (Mellor and Yamada 1982) and constant viscosities (Legg and Klymak 2008) at yielding energetically consistent estimates of dissipation for the parameter regimes explored here.

3. Phenomenology

As with barotropic generation (discussed in detail in KLP10), the dissipation of an incoming tide from a Gaussian ridge is dominated by an arrested lee wave that forms during each phase of the tide that has strong flow (Fig. 1). For a given topography and stratification, the size of this wave and the turbulence that it generates depend on the incoming tide. For a mode-1 internal tide this dependence is relatively straightforward (Figs. 1a,b), similar to the barotropic case. A lee wave forms on each side of the ridge during alternating phases of the incoming tide with slightly more dissipation on the side of the topography facing the incoming wave (Fig. 1b).

For a wavefield with more than one mode, the response depends on the relative phasing of the modes, which complicates determining the dissipation a priori (Figs. 1c–h). Even for just a mode-1 and mode-2 incoming wave, the phasing of the modes changes the response at the crest of the topography and can vary the dissipation by almost
a factor of 3. Lee waves still form every cycle, but their size depends on whether the flow is reinforced or interfered with at the crest of the ridge.

Of recent interest because of work in the South China Sea, the generation and resonance of a two-ridge system is also considered. The generation and dissipation problem is similarly complex. The dissipation can vary significantly for the same forcing depending on the distance between the ridges (Fig. 2). As we show below, this has to do with whether the “beams” from the ridges constructively or destructively interfere with one another. However, as with the other supercritical topography cases, the dissipation is dominated by near-crest turbulence generated in breaking trapped lee waves, so the same approach as for a baroclinic incoming tide is suggested below.

4. Recipe and tests

The new, more general, recipe has the same ingredients as the recipe for the barotropic case (KLP10): a theoretical generation model gives the rate that energy is generated or scattered into radiated modes $F_n$. Then the modes that have slower deep water phase speed ($c_n$) than a characteristic ridge-top speed ($U_n$) (i.e., $c_n < U_n$) are assumed to dissipate. The function $F_n$ was calculated from a linear knife-edge model following St. Laurent et al. (2003).

For a WKB-stretched ocean with depth $H$, constant stratification $N$, and a knife-edge ridge of height $h$ (and ridge-top water depth $H-h$, see Fig. 3) we decompose the forcing and response into vertical modes that obey the eigenvalue problem:

$$\frac{d^2\phi}{dz^2} + \frac{N^2}{c_e^2}\phi(z) = 0,$$

where it is found from the boundary conditions at the seafloor and surface that $c_e(m) = NH/m\pi, m = 1, 2, \ldots,$ and

$$\phi_m(z) = \cos\left(\frac{m\pi z}{H}\right).$$

Suppose we have an isolated piece of topography with a tide coming in from the positive-x direction. We assume a forcing comprised of incoming vertical modes:

$$u_i = \Re\left[\sum_{m=0}^{M} a_i(m)\phi_m(z)e^{i(k_m x + \omega t)}\right],$$

where $a_i(m)$ is the complex amplitude of each vertical mode with shape $\phi_m(z)$, $k_m$ is the horizontal wavenumber, and $\omega$ is the frequency of the tide. The horizontal wavenumber is determined by $k_m = (\omega^2 - f^2)^{1/2}/c_e(m)$. The horizontal phase and group speeds are related to the eigenspeeds by

$$c_g = c_e \sqrt{\frac{\omega^2 - f^2}{\omega}}$$

and

$$c_e = \frac{\sqrt{\omega^2 - f^2}}{\omega}.$$
\[ c_p = c \frac{\omega}{\sqrt{\omega^2 - f^2}} \]  

(6)

The internal response to this forcing is assumed to comprise a transmitted internal wave signal and a reflected one:

\[ u_i = \Re \left[ \sum_{m=1}^{M} a_i(m) \phi_m(z) e^{i(k_m x + \omega t)} \right] \]  

and

\[ u_r = \Re \left[ \sum_{m=1}^{M} a_r(m) \phi_m(z) e^{i(k_m x - \omega t)} \right] \]  

(7)

(8)

If we assume a knife-edge topography and a linear response, the amplitudes \( a_i(m) \) and \( a_r(m) \) can be determined by matching the velocities at the topography so that \( u = 0 \) at depths deeper than the ridge crest \((z < -H + h)\) and \( u \) and \( w \) are matched above the ridge crest \((z > -H + h)\). This leads to a matrix that can be inverted for the modal amplitudes, \( a_i(m) \) and \( a_r(m) \), as described in the appendix. These modal amplitudes can be expressed as energy fluxes by the relation:

\[ F_n = \frac{H^2}{n\pi} g(\omega) \frac{|a_i|^2}{2}, \]

(9)

where \( g(\omega) \) is

\[ g(\omega) = \rho \left[ (N^2 - \omega^2)(\omega^2 - f^2) \right]^{1/2}. \]

(10)

The recipe for the turbulence requires knowing for what values of \( m \) the modes are arrested by ridge-top velocities and, thus, are trapped and dissipate as part of the lee wave. For the barotropic situation in KLP10 the ridge-top speed was simply given by the barotropic speed at the crest of the obstacle:

\[ U_n = U_T \frac{H}{H-h}, \]

(11)

where \( U_T \) is the deep water barotropic tide, \( H \) is the depth of the deep water, and \( h \) is the height of the obstacle.

Such a simple scaling does not work for the baroclinic case, as should be readily apparent from the examples given above (Figs. 1 and 2); the phasing of the forcing and response matters and must be taken into account when determining the trapped mode that will form the lee wave. To account for the response, we propose a modified recipe as follows.

First, presupposing that the critical mode is \( M \), the cross-ridge velocity response at the top of the ridge(s) is calculated from the linear solution made up only of modes lower than or equal to \( M \); that is,

\[ u_M(z,t) = \sum_{m=0}^{M} a_m(z) e^{i(k_m x - \omega t)} \]

(12)

Here \( a_m \) is meant to represent the solution on either side of the ridge crest, so for our single-ridge example \( a_m = a_r = a_i + a_r \).

Second, the velocity response is averaged for half a vertical wavelength of the critical mode, \( \lambda_M = H/M \) over the crest of the sill, and the maximum taken over the tidal cycle:

\[ U_M = \max \left| \langle u_M(z,t) \rangle \right|_{z=-H+h} \]

(13)

The vertical scale to average over is chosen following Klymak et al. (2010a), where the size of the lee wave is shown to be on the order of half a vertical wavelength of the arrested mode. This half-wavelength-averaged velocity scale is recalculated starting with the first mode and moving to higher modes until a mode \( M \) is found such that \( c_M \approx U_M \). This mode \( M \) is the first arrested mode for which \( c_M = c_c(M) \) is the eigenspeed of the \( M \)th mode. Once we determine the first arrested mode \( M \), the dissipation is calculated as \( D = \sum_{n=M}^{\infty} F_n \), where \( F_n \) is the rate that energy is predicted to be put into each mode.

So, to summarize,

(i) we determine the linear response due to the forcing represented by the modal amplitudes \( a_i(n) \), and thus the coefficients \( a_i(n) \), \( a_r(n) \) and the energy fluxes \( F_n \), and

(ii) iterate through all modes \( M \) and smooth the response at the top of the sill by \( H/M \), to determine a velocity scale at the top of the ridge \( U_M \);

(iii) the lowest mode with eigenspeed slower than the corresponding \( U_M \) (i.e., \( c_M < U_M \)) is chosen as the critical mode.

(iv) the dissipation is the sum of the rate of energy input into modes \( M \) and higher: \( D = \sum_{n=M}^{\infty} F_n \).

a. Test 1: Scattering of mode 1 or mode 2 from an isolated ridge

The recipe requires the expected velocity profile at the top of the ridge. To start, we consider the scattering of an incoming internal tide from a single isolated ridge. The problem can be solved numerically using linear algebra in a manner analogous to the barotropic generation
problem (St. Laurent et al. 2003; KLP10) and the scattering problem from a continental shelf (Chapman and Hendershott 1981; Klymak et al. 2011). If an incoming tide is specified by modal amplitudes $d_m$, the net can calculate the transmitted internal tide $a_m$, and the reflected $b_n$, by assuming the velocities match above the ridge and are zero below the ridge. Details follow the above papers and are presented briefly in the appendix. From this, we can construct the velocity profile at the top of the ridge.

First, we illustrate the iterative procedure to determine the critical mode, as described above. An example velocity profile is considered for a ridge with $h/H = 0.61$, $N = 5.2 \times 10^{-3}$ s$^{-1}$, and incoming mode-1 tide of amplitude $d_1 = 0.2$ m s$^{-1}$. For these runs, supercritical ridges were used with $\sigma = 10$ km. For all modes (Fig. 4, gray lines, which are the same in all the panels), the velocity has a very sharp maximum at the ridge crest, and then a zero crossing approximately 200 m above (the high wavenumber oscillations are due to choosing a finite number of modes to represent the solutions). Figure 4a shows what happens if we guess that the critical mode is $M = 4$: $c_4 = 0.84$ m s$^{-1}$ (thin dashed line), and the black curve is the solution composed of only the first four modes. The mean of this $M = 4$ curve for half a wavelength above the ridge crest is much less than $c_4$: $U_4 = 0.04$ m s$^{-1}$ (black dashed line), so mode-4 is not “critical” and can propagate away from the ridge. Trying the procedure on higher modes shows that $M = 12$ is still too low, $M = 20$ is too high, but $M = 16$ is the first mode that is critical.

The same procedure applies if the incoming mode-1 tide is stronger in amplitude, with a corresponding drop in the critical mode as amplitude increases (Fig. 5). Similarly, the response for different ridge heights changes nonmonotonically as the ridge and the incoming mode shapes interfere (Fig. 6). However, in general, very tall ridges do not dissipate as much as shorter ridges because a large fraction of energy reflects as low-mode waves. This is in contrast to the situation for a barotropic flow over a ridge where the flow is forced through the constriction, developing large velocities, and therefore turbulent velocities.

The dissipation predicted by the recipe ($D_{Th}$) agrees very well with a suite of two-dimensional numerical experiments ($D_{model}$; Fig. 7; see Table 1 for parameter space). There does tend to be some overestimation of the numerical model turbulence by the theory, though
just by a small factor. The strongest dissipations are also somewhat poorly constrained, likely as much because of the difficulty in estimating the internal tide amplitudes in a strongly nonlinear environment as to a problem with the recipe.\footnote{The model is forced by boundary nudging, and thus the response away from the boundary is hard to specify precisely and must be estimated from the model state, rather than be known a priori.} Low dissipations predicted by the knife-edge model start to be overpredicted significantly. This is because the numerical model’s vertical resolution is too low to properly resolve the turbulence in the lee waves as their vertical scale approaches the model resolution. In all, the recipe above could perhaps be tuned slightly to make the predicted dissipation smaller, but given the relative naiveté of the recipe, such tuning is not particularly warranted.

\textbf{b. Test 2: Barotropic generation and mode-1 scattering from an isolated ridge}

If there is a piece of topography that interacts with a remote incoming low-mode tide and the local barotropic tide, the combination of internal wave scattering and generation can have a significant impact on the internal tide response. This has been recently pointed out by Kelly and Nash (2010). The effect shows up profoundly for isolated topography, even in the simple linear model, as we discuss here. We then consider the effect on the dissipation.

Using the linear methods described in the appendix, we can consider the case of a barotropic tide with an incoming tide over an isolated knife edge. This system has an interesting set of interactions that depend strongly on the phase between the two tides when they impact the topography (Fig. 8). In all cases, the same amount of internal energy transmits past the ridge. However, the fraction of that energy that is in mode-1 changes with the phase between the two forcing waves: at zero phase almost all the transmitted energy is mode 1, while at 180° out of phase that fraction drops to almost zero. The difference in the reflected energy is even more pronounced, with almost no energy reflected in the zero-phase difference case (Fig. 8a), and a substantial increase for the out-of-phase case (Fig. 8e). All of the transmitted/reflected asymmetry in fluxes is in mode 1, as high modes must match at the ridge crest by the boundary conditions there (i.e., the length of the black portions of the bars in Fig. 8 is the same in both directions).

The full response of this simple linear system is surprisingly complex, as we can see if we hold the ridge height constant and vary the ratio of the baroclinic to barotropic forcing (\(y_1/V_0\)) and their phase (Fig. 9). The total transmitted energy flux is simply the linear sum of the flux created by the barotropic generation and the baroclinic flux and does not change with phase between the two forcings (Fig. 9a). However, the modal content of the transmitted flux changes significantly, with less high-mode energy when \(y_1 = V_0\) and the phase difference is low (Fig. 9b). This difference is because the individual modal responses of the two forcings are slightly different and constructively or destructively interfere. It is interesting that the effects balance, to produce a constant transmitted flux as a function of phase difference. The reflected flux is much more variable and can be much stronger (Figs. 9b,d) because of the strong interaction between the barotropic wave, the incoming mode-1 wave, and the reflected wave. It is a curious result of this system that the asymmetry in the strength of the internal response is all on the side of the obstacle impacted by the incoming internal tide.

Just as the high-mode linear response strongly depends on the phase of the barotropic and baroclinic forcing, so does the dissipation predicted by the recipe (Fig. 10, solid line). When the phase differences between the barotropic and baroclinic tides are closer to 180°, the response has more high-mode energy and, thus, more dissipation. This effect is seen in the numerical model dissipation (Fig. 10, symbols), which agree very well with
the recipe dissipations. We ran the simulations over a range of forcings (Table 2) and found very good agreement between the recipe and the simulations (Fig. 11).

This mixed-forcing case, and the two that follow, indicate why the recipe needed to be more complicated than the barotropic-generation case discussed in KLP10. As soon as there are two different modes, there is no single characteristic speed at the ridge crest that we can appeal to in order to determine criticality because the response changes significantly with the phase of the forcings.

c. Test 3: Scattering of a mode 1 combined with a mode-2 incoming tide from an isolated ridge

The behavior can become even more complex if there is more than one internal mode in the incoming tide because the phasing of the information at the two tides now depends on depth and thus the height of the ridge. As an example, consider the case of a mode-1 and mode-2 incoming tide, both tides having the same amplitude, so that the energy flux in mode 1 is twice that in mode 2. The results of applying the recipe are quite complex (Fig. 12), with different ridge heights having distinctly different responses. For instance, short ridges \((h/H = 0.2)\) have a response that is 180° out of phase with taller ridges because the high velocities are near the seafloor rather than near the surface (see below).

Testing the recipe in the numerical model yields promising results (Fig. 13). Just considering the case of a ridge with \(h/H = 0.25\), and mode-1 and mode-2 incoming tides each with \(0.3\) m s\(^{-1}\), we see a similar relationship between the numerically determined dissipation and the theory.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modes</td>
<td>1 and 2</td>
</tr>
<tr>
<td>(h/H)</td>
<td>0.75, 0.55, 0.25</td>
</tr>
<tr>
<td>(u_i(m)) (m s(^{-1}))</td>
<td>0.10, 0.25, 0.55, 1.00</td>
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The nulls and peaks as the phase of the internal tide changes can easily be understood from the location of strong flows in the interfering tides (Fig. 14). For the in-phase mode-1 and mode-2 incoming tide, the location of the ridge is almost a null in the tidal velocity (Fig. 14a), leading to a weak response. Conversely, when mode 1 and mode 2 are exactly out of phase, the ridge has a strong lobe of velocity (Fig. 14b)—so much so that a lot of energy is reflected upstream (Fig. 14d) and strong dissipation is found at the ridge (Fig. 14f). Overall, the recipe presented here does a good job over a range of forcings (Table 3, Fig. 15) where both the amplitude of the baroclinic tides was varied and the phase between mode 1 and mode 2 was varied.

d. Test 4: Dissipation at a pair of ridges

The final application is for a pair of ridges, a problem motivated by the situation in Luzon Strait. The two-ridge generation problem is solved with a similar linear method (see the appendix), and the turbulence diagnosed at each ridge as was done above for a single ridge. This diagnosis is then compared to numerical model runs.
Before discussing the turbulence, it is worth pointing out that this simplified system has some complex, but classifiable behavior. The most striking effect comes if \( h_1/H + h_2/H > 1 \); then there is a resonance when the ridge separation \( \Delta x \) approaches half of a mode-1 horizontal wavelength \( \Delta x/\lambda_1 = 0.5 \). In this case, any characteristics emanating inward from the ridge crests are trapped between the ridges, leading to a standing wave pattern that self-reinforces upon returning to the emanating ridge. The radiating flux becomes much greater than the radiating flux from just one ridge, and a lot of energy is trapped between the two ridges (Fig. 16), similar to so-called attractors (Echeverri et al. 2011). If the barotropic forcing is held constant, then the internal flux will go infinite, though of course in a real system the finite energy in the barotropic tide would prevent this, even in the absence of dissipation. Two tall ridges also can have slightly weaker response than a single ridge if the two ridge tops line up in such a way that their tops are connected by a characteristic (Fig. 16e).

If the ridges are short enough that \( h_1/H + h_2/H < 1 \), then the perfect resonance does not occur. However, there is still a peak in the response where the two ridges positively reinforce one another by having intersecting characteristics after a bounce, either on the seafloor or the sea surface (Fig. 16c). Mathematically the seafloor bounce occurs for \( \Delta x/\lambda_1 = (h_2 + h_1)/2H \), and the surface bounce for \( \Delta x/\lambda_1 = (2 - h_2 - h_1)/2H \). Again, a null occurs when a characteristic joins the two peaks (Fig. 16a, \( \Delta x/\lambda_1 = (h_1 - h_2)/2H \), if \( h_1 > h_2 \)).

The overall complexity of the system can be judged by considering a fixed ridge height for one of the ridges, \( h_1/H = 0.6 \), and varying the other ridge height \( h_2/H \) and ridge spacing \( x/\lambda_1 \) (Fig. 17). First, if ridge 2 is “shadowed” by ridge 1, then the response is just the same as a single-ridge of height \( h_1/H \), that is, when \( h_2 < 0.6 - 2\Delta x/\lambda_1 \).

![Fig. 10. Comparison of theoretical dissipation (solid line) and dissipation observed in model (symbols) for a ridge with \( h/H = 0.6 \), \( v_0 = 0.1 \text{ m s}^{-1} \), and \( v_1 = 0.11 \text{ m s}^{-1} \).](image)

**Table 2.** Parameter matrix for numerical runs used in case 2, barotropic tide \( v_0 \) and incoming baroclinic tide \( v_1 \) combined. There were 32 runs in total, as summarized in Fig. 11. For all runs, \( N = 5.2 \times 10^{-5} \text{ s}^{-1} \) and \( f = 5.23 \times 10^{-5} \text{ s}^{-1} \); \( H = 2000 \text{ m} \), and \( h/H = 0.6 \). Each forcing was run with phase differences at the topography of 0°, 45°, 90°, and 135°.

<table>
<thead>
<tr>
<th>( v_0 ) (m s(^{-1}))</th>
<th>( v_1 ) (m s(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.025</td>
<td>0.15, 0.25</td>
</tr>
<tr>
<td>0.10</td>
<td>0.02, 0.05, 0.10, 0.25</td>
</tr>
<tr>
<td>0.20</td>
<td>0.15, 0.55</td>
</tr>
</tbody>
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![Fig. 11. Comparison of the recipe dissipations \( D_{\text{th}} \) and the numerical model dissipations \( D_{\text{model}} \). Runs here were made with \( h/H = 0.6 \), and eight combinations of \( v_0, v_1 \), as described in Table 2.](image)

![Fig. 12. Dissipation predicted from the recipe for an internal tide consisting of both mode-1 and mode-2 waves with equal amplitudes impacting a ridge, presented as a function of the phase between the mode-1 and mode-2 tides when they arrive at the ridge crest. The four curves represent four different ridges heights.](image)
Peaks in the response occur when the ridge crests line up after an odd number of bounces from either the surface, the seafloor, or the side of the topography (dotted lines Fig. 17). As noted above, resonance occurs at $D_x/l_1 = 0.5$ for $h_2/H > 1 - h_1/H$. Off this resonance, tall $h_2$ still traps the energy for a number of bounces before it radiates away, so energy builds up between the ridges, and there is an enhancement of radiation.

The numerical model tests of the dissipation recipe were run with two ridge heights and a number of separations. For all of the runs, $h_1/H = 0.6$. For half of the runs, $h_2/H = 0.17$; this geometry is similar to the geometry of the Luzon Straits in WKB-stretched coordinates. Sixteen simulations were then made with the ridges separated by $\Delta x/l_1 = 0.1, 0.26, 0.55,$ and 0.63, and barotropic forcing of $U = 0.03, 0.3, 1.0$, and 2.0 m s$^{-1}$. As before, total depth was $H = 2000$ m, and stratification is constant at $N = 5.2 \times 10^{-3}$ s$^{-1}$. A smaller set of runs was made with $h_1/H = 0.6$ and $h_2/H = 0.27$ and spacing $\Delta x/l_1 = 0.2, 0.3, 0.43,$ and 0.8. These were run only at $U = 0.25$ m s$^{-1}$. Narrow ridges were used ($\sigma = 2$ km), though the importance of ridge width was tested below.

The recipe given here is to calculate the linear response and then determine what portion of that response dissipates local to the topography. Of course, in so doing, the response of the topography itself is changed; that is, if mode 10 is dissipated at the right-hand ridge, it never reaches the left-hand ridge to create part of the response. This affects the high-mode response, and thus the amount of dissipation predicted, typically leading to an overestimate. To account for this, we run the linear model described above twice. The first time, we determine what the critical mode is at each ridge from the full solution. We then recalculate the linear response, but do not allow the supercritical modes to be part of the solution at the other ridge. For the runs presented below, this tends to reduce the predicted dissipation by approximately a factor of 2.

![Graph](image-url)
Again, the dissipation recipe does well in predicting the dissipation of the ridge system (Table 4, Fig. 18) usually well within a factor of 2, over three orders of magnitude of turbulence. With a bit more scatter, the dissipation at the individual ridges is also relatively well predicted (Fig. 18d), with the exception of three of the left-hand ridges (squares). These three exceptions are from when the short ridge is sheltered from the internal tide by the taller ridge, so there is a lot of dissipation at the tall ridge (which is well predicted) and relatively little at the short ridge. Thus the theory is working with a particularly poor guess at the how the wave field is modified by turbulence, and does a relatively poor job at predicting the dissipation at the smaller, inconsequential, ridge.

5. Varying width

There are many caveats to a recipe like the one presented here, but perhaps the most important is the role of varying topographic slopes. As discussed in KLP10, the knife-edge approximation is really only very good if the width of the topography is narrow enough that \( \frac{dh}{dx} > 2a \), where \( a \) is the slope of internal tide rays. For gentler slopes, lee waves are no longer the dominant dissipative mechanism, with near-critical bores becoming more important until the flow

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modes</td>
<td>1 and 2</td>
</tr>
<tr>
<td>( h/H )</td>
<td>0.75, 0.55, 0.25</td>
</tr>
<tr>
<td>( u(m) ) (m s(^{-1}))</td>
<td>0.10, 0.25, 0.55, 1.00</td>
</tr>
</tbody>
</table>

Table 3. Parameter matrix for numerical runs used in Case 3, mode-1 and mode-2 tides combined scattering from an isolated ridge. There were 24 runs in total, as summarized in Fig. 15. For all runs, \( N = 5.2 \times 10^{-3} \) s\(^{-1}\) and \( f = 5.23 \times 10^{-5} \) s\(^{-1}\); \( H = 2000 \) m.

![Image](http://journals.ametsoc.org/jpo/article-pdf/43/7/1380/4559219/jpo-d-12-061_1.pdf)
becomes subcritical, after which the dissipation drops off sharply.

For the baroclinic incoming tide case discussed here, the same dependence on slope applies (Fig. 19). For supercritical slopes, the recipe does quite well as the lee-wave physics dominates the dissipation. As $\frac{dh}{dx}$ is decreased below $2\alpha$ (i.e., $\sigma / \sigma_c$ increases, where $\sigma$ is the ridge width and $\sigma_c$ the ridge width where $\frac{dh}{dx} = \alpha$) the dissipation in the model increases to a peak at $\frac{dh}{dx} = \alpha$, and thus the theory ($D_{th}$) underpredicts.

Similar infidelity in the model can be seen for the two-ridge case (Fig. 20). Here, the mismatch surprisingly reaches a factor of 2 for even a moderately wide ridge and oscillates much less deterministically than the one-ridge case. Similarly confusing results were obtained for other geometries and forcings. As discussed below, we feel this indicates that significant care should be taken in applying this recipe in a complicated situation like a two-ridge system where the response at one generation site depends strongly on the result at another.

6. Summary and discussion

In the above we have demonstrated that turbulence dissipation at supercritical isolated features due to a baroclinic wavefield is concentrated at the crest of the features and takes the form of lee waves as we found for barotropic generation (Legg and Klymak 2008; Klymak et al. 2010b). The dissipation in these lee waves can be predicted with reasonable fidelity by considering the linear generation from a knife edge, and then assuming that all modes that move slower than an appropriately constructed ridge-crest speed are arrested and dissipate locally. The ridge-crest speed in this recipe is the mean speed half a wavelength above the ridge crest of the mode being presumed critical, comprising only the faster modes. To find the critical mode we therefore must iterate this procedure through all modes, but the linear model is relatively inexpensive, and this is easy to do on a desktop computer.

We tested this recipe on numerical simulations using an isolated ridge with barotropic, mode-, and mode-2 incoming waves. When these waves are combined, the dissipation response at the ridge changes significantly depending on the phase difference between the waves, and the recipe replicates this very well (Fig. 13). We also tested the recipe on a two-ridge system with barotropic forcing, with very good results if the ridges were very thin in the numerical model (Fig. 18), again with very good predictive ability.

The same caveats apply to this analysis as applied to the barotropic generation case. If the obstacles are too wide in the along-wave direction such that the slope is not sufficiently supercritical ($\frac{dh}{dx} > 2\alpha$), dissipation starts to be much larger in the simulations than...
predicted by the knife-edge theory. The problem with widening ridges is worse, and not fully understood, for the two-ridge case. Two ridges interact to create the internal response. Even mild widening of the ridges appears to change the lee wave response significantly enough that the recipe can be off by over a factor of 2, even if the ridges are still sufficiently supercritical. To us, this indicates that even more substantial caution should be used when applying this recipe to complicated bathymetry like a two-ridge system.

A host of other caveats should be borne in mind before applying this recipe. Topography with a lot of “medium scale” roughness should be treated with caution as subsidiary lee waves can develop (Nash et al. 2007). Large regions of near-critical slope have a similar problem. The effects of three-dimensionality mean that applying this recipe to complicated topography will be suspect (Buijsman et al. 2012).

Finally, it is important to determine how significant the dissipation in this problem actually is. To consider this, we examine the response of changing the height of the ridge and the incoming mode-1 tidal amplitude (Fig. 21). As with the barotropic case, it requires quite
strong tides to obtain the fraction of energy dissipated above 10% of the incoming energy flux (Fig. 21b). Indeed, for the scattering problem it is difficult for more than 20% of the energy to leave mode-1 (not shown). For tall ridges most of the mode-1 energy reflects as a mode-1 tide, whereas for short ridges most of it is transmitted. The greatest high-mode scattering and dissipation occurs at $h/H = 1/3$.

These findings still leave open the question of what happens to most of the low-mode tidal energy that radiates from supercritical ridges. The supercritical scattering process does not seem very dissipative, nor very efficient at moving energy into higher modes. This leaves open the possibility that mode-1 waves move through ocean basins relatively un molested. As noted here and for other efforts (Kelly and Nash 2010), remote mode-1 internal tides can interfere with the generation of mode-1 internal tides by the barotropic tide, leading to the complicated picture of an ocean full of mode-1 energy (i.e., Cummins and Oey 1997; Ray and Cartwright 2001) that is constructively and destructively interfering with local mode-1 generation, having a nontrivial pathway to turbulence. Turbulence pathways are likely smaller-scale rough topography, near-critical slopes, and dissipation in shallow water (where the turbulence will not drive deep-ocean mixing). This leads to the speculation that the low-mode internal tide needs to be treated as a basin-scale phenomena for which the response needs to be calculated for the whole basin, rather than as local generation problems, in a manner similar to the surface tide.

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APPENDIX

More Scattering and Generation Problems from Knife-Edge Topography

a. Scattering and generation from a single knife edge

The knife-edge problem of tidal generation (St. Laurent et al. 2003; KLP10) can be readily extended to a more general case that also includes an incident wavefield radiating from off ridge (Fig. 3). We solve the problem in a WKB stretched coordinate system, with constant stratification $N$, water depth $H$, and ridge height $h$. A barotropic tide is imposed with $U = U_o \cos \omega t$. The resulting baroclinic flows are decomposed into vertical modes:

$$u_i = \Re \left[ \sum_{n=1}^{N} d_n \cos(n\pi z)e^{i(k_n x + \omega t)} \right],$$  \hspace{1cm} (A1)

$$u_r = \Re \left[ \sum_{n=1}^{N} b_n \cos(n\pi z)e^{i(-k_n x + \omega t)} \right],$$  \hspace{1cm} (A2)

$$u_t = \Re \left[ \sum_{n=1}^{N} a_n \cos(n\pi z)e^{i(k_n x + \omega t)} \right],$$  \hspace{1cm} (A3)

$$w_i = \Re \left[ \sum_{n=1}^{N} d_n \sin(n\pi z)e^{i(k_n x + \omega t)} \right],$$  \hspace{1cm} (A4)

$$w_r = \Re \left[ - \sum_{n=1}^{N} b_n \sin(n\pi z)e^{i(-k_n x + \omega t)} \right],$$  and \hspace{1cm} (A5)

$$w_t = \Re \left[ \sum_{n=1}^{N} a_n \sin(n\pi z)e^{i(k_n x + \omega t)} \right],$$  \hspace{1cm} (A6)

where $k_n = a\pi n H$ are the horizontal wavenumbers and $\alpha$ is the internal tide propagation angle.

Fig. 21. Transmission, dissipation, and reflection for a mode-1 wave impacting an isolated ridge, as predicted by the recipe in this paper. The amplitude of the mode-1 wave $u_o$ is normalized by the mode-1 wave speed; $c_1 = 3.3 \text{ m s}^{-1}$. 

Downloaded from https://journals.ametsoc.org/jpo/article-pdf/43/7/1380/4559219/jpo-d-12-061_1.pdf by guest on 16 June 2020
\[ \alpha = \left| \frac{k}{m} \right| = \left( \frac{\omega^2 - f^2}{N^2 - \omega^2} \right)^{1/2}. \] (A7)

Note that the modal amplitudes \( a_n \), etc. are allowed to be complex in general. This allows the incoming baroclinic tide to have a different phase at the ridge than the barotropic tide and for the different modes to have varying phase.

If we define \( \gamma = (H - h)/H \) and require that the wavefields are matched at the ridge, \( x = 0 \) so that we find
\[ u_t = u_i + u_r - 1 \leq z/H \leq 0, \] (A8)
\[ 0 = U + u_i + u_r - 1 \leq z/H \leq -\gamma, \] and (A9)
\[ w_i = w_j + w_r - \gamma \leq z \leq 0. \] (A10)

The first condition says that \( a_n = b_n + d_n \). We find the modal amplitudes by Fourier expansion about \( \cos(m\pi z)/H \) for \( m = 0, 1, \ldots, N - 1 \), giving us \( N \) linear equations in \( N \) unknowns for each element,
\[ (A_{mn} + B_{mn})b_n + A_{mn}d_n = c_m, \] (A11)
where \( c_m \) is a column vector as in St. Laurent et al. (2003, hereafter S03)
\[ c_m = \frac{U_o}{m} \sin(m\pi \gamma), \] (A12)
where \( \gamma = (H - h)/H \), and \( c_o = -U_o \pi(1 - \gamma) \) for \( m = 0, 1, \ldots, N - 1 \). The matrices \( A \) and \( B \) are the two components of S03’s \( A \) matrix:
\[ A_{mn} = \frac{n \sin\pi \gamma \cos m\pi \gamma - m \cos\pi \gamma \sin n\pi \gamma}{m - n}, \] (A13)
and
\[ B_{mn} = \frac{-n \cos\pi \gamma \cos m\pi \gamma - m \sin\pi \gamma \sin n\pi \gamma}{m - n}, \] (A14)
and for the singularities
\[ A_{mn} = \frac{m\pi(1 - \gamma) - \sin m\pi \gamma \cos n\pi \gamma}{2m\pi} \] (A15)
\[ B_{mn} = \frac{-\sin^2 m\pi \gamma}{2m\pi}. \] (A16)

Note that the above reduces to S03’s knife-edge when \( d_n = 0 \). Equation (A11) is easily inverted to solve for the vector \( b_n \) (so long as \( \gamma \) is not allowed to be too close to an integer division of 1, so instead of using \( \gamma = 0.5 \), we use \( \gamma = 0.50001 \), otherwise singularities result).

b. Generation from two knife edges

The generation problem from two knife edges proceeds in a very similar manner (Fig. A1). Here one ridge is supposed to be at \( x = 0 \) with height \( h_o \) and the other at \( x = L \) with height \( h_L \).

Again, we can divide the velocities into modes:
\[ u_a = U_o \sum_{n=1}^{N} a_n \cos(n\pi z)e^{ik_n x + \omega t}, \] (A17)
\[ u_b = U_o \sum_{n=1}^{N} b_n \cos(n\pi z)e^{-ik_n x + \omega t}, \] (A18)
\[ u_c = U_o \sum_{n=1}^{N} c_n \cos(n\pi z)e^{ik_n x + \omega t}, \] (A19)
\[ u_d = U_o \sum_{n=1}^{N} d_n \cos(n\pi z)e^{-ik_n x + \omega t}, \] (A20)
\[ w_a = U_o \sum_{n=1}^{N} a_n \sin(n\pi z)e^{ik_n x + \omega t}, \] (A21)
\[ w_b = -aU_o \sum_{n=1}^{N} b_n \sin(n\pi z)e^{-ik_n x + \omega t}, \] (A22)
\[ w_c = aU_o \sum_{n=1}^{N} c_n \sin(n\pi z)e^{ik_n x + \omega t}, \] (A23)
\[ w_d = -aU_o \sum_{n=1}^{N} d_n \sin(n\pi z)e^{-ik_n x + \omega t}, \] (A24)
where \( k_n = \alpha n \pi \) are the horizontal wavenumbers and the amplitudes are complex.

So, the matching conditions at \( x = 0 \) are as before:
\[ u_a = u_b + u_c - 1 \leq z \leq 0, \] (A25)
At \( x = L \) they are
\[
-1 = u_b + u_c \quad -1 - \gamma_L \leq \gamma_L \quad \text{and} \quad (A26)
\]
\[
w_a = w_b + w_c \quad -\gamma_0 \leq \gamma \leq 0. \quad (A27)
\]

The first implies that \( a_n = b_n + c_n. \) The second implies that \( d_n e^{-i\kappa_0 L} = b_n e^{-i\kappa_0 L} + c_n e^{i\kappa_0 L}. \) We will shorten \( d_n e^{-i\kappa_0 L} = b_n e^{-i\kappa_0 L} + c_n e^{i\kappa_0 L}. \) This eliminates \( a_n \) and \( d_n, \) so we just need to solve for \( b_n \) and \( c_n \) summed over the elements:
\[
-1 = \sum_{n=1}^{N} (b_n + c_n) C_n \quad z \leq -\gamma_0. \quad (A31)
\]
\[
\sum_{n=1}^{N} a_n S_n = \sum_{n=1}^{N} (c_n - b_n) S_n \quad z \geq -\gamma_0. \quad (A32)
\]

becomes
\[
-1 = \sum_{n=1}^{N} (b_n + c_n) C_n = -1, \quad z \leq -\gamma_0 \quad \text{and} \quad (A33)
\]
\[
\sum_{n=1}^{N} b_n S_n = 0, \quad z \geq -\gamma_0. \quad (A34)
\]

Similarly at \( x = L: \)
\[
\sum_{n=1}^{N} (e_n b_n + e_0 c_n) C_n = -1, \quad z \leq -\gamma_L \quad \text{and} \quad (A35)
\]
\[
\sum_{n=1}^{N} (e_n c_n) S_n = 0, \quad z \geq -\gamma_L. \quad (A36)
\]

Again integrating by \( \cos(m \pi L) \) and using matrix notation, where \( A_0 = A_{mn} \) for the left ridge:
\[
(A_0 + B_0) b + A_0 c = C_0 \quad \text{and} \quad (A37)
\]
\[
(A_L + B_L) (E^+ c) + A_L (E^- b) = C_L. \quad (A38)
\]

Solving, we get
\[
b = D_0^{-1} (C_0 - A_0 c) \quad \text{and} \quad (A41)
\]
\[
(D_L E^+ - A_L E^- D_0^{-1} A_0) c = C_L - A_L E^- D_0^{-1} C_0, \quad (A42)
\]

where \( D_0 = A_0 + B_0 \) and \( D_L = A_L + B_L. \) The last equation is invertible to get \( c = c_n. \)

REFERENCES


