The Influence of Stratification on the Instabilities in an Idealized Two-Layer Ocean Model

R. L. IRWIN AND F. J. POULIN

Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada

(Manuscript received 17 December 2013, in final form 9 July 2014)

ABSTRACT

This work investigates the instability of a two-layer Bickley jet in the context of the rotating shallow water (RSW) model. This provides a general context in which the instability of oceanographic jets with simple stratification can be investigated. The three objectives of this work are as follows: First, the study investigates the morphology of unstable modes that can occur in this two-layer model. This is done by performing a linear stability analysis to investigate different types of flows both with and without vertical shear. Second, the authors study how the growth rates of the unstable modes are affected by changes in the stratification. Third, this study looks at the nonlinear evolution of some of these instabilities to determine how easy it is for nonprimary instabilities to develop. This is motivated by the fact that in the literature there have been many investigations that have found a multitude of unstable modes in this model, and it is not evident as to how easily they can be generated in oceanographic flows.

1. Introduction

Strong, energetic shear flows are ubiquitous features in both the oceans and the atmosphere. Two prominent examples are western boundary currents, such as the Gulf Stream, and the jet stream. All become unstable and generate vortical motions that can then transport fluid properties over vast distances and long periods. The mechanisms through which these instabilities occur are inherently nonlinear, but it has been demonstrated repeatedly that linear theory can be very important in predicting some characteristics of how the flows evolve. The process initiates with small-amplitude perturbations that extract energy from the background shear flow and grow exponentially in time before nonlinear processes saturate their growth. Subsequently, the meanders cause separation and then generate isolated vortical structures. Flierl (1975) showed how linear stability theory can be very useful in understanding the stability characteristics of the Gulf Stream. More recent studies that shed insight on the stability of stratified shear flows in the world’s oceans can be found in Xue and Mellor (1993), Lozier et al. (2002), Lozier and Reed (2005), and Tai et al. (2010).

Flierl et al. (1987) did a thorough analysis of the instability of a barotropic (BT) Bickley jet profile in the context of the one-layer quasigeostrophic (QG) model on a $\beta$ plane. This consisted of both a linear stability analysis that allowed them to explore parameter space very thoroughly and also perform a series of nonlinear simulations to illustrate the morphology of vortical structures that develop in a vortex street. Subsequently, Poulin and Flierl (2003) extended this study by looking at BT instability in the context of a one-layer reduced-gravity rotating shallow water (RSW) model on an $f$ plane to learn how these instabilities develop in non-QG regimes. This investigation was extended by Perret et al. (2011) who classified the different ways in which the QG assumptions can be violated. The first is the cyclogeostrophic regime; the Rossby number is order one, but the isopycnal deformations are small. The second is the frontal geostrophic regime; the Rossby number is small, but there are large deformations in the depth of the RSW layer. More details of the latter can be found in Cushman-Roisin (1986), Cushman-Roisin and Tang (1990), and Perret et al. (2006).

Baroclinic (BC) instability, which is more important on larger scales, is intimately connected to the meridional transport of heat from the tropics toward the poles.
Two classical works of BC instability in a continuously stratified fluid are Eady (1949) and Charney (1947). Soon afterward, Phillips (1954) presented a much simpler two-level model that captured the essential details of BC instability. This has since been extended to the two-layer Phillips model for BC instability (Flierl 1978). The fact that the two-layer model is able to represent both the time and horizontal length scales of the most unstable mode makes it a very powerful tool to study geophysical flows (Vallis 2006).

The two-layer RSW model is an excellent paradigm to study the instability of oceanographic shear flows because it allows for both BT and BC instabilities, and it is not restricted by the QG assumptions. One pioneering paper that investigated this problem is Boss et al. (1996). They determined that geophysical instabilities that were previously believed to be ageostrophic were actually only slightly modified QG instabilities. Subsequently, many studies have used numerical methods to study this type of problem. For example, Gula et al. (2009), building on the pioneering works of Hayashi and Young (1987), studied a generalization of the Phillips model to two-layer RSW flows. This study found a variety of interesting resonances that can cause instabilities involving both vortical and gravity waves. These ageostrophic instabilities are generally much weaker than the QG ones and even though they could appear in the ocean it is not clear what role they play.

The authors built upon this work to study the dynamics of coastal flows with both a one-and-a-half-layer model (Gula and Zeitlin 2010) and a two-layer model (Gula et al. 2010). Both of these studies focused on the frontal geostrophic regimes, as they studied basic states where the surface layer depth vanished. The stability of two-layer RSW models was compared to laboratory experiments and high-resolution numerical experiments in Pennel et al. (2012). Also, the linear stability problem for the two-layer RSW model has been used in conjunction with observations to better understand that the Bransfield Current is in part stabilized due to the steep underlying topography (Poulin et al. 2014).

Bouchut et al. (2011) investigated the stability of a BT flow in a two-layer RSW model where the horizontal shear was a Bickley jet. This study, which addressed the cyclogeostrophic regime, explored both the linear stability of the flow as well as the resulting nonlinear evolution. In particular, they focused on studying the importance of inertial instability, a mechanism that arises when the potential vorticity (PV) becomes negative, but cannot be adequately resolved without continuous stratification in the vertical (Ribstein et al. 2014). Lambaerts et al. (2011) used the one-layer RSW to study moist versus dry BC instability in the context of atmospheric flows, and then they extended their work to two-layers in Lambaerts et al. (2012). The basic state velocity consisted of a Bickley jet in the lower layer underlying an ambient (but active) upper layer.

Here, we investigate the instability of a Bickley jet in a two-layer RSW to map out the distinct types of unstable modes that can occur. We focus on the parameter regime that is outside of the QG limit with relatively weak stratification with three distinct types of shear flows. We classify the various unstable modes that can occur and discuss what modes we expect to be dominant depending on the stratification. Subsequently, we investigate the nonlinear evolution of some of these instabilities to determine how easy it is for nonprimary instabilities to develop. This is all done using parameters that are appropriate for the ocean and the types of instabilities are all due to vortical–vortical wave interactions.

The structure of this manuscript is as follows: In section 2, we present details on the linear stability problem, the nonlinear numerical model, as well as an analytical, two-layer, QG model with five patches of constant PV to study the unstable modes. Then, in section 3, we present the results obtained by solving the linear stability problem for four different cases: one-layer, pure BT shear, and two distinct types of mixed BT–BC shear in the two-layer RSW model. We also map out the modes in detail in the context of a five-patch analytic QG model. In section 4, we present results from the nonlinear simulations, and then in section 5 we conclude our findings.

2. Model equations and numerical methods

The fully nonlinear, two-layer RSW model (Vallis 2006) with two active layers can be written compactly as

$$\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i - f \mathbf{v}_i = - \frac{\partial}{\partial x} (g \eta_1 + \delta_{ij} g \eta_2),$$  \quad (1a)

$$\frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{v}_i + f \mathbf{u}_i = - \frac{\partial}{\partial y} (g \eta_1 + \delta_{ij} g \eta_2), \quad \text{and}$$  \quad (1b)

$$\frac{\partial h_i}{\partial t} + \nabla \cdot (h_i \mathbf{u}_i) = 0,$$  \quad (1c)

for $i = 1, 2$, the upper and lower layers, respectively, and $\delta_{ij}$ is the Kronecker delta function. The reduced gravity between the two layers is defined to be $g' = g(\rho_2 - \rho_1)\rho_1$, where $\rho$ is the velocity in each layer. The primitive variables are $u_i, v_i$, and $\eta_i$ for the along- and across-channel velocities and interfacial displacements. Note that the total depths $h_i$ for $i = 1, 2$ satisfy the relationships $\eta_1 + H = h_1 + h_2$ and $h_2 = H_2 + \eta_2$, where $H_i$
is the mean layer depth of each layer, and \( H = H_1 + H_2 \) is the total depth. We consider the case of equal mean depths but recognize that very different behavior can occur if the upper layer is much shallower.

### a. Nondimensional parameters

One aim of this study is to determine how to quantify the presence of unstable modes in the nonlinear context. Even though this is very difficult in general, if modes are barotropically or baroclinically intensified (or surface or bottom intensified), then it is possible to exploit this structure to determine the growth rate of different simultaneously growing modes.

With this goal in mind, it is convenient to define BT and BC velocity scales (based on the linear theory):

\[
\mathbf{u}_{BT} = \frac{H_1 \mathbf{u}_1 + H_2 \mathbf{u}_2}{H_1 + H_2}, \quad \mathbf{u}_{BC} = \mathbf{u}_1 - \mathbf{u}_2. \tag{2}
\]

An alternative definition for the barotropic velocity uses the full-layer depths, but we have decided instead to use the linear version as indicated above.

If we use \( U_i \) to denote the velocity scale in each layer, we can similarly define the BT and BC velocity scales to be

\[
U_{BT} = \frac{H_1 U_1 + H_2 U_2}{H_1 + H_2}, \quad U_{BC} = U_1 - U_2. \tag{3}
\]

With this we define the BT–BC Rossby and Burger numbers as

\[
\begin{align*}
\text{Ro}_{BT} &= \frac{U_{BT}}{fL}, & \text{Ro}_{BC} &= \frac{U_{BC}}{fL}, \\
\text{Bu}_{BT} &= \frac{gH}{f^2 L^2}, & \text{Bu}_{BC} &= \frac{g'H}{f^2 L^2}. \tag{4}
\end{align*}
\]

From the geostrophic relations, we require the following relationship between the Rossby and Burger numbers:

\[
\frac{\text{Ro}_{BT}}{\text{Bu}_{BT}} \sim \frac{\Delta \eta_1}{H}, \quad \frac{\text{Ro}_{BC}}{\text{Bu}_{BC}} \sim \frac{\Delta \eta_2}{H}. \tag{5}
\]

where \( \Delta \eta_i \) is the size of the interfacial displacements above layer \( i \), and \( L \) is the horizontal length scale. This shows that geostrophic balance imposes a constraint on the size of the Rossby and Burger numbers. This is the two-layer analog of what was previously established in the one-layer case (Poulin and Flierl 2003). We focus on the non-QG effects in the cyclogeostrophic regime as compared to the frontal regime but recognize that in the case of weak stratification \( \Delta \eta_i \) is no longer small in comparison to the mean depth.

### b. Basic state

The geostrophically balanced Bickley jet with a free surface and flat bottom can be written in dimensional form as

\[
\eta_i = U_i \text{sech}^2 \left( \frac{y}{L_j} \right), \tag{6}
\]

\[
\eta_1 = -H \left( \frac{\text{Ro}_1}{\text{Bu}_{BT}} \right) \tanh \left( \frac{y}{L_j} \right), \quad \text{and} \tag{7}
\]

\[
\eta_2 = H \left( \frac{\text{Ro}_1}{\text{Bu}_{BC}} - \frac{\text{Ro}_2}{\text{Bu}_{BT}} \right) \tanh \left( \frac{y}{L_j} \right). \tag{8}
\]

Note that \( U_i \) sets the magnitude of the velocity in each layer, and \( L_j \) determines the width of the jet. Also, we used \( \text{Ro}_i = U_i/(fL_j) \) with \( i = 1, 2 \) to denote the Rossby number for each layer, respectively. Because the dynamics are induced by the jet, we use the width scale of the jet \( L_j \) as the length scale in our nondimensional parameters rather than the size of the domain.

A given basic state profile can be classified in terms of the four parameters in Eq. (4). They are \( \text{Ro}_{BT} \) and \( \text{Ro}_{BC} \), the strength of the BT and BC shears, respectively, and \( \text{Bu}_{BT} \) and \( \text{Bu}_{BC} \), the strength of the stratification at the surface and the interior interface, respectively. We keep \( \text{Bu}_{BT} \) fixed throughout \( (\text{Bu}_{BT} = 5000) \) [see Poulin and Flierl (2003) for how it affects BT instability] and consider changes in \( \text{Bu}_{BC} \) to see what instabilities develop as a result of the stratification between the two dynamics layers. We consider regimes of relatively weak \( (\text{Bu}_{BC} = 5) \) and strong \( (\text{Bu}_{BC} = 500) \) stratifications that are often observed in the world’s oceans (Talley et al. 2011). Furthermore, we consider a range of BT and BC Rossby numbers that are either small or order one.

### c. Linear stability problem

The numerical method that we use is very similar to that of Poulin et al. (2014) except that we do not include any dissipative terms and we neglect topography. The dimensional variables are decomposed into a mean flow and perturbations that are Fourier in both the along-channel direction and time:

\[
\begin{align*}
u_i(x, y, t) &= \mathcal{U}_i(y) + u_i(y)e^{ik(x - \xi t)} + \text{c.c.}, \tag{9a} \\
v_i(x, y, t) &= iku_i(y)e^{ik(x - \xi t)} + \text{c.c.}, \quad \text{and} \tag{9b} \\
h_i(x, y, t) &= \mathcal{H}_i(y) + h_i(y)e^{ik(x - \xi t)} + \text{c.c.}. \tag{9c}
\end{align*}
\]

Where \( \xi \) is the phase speed, and \( \mathcal{U}_i(y) \) and \( \mathcal{H}_i \) are the velocity and depths of each layer, \( i = 1, 2 \). Substituting
the above into the linearized equations yields an eigenvalue problem:

$$c u'_i = \mathcal{U}_i u'_i + \frac{1}{\rho_i} \mathcal{U}_i \left( \frac{\rho_i}{\rho_i} Bu_i h'_i + Bu'_{z_2} h'_i \right),$$

(10a)

$$c v'_i = -\frac{u'_j}{k^2} + \mathcal{U}_i v'_i - \frac{1}{k^2 \rho_i} \frac{d}{dy} \left( \frac{\rho_i}{\rho_i} Bu_i h'_i + Bu'_{z_2} h'_i \right), \text{ and}$$

(10b)

$$c h'_i = \mathcal{H}_i u'_i + \frac{d}{dy} \left( v'_i \mathcal{H}_i \right) + \mathcal{U}_i h'_i.$$  

(10c)

Note that the $Ro = \max(U_1, U_2)/(fL_x)$, where $L_x$ is the meridional length scale of the domain, and the Burger numbers are $Bu = (gH_i)/(f^3L_x^2)$.

To ensure numerical accuracy, we have solved this problem using a two-step approach. First, a spectral collocation method on a Chebyshev grid (Trefethen 2000) is used to build a discrete approximation to the system, and the eigenspace is computed in MATLAB using a direct method (eig). This is the highest order of accuracy that we know to have spurious roots that can be misleading (Boyd 1992). One way to ensure the robustness of each method (eig) is to increase the spatial resolution of the spectral method. However, this quickly becomes rather expensive as predicted from the spectral method, and use it as a guess in a much higher-resolution low-order finite-difference method with a much higher resolution (T. Dubos 2013, personal communication). In particular, we take an eigenmode with a nonzero growth rate, as predicted from the spectral method, and use it as a guess in a much higher-resolution low-order finite-difference method that is solved using an indirect Krylov method (eigs) to obtain an improved refinement. Typically we used 500 points in the spectral method and 10^7 points in the finite-difference method. The two-step method helped to improve the quality of some of our results. We note that we were unable to do a linear stability calculation in the limit of no stratification ($Bu_{SC} = 0$) since the numerical method failed to give correct results in this limit.

### d. Nonlinear numerical model

We developed a new numerical code to evolve the nondimensional, nonlinear RSW equations on an $f$ plane in terms of the transport velocities $U_i = u_i h_i$:

$$\frac{\partial \mathcal{U}_i}{\partial t} = -\frac{\partial}{\partial x} \left( \frac{U_i^2}{h_i} + \frac{Bu_i}{2Ro^2} h_i^2 \right) - \frac{\partial}{\partial y} \left( \frac{U_i V_i}{h_i} \right),$$

(11a)

$$\frac{\partial V_i}{\partial t} = -\frac{\partial}{\partial x} \left( \frac{U_i V_i}{h_i} \right) - \frac{\partial}{\partial y} \left( \frac{V_i^2}{h_i} + \frac{Bu_i}{2Ro^2} h_i^2 \right),$$

(11b)

$$\frac{\partial h_i}{\partial t} = \frac{\partial \mathcal{L}_i}{\partial x} - \frac{\partial \mathcal{V}_i}{\partial y}, \text{ with } i = 1 \text{ or } 2, \text{ and}$$

$$j = \begin{cases} 2, & i = 1 \\ 1, & i = 2 \end{cases}.$$  

(11c)

We impose no normal flow and Neumann boundary conditions on the along-channel velocities and the interfacial deformations. These parameters differ slightly from the linear problem, but before we compare the two results we ensure that we convert everything to dimensional form.

The numerical model is Fourier spectral based and uses a uniform staggered grid. The derivatives are computed using FFTW and extending the fields appropriately based on the boundary conditions. Time stepping is done using third-order Adams-Bashforth except that we use an explicit Euler method for the first step, then an AB2 step for the second. Also, we use an exponential filter in spectral space to remove the small-scale noise that naturally develops throughout the evolution of the flow. We use 256 grid points in each direction with a time step on the order of 30 s to adequately resolve the fastest gravity waves.

### e. Simplified QG model

The linear stability analysis described previously allows us to explore the stability characteristics of the flow much more efficiently than focusing only on the nonlinear dynamics. However, it is still computationally expensive because of the large number of degrees of freedom that we need to guarantee numerical accuracy. Consequently, we have created an analytical model that assumes a two-layer QG fluid is divided into five regions of piecewise constant PV. This has the advantage of being computationally inexpensive and, as we shall see, it can well idealize the unstable modes that occur in oceanographic flows in the QG regime. The model builds on previous works such as Bowen and Holman (1989) and Talley (1983a,b). To simplify matters slightly we assume the rigid-lid limit, thereby removing one of the
parameters completely, which is justifiable considering the magnitude of the BT Burger number in the oceanographic context.

The governing equations for each layer are

\[
\frac{\partial}{\partial t} + U_n \frac{\partial}{\partial x} \left[ \nabla^2 \psi_n - (-1)^n L_d^{-2} (\psi_2 - \psi_1) \right] + \frac{dQ_n}{dy} \frac{\partial \psi_n}{\partial x} = 0, \quad n = 1, 2, \tag{12}
\]

where \(Q_n\) satisfies

\[
\frac{dQ_n}{dy} = -\frac{d^2 U_n}{dy^2} - (-1)^n L_d^{-2} (U_1 - U_2), \quad n = 1, 2, \tag{13}
\]

and in particular its derivative vanishes and \(L_d = \sqrt{g^3 H f_0}\) is the internal Rossby radius of deformation. Seeking a normal mode decomposition in \(\psi_n = \psi_n(y) e^{i(kx - ct)}\) yields the equations

\[
(U_n - c) \left[ \frac{\partial^2 \psi_n}{\partial y^2} - k^2 \psi_n - (-1)^n L_d^{-2} (\psi_2 - \psi_1) \right] + \frac{dQ_n}{dy} \psi_n = 0. \tag{14}
\]

After determining the general form of the solution in each patch we then need to impose boundary conditions at the interfaces. We assume the domain is infinite, and therefore we require that the perturbations vanish in the farfield. At the actual interfaces we impose that the flow perpendicular to the interface is continuous and the flow parallel to the interface has a jump that is dependent on the magnitude of the PV jump (Talley 1983b):

\[
\left( \frac{\psi_n}{U_n - c} \right)_{y_1} = 0, \quad \text{and} \quad \left[ (U_n - c) \left( \frac{\partial \psi_n}{\partial y} - \frac{\partial \psi_n}{\partial y} \right) \right]_{y_0} = 0. \tag{15}
\]

The governing equation and boundary conditions result in a generalized eigenvalue problem for \(c\) that consists of a \(16 \times 16\) system, which can be solved very rapidly for a wide range of parameters. The results will be presented at the end of the next section.

3. Linear stability analysis

We have studied the stability characteristics for a variety of different nondimensional parameters. Here, we present the results from a few interesting cases and present the nondimensional growth rates (scaled by an advective time scale) and phase speeds (scaled by the maximum speed) as a function of along-channel wavenumbers and also the spatial structure of the modes. We present the structure of the fields in each layer using either layerwise or BT/BC components, depending on what is more appropriate for a given mode.

a. One-layer model

The first scenario we investigate is that of a one-layer RSW model with an unstable Bickley jet as was studied in Poulin and Flierl (2003). There are no planetary Rossby waves present on an \(f\) plane, but there are low-frequency waves that exist because of the ambient vorticity gradient of the basic state itself. Following Boss et al. (1996), we refer to them as vortical modes. Figure 1a presents the growth rates as a function of the nondimensional along-channel wavenumber \(k\) for the case with \(R_{0BT} = 1\) and \(B_{UT} = 5000\), which is similar to what is presented in Perret et al. (2011). There are two unstable modes in the system, and each has a high-wavenumber cutoff but no low-wavenumber cutoff. Given that these modes occur in a one-layer model, they are clearly BT in nature in that they exist as a result of horizontal shear in the jet. In this particular case, the most unstable mode is about 3 times stronger, has a wider range of unstable wavenumbers, and produces smaller length scale instabilities in comparison to the secondary mode.

The associated phase speeds are plotted in Fig. 1b. The convention used here and throughout is that a dot indicates the phase speed of the neutral vortical modes, and a circle reflects that the vortical mode is unstable. We readily observe that the phase speeds of the two unstable modes saturate at high wavenumbers to very similar values. However, at large scales the weaker mode is much faster with a nondimensional speed of approximately 0.9, whereas the most unstable mode has a vanishing phase speed. This difference is one of the defining features between these two modes. This figure shows, as was shown in Hayashi and Young (1987) and Sakai (1989), that the instabilities are due to a coalescence of low-frequency vortical waves. There are surface gravity waves present, but they have much higher phase speeds and are all stable and therefore not depicted in this figure. Wang et al. (2012) present other examples where vortical–gravity wave resonances occur in the one-layer context, but that is beyond the scope of this work.

By plotting the structure of the unstable modes in Fig. 2, we readily observe that the first (second) mode is symmetric (antisymmetric) in the across-channel or spanwise velocity and referred to as the sinuous (varicose) mode (Lipps 1962). Therefore, the two characteristics that allow us to differentiate between sinuous and varicose modes are that the former has a symmetric across-channel velocity and has a phase speed that
vanishes at large scales (when it is not damped). In contrast, the varicose mode has an asymmetric across-channel velocity and is much faster at large scales.

b. Two-layer model with a barotropic shear

Next, we focus on the growth of the unstable modes in the two-layer RSW where the jet velocity is completely BT, using parameters $R_{0BT} = 1$, $R_{0BC} = 0$, and $B_{uBT} = 5000$. Figure 3 shows the growth rates in Figs. 3a and 3c and phase speeds in Figs. 3b and 3d for the unstable modes with weak ($B_{uBC} = 5$) and strong ($B_{uBC} = 50$) stratifications, respectively. Both sets of parameters can be readily observed throughout the oceans (Talley et al. 2011). There are certainly other examples of oceanographic flows that are beyond this range but they can be understood to a certain extent by extrapolating our results. The sinuous and varicose modes from the one-layer context are not significantly affected by either

Fig. 1. Normalized (a) growth rates and (b) phase speeds of the two unstable modes as computed for the linear stability calculation of the one-layer RSW. The sinuous mode has the largest growth rate and the varicose mode is the weaker of the two instabilities.

Fig. 2. Spatial structure of the two unstable modes as computed for the linear stability calculation of the one-layer RSW with $R_{0BT} = 1$, $B_{uBT} = 5000$: (a)–(c) $u_1'$, $v_1'$, $\eta_1'$ for the most unstable sinuous mode and (d)–(f) $u_1'$, $v_1'$, $\eta_1'$ for the unstable varicose mode.
weak or strong stratification. This is because the gravity at the surface is fixed, and this is the parameter that is more important in determining their stability characteristics. We refer to these two modes as BT sinuous (BTsin) and BT varicose (BTvar) modes, respectively, and they are labeled in Fig. 3.

Figures 3a and 3b are similar to what was found in Bouchut et al. (2011) in that they show the growth rate and phase speeds of the unstable modes, but Fig. 3b also shows the phase speed of the neutral vortical modes. This is to illustrate that the mechanism of instability is due to a resonance of subinertial waves. As previously found in Bouchut et al. (2011), we see that in the presence of weak stratification, there is an additional unstable mode, which they refer to as BC. However, with stronger stratification, there are two modes that do not appear in the BT case and are both BC in nature. With strong stratification, the growth rates of the two BC modes tend to those of the BT modes, but a weakening of the stratification stabilizes these BC modes. In Fig. 3a one has already been completely stabilized.

Figure 4 presents the BC and BT spatial structure of the two most unstable modes in the case of stronger stratification. Figures 4a(1)–4a(6) depict the BT and BC profiles, respectively, for the BTsin mode. The fact that the BT velocities are clearly dominant and almost identical to the one-layer case supports the idea that this mode is indeed BT. However, we note that there is a BC signature that is present and remark that the relative importance of this BC component is dependent on the strength of the Rossby number and the stratification. Similarly, Figs. 4b(1)–4b(6) are the structures for the BC mode that appears in both weak and strong stratification. The symmetry in the cross-channel velocity indicates that it is sinuous, and the fact that it is mostly BC in nature and only slightly BT motivates us to call it a BCsin mode.

In Fig. 5, we see the BT and BC structure of the two weaker unstable modes for the same parameters as mentioned above. Following the convention in the previous figure, the first (last) six subplots are for the third (fourth) most unstable mode. Because these modes all have asymmetric cross-channel velocity, they are varicose. The first of these two has a dominant BT structure, whereas the second is mostly BC and therefore we refer to them as BTvar and BCvar modes, respectively. The
FIG. 4. BT and BC structure of the two most unstable modes in the two-layer model with $R_oBT = 1$, $R_oBC = 0$, $B_iBT = 5000$, and $B_iBC = 50$. (a) The first mode is the BT sinuous mode that occurs in the one-layer case whereas (b) the second is a BC sinuous mode that arises as a result of the stratification.
FIG. 5. BT and BC structure of the two weaker unstable modes in the two-layer model with $Ro_{BT} = 1$, $Ro_{BC} = 0$, $Bu_{BT} = 5000$, and $Bu_{BC} = 50$. (a) The first mode is the BT varicose mode that occurs in the one-layer case, whereas (b) the second is a BC varicose mode that arises in the presence of strong stratification.
BCvar mode did not appear in Bouchut et al. (2011) because they considered a weak enough stratification that this mode was completely stabilized.

The relation of these BT and BC modes can be understood as follows. Each layer has an unstable jet profile that has two unstable modes: one sinuous and the other varicose. The two unstable BT modes arise when the unstable sinuous or varicose modes in each layer are in phase, whereas the BC modes occur when the two layers are completely out of phase. In the subsequent subsections, we will explore how these instabilities are altered when there is an asymmetry in the strength of the velocities between the two layers.

In section 3, we will present results of numerical simulations that confirm that in the regime of weakly stratified BT flow the small-amplitude perturbations grow exponentially and the BTsin mode dominates. However, with random initial conditions, all unstable modes will develop given enough time and will subsequently grow exponentially in time. By decomposing the flow into BT and BC components, we will be able to approximate the growth rate of each component in the nonlinear regime.

c. Mixed barotropic–baroclinic shear with zero net transport

In this subsection, we focus on the stability characteristics of a mixed BT–BC mean flow in the cyclogeostrophic regime; $\alpha_{\text{BT}} = 0$, $\alpha_{\text{BC}} = 1.0$, and $\mu_{\text{BT}} = 5000$, for the two different stratifications, and $\mu_{\text{BC}} = 5, 50$. Figures 6a and 6c show the growth rates, and Figs. 6b and 6d plot the phase speeds for the cases of weak and strong stratification, respectively. The plots of the growth rates have three sets of unstable modes: a pair of two overlapping BTsin modes, a pair of two overlapping BTvar modes, and a BC mode that arises in the classic QG Phillips problem (Pedlosky 1987), which we refer to as BCPhil. In each pair there is one mode that is mostly trapped in either the upper or lower layer, and therefore we refer to them as surface or bottom intensified, respectively. These four modes are the direct extensions of the four modes we saw in the case of pure BT shear; however, in the case of strong BC shear we see that they can only exist as surface- or bottom-intensified modes. A qualitative difference in the stability characteristics in this case is that the BTsin mode develops a low-wavenumber
cutoff with BC shear, analogous to how a \( \beta \) plane stabilizes the classical BC mode.

By comparing the two plots of growth rates, we conclude that as the stratification weakens the low-wavenumber cutoff increases, the high-wavenumber cutoff decreases, and the growth rate of the modes decreases. The BTvar mode is similarly affected by the stratification but it does not develop a low-wavenumber cutoff. Therefore, in the presence of BC shear, a weakening in the stratification stabilizes these inherently BT modes. In contrast, the BC mode becomes more effective in extracting potential energy the weaker the stratification and consequently its growth rate increases in this limit. However, we point out that this mechanism relies on the stratification being weak enough for this to dominate over the BT modes. We observe in the plots of the phase speeds that the BTsin modes have a vanishing phase speed at large scales, and the BTvar modes have a nonzero phase speed as the along-channel wavenumber tends to zero. The phase speed for the classic BC mode is much smaller in magnitude and not apparent in the figures. When comparing the growth rates of these cyclostrophic flows with their corresponding geostrophic counterparts, RoBC \( \ll 1 \), we find that the nearly QG flows have larger relative growth rates (growth rates normalized by RoBC in this case). The behavior that an increase in Rossby number is relatively stabilizing is consistent with what was found in the one-layer limit in Poulin and Flierl (2003).

Figure 7 shows the horizontal BT and BC structures of the BCPhil mode with RoBC = 1, BuBC = 5, RoBT = 0, and RoBC = 1.

Fig. 7. BT and BC structure of the most unstable BC mode with BuBT = 5000, BuBC = 5, RoBT = 0, and RoBC = 1.
the BCPhil mode reveal that the along- and cross-channel velocities in each of the two layers are out of phase by close to 90°. It is this partial phase shift that enables the perturbations to extract potential energy from the jet, unlike the BCsin and BCVar modes that can only extract kinetic energy. Our calculations indicate that for larger (smaller) stratifications, there was a slight increase (decrease) in the phase shift, which corresponds to a decrease (increase) of the growth rates.

Next we look at the nonprimary instabilities in the system at \( R_{\text{BT}} = 1 \). Figure 8a (first six subplots) and Fig. 8b (last six subplots) show the spatial structure of the different fields in terms of each layer for surface-trapped sinuous and varicose modes, respectively. Each mode is surface intensified in terms of its cross-channel velocity, and the modes are symmetric and antisymmetric, respectively. This confirms our hypothesis that the unstable modes at small scales are the sinuous modes, whereas the unstable modes at large scales correspond to the varicose modes. We have determined that as the BC Rossby number increases, the relative magnitude of the nonintensified layer decreases. The lower-layer equivalents are not shown but are completely analogous.

d. Mixed barotropic–baroclinic with net transport

Next, we focus on the stability characteristics that arise with both BT and BC shear. There is a much wider range of possibilities in this regime, and we do not aim to do an extensive analysis but instead to learn how the modes transition and what modes tend to dominate with a net transport. In Fig. 9, we present the growth rates and phase speeds for the two sets of parameters, both with \( R_{\text{BT}} = 1.0, R_{\text{BC}} = 1.0, \) and \( B_{\text{BT}} = 5000 \); Figs. 9a and 9b correspond to \( B_{\text{BC}} = 5 \), and Figs. 9b and 9d have \( B_{\text{BC}} = 50 \).

In the lower row of Figs. 9c and 9d we see the growth rates and phase speeds in the case of relatively stronger stratification. There are four unstable modes and they are easily identified by looking at the modal structures, which we do not present for the sake of brevity. In order of decreasing maximum growth rates the modes are surface-trapped BTsin mode, bottom-trapped BTsin mode, surface-trapped BTvar mode, and bottom-trapped BTvar mode. The surface is more unstable than the lower layer because it has larger maximal velocities. The absence of the classical BC instability demonstrates that the presence of a nonzero along-channel transport tends to stabilize the flow. The growth rates for this two-layer case are well captured by computing the corresponding growth rates of the given velocity profiles in a one-layer context and then superimposing the different curves.

In contrast, Fig. 9a shows the growth rates for the same shear flow but now in the context of weaker stratification. There are three unstable modes that are well resolved. The most unstable mode is a surface-trapped BTsin mode, which possess two local maxima both of which have similar spatial structures. The remaining two modes are mostly sinuous and varicose, respectively, but the symmetry is distorted, again due to the net transport. They each have a strong presence in each layer so they are neither surface nor bottom intensified. However, we suspect that the sinuous mode is mostly due to the shear in the lower layer and the varicose mode as due to the shear in the upper layer. The lower-layer varicose mode has already been stabilized, and therefore a fourth curve is not visible. Given that the surface intensification is much weaker and the symmetry/asymmetry in the velocity fields are not as clear, it is a lot harder to classify the modes in this regime. However, in the next subsection we will extend this classification to a range of relative velocities in the context of our five-patch QG model.

e. Comparing the five-patch QG model with RSW

Next, we present some findings from our analytic QG model. Figure 10 compares different fields of the five-patch QG model with the piecewise constant PV case with those from the nearly continuous shallow water (SW) model with order one Rossby number and \( B_{\text{BC}} = 5 \). We remark that we needed at least five patches to resolve both the sinuous and varicose modes, otherwise the latter mode never appeared. To build a piecewise constant velocity profile to match the Bickley jet we pick the maximum velocities to be the same in the two models; however, there is still a degree of freedom in terms of how to pick the width of the five-patch profile. We picked the length scale so that the maximum growth rates were similar. We see that a major difference is that the QG model predicts much larger length scales than it should. That aside, it is rather remarkable how well this idealized model can capture the growth rates and phase speeds of the different modes. We plot the results in the first case for weak stratification but remark that for stronger stratification it did capture all the modes.

In the case of weakly stratified pure BC instability, we find that the QG model overestimates the BTsin mode quite drastically. It captures the BTvar mode more accurately presumably because it appears at larger length scales, and it is well known that these piecewise constant PV profiles are known to deviate more significantly from the actual solution when we move to smaller length scales. The case of mixed BT–BC shear with a net transport looks quite different and even generates additional modes that do not arise in the RSW model. However, at the large scales the stability characteristics can agree quite well.
FIG. 8. BT and BC structure of the two unstable BC modes with $Bu_{BT} = 5000$, $Bu_{BC} = 50$, $Ro_{BT} = 0$, and $Ro_{BC} = 1.0$ that correspond to the higher (lower)-wavenumber instabilities, respectively.
f. Mapping the different modes of instability

Besides simply confirming what we already know in the SW context, this five-patch QG model has been useful in allowing us to explore the parameter space much more rapidly. We have used this relatively simple model to compute the stability characteristics of the two-layer jet profiles over a range of velocities. For this problem, we found it convenient to deal with dimensional parameters and have chosen $f = 10^{-4}$ s$^{-1}$, $g = 9.81$ m s$^{-2}$, $H_1 = H_2 = 250$ m, and $L_j = 15$ km. The maximum speed in the upper layer is fixed at $U_{1m} = 1$ m s$^{-1}$, and the speeds in the lower layer are $U_{2m} \in [-1.5, 1.5]$ m s$^{-2}$. When $U_{2m} = -1$ and 1 (we drop the units for convenience), we have the mixed BT–BC shear flows with no transport and the pure BT shear case, respectively, and these are denoted in plots of the phase speeds with vertical dotted lines. The other values of $U_{2m}$ correspond to mixed BT–BC shear flows with a net transport. The three columns of Fig. 11 from left to right have very strong, strong, and weak stratifications ($g' = 1.0, 0.1$, and 0.01 m s$^{-2}$), respectively. The rows depict the growth rates, phase speeds, and the phase differences between the two layers. The blue and green lines are for the two sinuous modes, the red and yellow lines represent varicose modes, and the cyan line is the BC Phillips mode. The surface-intensified modes have constant growth rate, whereas those of the bottom-intensified modes increase linearly due to an increase in the lower-layer velocities. These parameters cover a wide range of oceanographic motions.

The first feature to recognize is that for strong stratifications and weak lower-layer velocities, $U_{2m} \in [-0.5, 0.5]$, the growth rates and phase speeds are linear. The surface-intensified sinuous and varicose modes have constant growth rates and phase speeds, whereas their bottom-intensified analogs have linearly increasing growth rates and phase speeds, as is to be expected. In the case of negative velocities and strong stratification, the trend continues similarly. The one change is that the Phillips mode is introduced near $U_{2m} = -1$, and the width and strength of this mode is inversely proportional to the reduced gravity. It is interesting to point out that the growth rate of this mode increases with the lower-layer velocities, and the transition to stability is as abrupt on either side of the mode. In the case where the lower-layer flows in the same direction as the upper layer and the two speeds are comparable we see that there is
a difference between the case with $g' = 1.0, 0.1$. With very strong stratification, there is perfect symmetry with the growth rates about $U_{2m} = 0$. However, with strong stratification, near the regime of pure BT shear, there is a bubble in the growth rates of the sinuous modes. Also, the varicose modes are more strongly affected in that the growth rates actually separate before $U_{2m} = 1$. This yields that the varicose modes with larger speeds in the lower-layer switch growth rates compared to the left but maintain the same phase speed. This is in contrast to the sinuous modes that, beyond the bubble, keep the same growth rates as before but their phase speeds switch. We verified that the varicose mode that is more unstable for $U_{2m} > 1$ is bottom intensified, in contrast to the left of this region where it was surface intensified. The case of weak stratification is significantly different in that the Phillips mode is much stronger over a wide range of parameters and the BT modes tend to be stabilized. For example, the surface-intensified sinuous mode is completely stabilized when the lower layer is fast enough, $U_{2m} > -0.5$, and the bottom-intensified varicose mode is much weaker where $U_{2m}$ is positive compared to where it is negative. The varicose modes are both stabilized by weak stratification but appear qualitatively the same for negative velocities. Also, near $U_{2m} = 1$, the modes coalesce into a single mode with consistent phase speeds and growth rates.

We note that to sort the distinct modes in Fig. 11, we found the best match by using projections of the eigenvectors. There are other ways of classifying the modes (e.g., based on surface–bottom intensification), but we see that some interesting structure arises through the use of modal projections.

The three figures in the bottom row of Fig. 11 depict the phase difference in the layers and how that changes with the velocity in the lower layer. We observe the well-known result that the classical Phillips mode has a strong vertical phase shift. However, it is not 90° but closer to 80° for strong stratification and even smaller for weaker stratifications. As this mode is stabilized, the phase shift decreases before eventually the mode can no longer extract potential energy from the basic state. The phase differences in the BT modes are strongly asymmetric in terms of the direction of the flow in the lower layer. For reverse shears the phase shifts seem all to flatten out rather quickly and achieve values close to 20°. These modes have a phase shift in the horizontal and that is how they extract their energy from the basic state. In the case when the two layers’ flow is in the same direction, which corresponds to stronger net transport, we see that the one varicose mode achieves a large phase shift, close to 180°, whereas the second is small, close to 20°. There are some oscillations in the phase differences for $U_{2m}$ near one, but they are not nearly as dramatic. In contrast, we have that the sinuous modes both increase rather steadily and near the regime of pure BT shear, and both have a phase shift of larger than 80°. Subsequently, they then diverge very rapidly. The two BT modes that have large phase shifts are the BCsin and

![Fig. 10. Comparison of the five-patch QG model with piecewise constant PV case with the nearly continuous SW model with order one Rossby number. The rows are for the (a) pure BT and (b),(c) mixed BT–BC cases with and without net transport, respectively. The columns are for the 1) velocity profiles, 2) growth rate of the RSW model, 3) phase speeds for the RSW model, 4) growth rates for the five-patch QG model, and 5) phase speeds for the five-patch QG model.](http://journals.ametsoc.org/jpo/article-pdf/44/10/2718/4537092/jpo-d-13-0280_1.pdf)
BCvar modes, whereas the other two are the BTsin and BTvar modes. The interval in velocity over which this transition occurs is proportional to the inverse of the reduced gravity. These results are derived from the five-patch QG model, but calculations we have done in the context of the RSW model are consistent with these observations. This is why we believe this morphology of unstable modes is relevant to oceanographic jets both in and beyond the non-QG regime.

4. Nonlinear evolution of the instabilities

In this penultimate section, we study the nonlinear evolution of unstable shear flows from the previous section. In particular, we wish to better understand how easy it is for weaker instabilities to develop and be detected in the presence of the primary instability in an oceanographic context. This is motivated by works such as Sakai (1989), Hayashi and Young (1987), and Gula et al. (2009) that find many nonprimary instabilities; however, we recognize that our modes are all vortical-vertical, and therefore our results are necessarily limited in scope.

a. Two-layer barotropic shear

First, we focus on the nonlinear dynamics that arises in the two-layer scenario with weak stratification and purely BT mean flow (\(\text{Ro}_{\text{BT}} = 1\), \(\text{Ro}_{\text{BC}} = 0\), and \(\text{Bu}_{\text{BC}} = 5\)). We decompose the two-layer flows into BT and BC components in order to tease out the relative importance of each different type of unstable mode. In all of the simulations, the basic state was initially perturbed with very small-amplitude, random, white noise localized around the jet.

In Fig. 12, we plot the time evolution of the Euclidean norm of the perturbation of the following fields: \(u_{\text{BT}}, v_{\text{BT}}, \eta_1, u_{\text{BC}}, v_{\text{BC}},\) and \(\eta_2\). Recall from section 2 that in
the case of weak stratification there were three unstable modes, BTsin, BCsin, and BTvar, in decreasing order of growth rates. This figure shows that the regime of exponential growth lasts for approximately 50 nondimensional time units. What is interesting is that there are two distinct slopes. The BT fields, which includes the free surface, have steeper slopes with an average nondimensional growth rate of 0.158, which has a relative error of 1.8% compared to what is predicted for the BTsin mode from the linear theory. The two BC velocities and the interfacial displacement have a lower mean growth rate of 0.121 with a relative error of 0.8% when we consider the growth rate predicted for the BCsin mode. Furthermore, when we compare the spatial structures of the unstable modes, the linear and nonlinear theory agree very well for both the sinuous modes. We remark that the BTvar mode should also be present during the phase of exponential growth but it is drowned out by the most unstable mode and therefore it is not evident in the fields. Therefore, if an unstable jet has two most unstable modes, one is predominantly BT and the other BC (as occurs with pure BT shear); this BT–BC decomposition can be advantageous to isolate the two modes.

To illustrate the spatial structure of the BT and BC vorticity of the simulation with $Ro_{BT} = 1$, $Ro_{BC} = 0$, $Bu_{BT} = 5000$, and $Bu_{BC} = 5$ in the nonlinear regime, we present snapshots at different times. This consists of the late stages of exponential growth and the beginnings of the early stages of nonlinear equilibration. Figures 13

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Fig. 12. Normalized growth rates on a log scale of the nonlinear evolution of the two-layer RSW with $Ro_{BT} = 1$, $Ro_{BC} = 0$, $Bu_{BT} = 5000$, and $Bu_{BC} = 5$ of $u_B^1$, $v_B^1$, $\eta_1$, $u_B^2$, $v_B^2$, $\eta_2$.

Fig. 13. The BT vorticity at times $t = 90, 110, 130, \text{and} 150$ in the two-layer nonlinear RSW simulation with $Ro_{BT} = 1$, $Ro_{BC} = 0$, $Bu_{BT} = 5000$, and $Bu_{BC} = 5$. 
and 14 show the BT and BC vorticities as defined in Eq. (3) at nondimensional times 90, 110, 130, and 150, respectively. In Fig. 13, we see the BT vorticity of the basic state early on and the initial development of the meanders. The final two frames show the meanders separate and yield a vortex street with both cyclones and anticyclones, as well as dipoles that are ejected on the periphery. This figure is essentially what arises in the one-layer case as was presented in Poulin and Flierl (2003). In some sense, Fig. 14 shows the part of the two-dimensional dynamics that is lost in the one-layer approximation. At the early stages, the unstable BC mode meanders and, in the third and fourth frames, forms a shadow of the BT vorticity: the vortex street and the dipoles that are ejected are readily observed. It is of interest to note that the regions of most intense BC vorticity reside in the dipoles and not the street itself. Indeed, the extreme values of the BC vorticity are similar in magnitude to the largest values of BT vorticity.

b. Two-layer mixed barotropic–baroclinic shear with zero net transport

In this subsection, we look at results from the nonlinear evolution of a two-layer Bickley jet with $\text{Ro}_{\text{BT}} = 0, R_{\text{BC}} = 1, \text{Bu}_{\text{BT}} = 5000, \text{and Bu}_\text{BC} = 5$. Figure 15 presents the growth rates of the six different fields under consideration in the context of two different simulations. The first run has a channel long enough to fit two wavelengths of the most unstable BC mode. The second has a much shorter channel that only fits the weaker unstable BTsin modes and is initially seeded with the spatial structure computed from the linear calculation to facilitate the evolution of this weak instability. Figure 15 shows that the six different fields all grow exponentially early on and have very similar slopes and therefore growth rates. The mean growth rates in the first simulation are 0.209 with a relative error of 0.9%. The second simulation that only allows the weak sinuous mode to grow has a much weaker average growth rate of 0.0550 with an error of 0.9%. This shows that there is excellent agreement between the nonlinear simulations and linear stability analysis.

In looking at the evolution of the most unstable BC mode in the first simulation there is no evidence that the BTsin modes are present, even though they presumably grow exponentially as well, albeit at a slower rate. This is more typical than what we saw in the previous subsection because the most unstable modes have both BT
and BC components, and therefore we cannot separate them easily. In another simulation, we used the same domain size as in the first but seeded the initial conditions with a BTsin mode. It did grow exponentially at the early stages, but the most unstable BC mode was soon excited, and it then proceeded to dominate the simulation. This demonstrates that it is very difficult to observe unstable modes that are not the most unstable mode because they are drowned out by the primary instability even in the most favorable of circumstances. In the world’s oceans, which are much more complicated, it would seem even harder to find evidence of these secondary modes of instability. Therefore, the physical importance of secondary unstable modes is very questionable in this context.

5. Conclusions

We have revisited the classic problems of BT and BC instabilities in the context of an idealized ocean using the two-layer RSW model. In the case of pure BT shear, we determined that there are four unstable modes that can occur: surface/bottom-intensified sinuous/varicose modes. The case of pure BT shear is somewhat particular in that the modes combine to form two BT modes that are in phase and two BC modes that are completely out of phase. The BC sinuous mode was previously observed in Bouchut et al. (2011), but we are not aware of other works that have witnessed both BC sinuous and BC varicose modes. BC modes can arise if (i) the stratification is sufficiently strong and (ii) the jet velocity is nearly uniform in the vertical. Otherwise, all the BT modes tend to be surface or bottom intensified. In the presence of a BC shear, there is an additional mode that is akin to the classical Phillips mode of BC instability that has a phase shift of close to 90° in the vertical. For it to be the most unstable mode of instability requires that (i) the stratification is sufficiently weak and (ii) the net transport along the channel is also weak. The five-patch QG model has been instrumental in getting a global view of how these five modes transition through the different parameter regimes and how that depends on stratification. This model will be less accurate in the regimes of large Rossby numbers and small BC Burger numbers but is consistent with our findings in the RSW model outside of this range.

By studying the nonlinear evolution of BT two-layer flow, we found that using a BT–BC decomposition allowed us to quantify the growth rates of the two sinuous modes before nonlinear equilibration is achieved. We have tried to generate nonprimary instabilities in the presence of a BC shear and have found that it is very difficult to see anything beyond the primary instability because most modes have both BT and BC components. This questions the physical importance of weak, small-scale instabilities that have been observed in the two-layer RSW in terms of causing an energy cascade from the mesoscale to the submesoscale and more generally loss of balance. Clearly, finding instabilities at small scales does not guarantee this mode is an effective means of generating unbalanced motions and care should be taken in determining the importance of weaker instabilities.

We have considered oceanographic shear flows with the simplest form of stratification and have been able to...
classify the different types of unstable modes that develop and also determine in what regimes of parameters we expect them to occur. There are a number of directions this will lead us in future research. One is to study the effect of topography on the classification of modes that arise in an oceanographic setting. A second is to include more degrees of stratification to see in part whether the BC sinuous and varicose modes persist, using either more layers or continuous stratification. Preliminary calculations in a three-layer RSW model with the same reduced gravities at both interfaces yield that there are still only four modes as we have found. It remains to be shown whether this persists to the continuously stratified flows. Third is to investigate a RSW model with patches of PV to see how unstable modes due to vortical–gravity wave resonances vary through parameter space. In general, we plan to add more components into our idealized models to better describe and understand shear instabilities that can occur in oceanic jets.

Acknowledgments. FJP thanks NSERC for financial support during the research and writing of this manuscript and CFI and SHARCNET for the computing resources to perform these calculations. We also thank Alexandre Stegner, Xavier Carton, Gordon Swaters, and an anonymous reviewer for constructive feedback in revising the manuscript.

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