Global Calculation of Tidal Energy Conversion into Vertical Normal Modes

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ABSTRACT

A direct calculation of the tidal generation of internal waves over the global ocean is presented. The calculation is based on a semianalytical model, assuming that the internal tide characteristic slope exceeds the bathymetric slope (subcritical slope) and the bathymetric height is small relative to the vertical scale of the wave, as well as that the horizontal tidal excursion is smaller than the horizontal topographic scale. The calculation is performed for the M\textsubscript{2} tidal constituent. In contrast to previous similar computations, the internal tide is projected onto vertical eigenmodes, which gives two advantages. First, the vertical density profile and the finite ocean depth are taken into account in a fully consistent way, in contrast to earlier work based on the WKB approximation. Nevertheless, the WKB-based total global conversion follows closely that obtained using the eigenmode decomposition in each of the latitudinal and vertical distributions. Second, the information about the distribution of the conversion energy over different vertical modes is valuable, since the lowest modes can propagate over long distances, while high modes are more likely to dissipate locally, near the generation site. It is found that the difference between the vertical distributions of the tidal conversion into the vertical modes is smaller for the case of very deep ocean than the shallow-ocean depth. The results of the present work pave the way for future work on the vertical and horizontal distribution of the mixing caused by internal tides.

1. Introduction

The deep-ocean stratification is largely maintained by the vertical mixing whose main cause is the breaking of internal waves and concomitant small-scale turbulence (Munk and Wunsch 1998; Wunsch and Ferrari 2004; Zhang et al. 1999). In the deep ocean, internal tides generated through the interaction of the barotropic tide with the underlying bathymetry are the main contributing factor to the internal wave field (St. Laurent and Garrett 2002; Garrett and Kunze 2007). This highlights the importance of calculating the energy conversion rate from the barotropic tide into the internal tide.

Based on assumptions that the internal tide characteristic slope is larger than that of the bathymetry, the bathymetric height is small relative to the vertical scale of the wave, and the tidal excursion is smaller than the horizontal topographic scale, Llewellyn Smith and Young (2002, hereinafter LY02) examined the generation problem in an ocean of finite depth, with a vertically nonuniform buoyancy frequency \(N(z)\). They decomposed the wave field into a linear superposition of vertical normal modes, which are determined through solving a Sturm–Liouville eigenvalue problem depending only on the buoyancy frequency profile \(N(z)\). If \(N(z)\) varies more slowly than the eigenfunction, the traditional Wentzel–Kramers–Brillouin (WKB) method can be used to obtain an approximate solution to the Sturm–Liouville problem. Based on the modal decomposition approach, LY02 showed that the essential effect of the finite depth is that topography with a scale larger than the wavelength of the first-mode internal wave cannot generate the internal tide.

Nycander (2005) used a semianalytical model based on the above-mentioned assumptions made by LY02, as well as the WKB method, to calculate the energy conversion rate over the global ocean. The solution was not decomposed into vertical modes; instead, the finite depth was taken into account in a heuristic way by using a filter to exclude the conversion by topographic scales larger than the horizontal wavelength of the first internal wave mode. The final result was in good agreement with an estimate of the M\textsubscript{2} tide dissipation in the deep ocean based on the inverse calculation using satellite altimetry data (Egbert and Ray 2000).
Green and Nycander (2013) used the computed energy conversion field to parameterize the wave drag in a barotropic tidal model, complemented by a performance evaluation against satellite altimetry. They found that the performance is significantly improved when using the parameterization based on Nycander (2005) as compared to the two other previously suggested ones based on simple scaling arguments.

Very recently, Falahat et al. (2014) compared the internal tide energy conversion rate based on an extension of the method of Nycander (2005) to observed dissipation data obtained over rough topographic features of the Mid-Atlantic Ridge (Polzin et al. 1997). They found a reasonable correlation between the calculated conversion and observed dissipation, and the magnitude agrees well.

Despite the satisfactory results obtained using Nycander’s (2005) formulation in the above-mentioned studies, this formulation has limitations stemming from linear wave theory and the WKB approximation. Our concern here is primarily directed toward the latter limitation. LY02 showed that the only properties of the buoyancy frequency that affect the WKB-based energy conversion rate into vertical modes are the bottom stratification $N_B$ and the vertical average $\overline{N}$. In this case, the energy conversion rate is in fact proportional to the bottom stratification $N_B$. As discussed by Zarroug et al. (2010), this dependence upon $N_B$ can yield erroneous results for the internal tide generation by the broad-scale topographic features. By using homogenization theory, they showed that the conversion rate is proportional to $N_b$, the average value of $N$ over a vertical region at the bottom having the same vertical scale as the vertical wavelength of the internal waves. This theory improved the results for the lowest modes, which are important since they can propagate a considerable distance from the generation site and hence contribute to the remote generation of the internal tide and the subsequent mixing (Alford 2003). This illustrates the importance of taking the whole depth structure of $N$ into account, which cannot be done by using the WKB method. To do so in a fully consistent way, it is necessary to employ modal decomposition.

On the global scale, waves have the same scale as the topography, up to the scale of mode 1. The low-mode internal tides are characterized by a large horizontal wavelength, as much as 250 km, and a fast propagation speed (Rainville and Pinkel 2006; Carter et al. 2012). The latter characteristic makes these modes less susceptible to local nonlinear processes leading to the eventual mixing (St. Laurent and Garrett 2002). How low modes eventually lose their energy to the small-scale mixing is still poorly understood. Several mechanisms are suggested and discussed by St. Laurent and Garrett (2002). Clearly, any progress along these lines cannot be made unless the generation of low modes is reasonably well understood. Our objective is to calculate the energy conversion rate into low vertical modes on the global scale.

In the present study, we investigate the modal energy conversion rate of internal tides for the global ocean using the physical space expression given by LY02. Two principle motivations to do so are as follows: First, the low modes can propagate over considerable distances, while high modes are more likely to dissipate locally near the generation site. Hence, information on the distribution of energy conversion rate into vertical modes is of paramount importance for studying the internal-tide-driven mixing in the ocean. Second, contrary to the WKB-based formulation used by Nycander (2005), the vertical variation of buoyancy frequency and the finite depth of the ocean are taken into account in a fully consistent way by projecting the internal tide generation problem onto the vertical modes.

We determine the energy conversion rate into modes 1–10, which account for the bulk of the energy radiation on the global scale. We also compare the results of the modal calculation with the one obtained using the formalism of Nycander (2005). The modal decomposition–based energy conversion rate given by LY02 is obtained by calculating a convolution integral. The kernel of this integral is a Green’s function that has a very slow asymptotic decay. In numerical calculations, this can cause a problem with convergence. This therefore necessitates considering the test case problem in which case the issue of the very slow asymptotic decay of the modal Green’s function can be investigated. Thus, we commence by examining the generation problem for the case of a unidirectional tide flowing over an idealized one-dimensional topography. In this simple test, the analytical solution for the energy conversion rate into vertical modes can be obtained, so that comparison with the numerical solution is possible. This also provides the basis for the global calculation performed afterward.

The semianalytical model used in this study to calculate the modal energy conversion rate is valid only when the internal tide characteristic slope is larger than the bathymetric slope, the bathymetric height is small relative to the vertical scale of the wave, and the horizontal tidal excursion is smaller than the horizontal topographic scale. To fully circumvent these limitations, one needs to use numerical models based on 3D primitive equations that do not require assumptions used in the semianalytical model (e.g., Zilberman et al. 2009; Niwa and Hibiya 2011). Nonetheless, these numerical models are computationally more expensive than the semianalytical model to be used over a large domain with fine horizontal grid spacing. Also, numerical models depend on how and
where internal tides are dissipated, since in these models internal tides propagate and they can alter a far distant internal tide generation. In contrast, the semianalytical model can be used to calculate the energy conversion rate without assumptions on the dissipation. In essence, the semianalytical model can be regarded as a complementary model to forward 3D time-stepping models to parameterize unresolved high-mode internal tide generation, for which high-resolution topography data are required.

This paper is organized as follows: The theory of the vertical normal modes followed by the formalism of the modal energy conversion rate, as well as Nycander's (2005) formulation, is briefly outlined in section 2. The numerical results for the modal energy conversion rate are compared with the exact solution and Nycander's (2005) formulation for the simple test case in section 3. The global calculation of the energy conversion rate into the first 10 vertical normal modes is presented in section 4 where a discussion on the validity of this calculation in regions with supercritical slopes is included. A summary and conclusions are offered in section 5.

2. Theory

a. Vertical normal modes

In an ocean of finite depth, the internal tide generation problem can be tackled by projecting the vertical structure of the physical variables onto vertical modes depending only on the vertical profile of the buoyancy frequency $N(z)$. Following LY02, these modes satisfy the following Sturm–Liouville problem subject to boundary conditions imposed at the sea surface ($z = 0$) and the sea floor ($z = -H$):

$$\frac{d^2 a_n}{dz^2} + c_n^{-2} N^2 a_n = 0, \quad a_n(0) = a_n(-H) = 0,$$  \hspace{1cm} (1)

where the index $n$ denotes the mode number, $c_n^{-2}$ is the eigenvalue, and $a_n$ is the eigenfunction. Physically speaking, $c_n$ is the eigenvalue of the mode-$n$ internal wave. Furthermore, the orthogonality relation for a set of $a_n$ terms is given by

$$\int_{-H}^{0} a_n(z) a_m(z) N^2(z) \, dz = |f| c_n \delta_{mn},$$  \hspace{1cm} (2)

where $f$ is the Coriolis parameter, and $\delta_{mn}$ is the Kronecker matrix. We define the horizontal wavenumber associated with the mode $n$ as follows:

$$\kappa_n = \sqrt{\frac{\omega^2 - f^2}{c_n}}.$$  \hspace{1cm} (3)

where $\omega$ is the frequency of the barotropic tidal constituent. Equation (1) can be solved numerically using a method presented by Chelton et al. (1998).

Finally, a dimensionless quantity $S_n$ is defined as

$$S_n = \frac{c_n d_n(-H)}{|f|},$$  \hspace{1cm} (4)

where $d_n'(z) = da_n/dz$.

For the case of uniform stratification ($N = \text{constant}$), $a_n$ can be readily evaluated analytically. In this case, $\kappa_n$ and $S_n^2$ are given by

$$\kappa_n = \sqrt{\frac{\omega^2 - f^2}{N H}}, \quad S_n^2 = \frac{2}{n \pi |f|}.$$  \hspace{1cm} (5)

In what follows, the horizontal wavenumber $\kappa_n$ and the nondimensional constant $S_n^2$ are used in the calculation of the energy conversion rate for the mode $n$, denoted by $C_n$.

b. The energy conversion rate into vertical modes

An expression for the energy conversion rate from the barotropic tide to the internal tide at the bottom of an ocean with finite depth was given by LY02. They considered the barotropic tidal velocity, with the amplitudes $U_0$ and $V_0$ as well as the temporal phase difference $\chi$, in the following form

$$U = [U_0 \cos(\omega t), V_0 \cos(\omega t + \chi)],$$  \hspace{1cm} (6)

and derived energy conversion rate for mode $n$ in the real space:

$$C_n = \iint P_n \, dx \, dy,$$  \hspace{1cm} (7)

where $P_n$ is the energy flux density (W m$^{-2}$). The expression for $P_n$ in the coordinate system aligned with the major and minor axes of the tidal ellipse in (6), denoted $U_+$ and $U_-$, respectively, is given by

$$P_n = \frac{1}{8} \rho_0 |f| \kappa_n^2 S_n^2 \sqrt{1 - \frac{f^2}{\omega^2}} \iint J_0(|\mathbf{x} - \mathbf{x}'|) \frac{\partial h}{\partial x'} \frac{\partial h}{\partial y'} \, d^2 \mathbf{x}' + U_+^2 \frac{\partial h}{\partial y} \iint J_0(|\mathbf{x} - \mathbf{x}'|) \frac{\partial h}{\partial y'} \, d^2 \mathbf{x}' \right].$$  \hspace{1cm} (8)

Here, $\rho_0$ is the mean density of the seawater, Green’s function $J_0$ is the zero-order Bessel function of the first kind, $h$ is the topography height, the $x$ axis is taken to lie along the major axis of $U_+$, and $U_+$ and $U_-$ are given by
\[ U_{\pm}^2 = \frac{1}{2} \left[ U_0^2 + V_0^2 \pm \sqrt{(U_0^2 + V_0^2)^2 - 4U_0^2V_0^2\sin^2(\chi)} \right]. \]  

(9)

The total energy conversion rate is the sum over the entire spectrum of vertical modes:

\[ C = \sum_{n=1}^{\infty} C_n. \]  

(10)

In deriving (8), LY02 employed the linearized hydrostatic Boussinesq equations subject to the linearized bottom boundary condition for the vertical velocity. The latter is justified provided that the slope of the internal tide characteristic, expressed as \( \alpha = \left[ (\omega^2 - f^2)/(N_B^2 - \omega^2) \right]^{1/2} \), is larger than the underlying bathymetric slope \( |\nabla h| \). The steepness parameter \( \epsilon \) is defined as the ratio between the latter and the former:

\[ \epsilon = \frac{|\nabla h|}{\alpha} = |\nabla h| \left( \frac{N_B^2 - \omega^2}{\omega^2 - f^2} \right)^{1/2}. \]  

(11)

If \( \epsilon \) is greater than unity, the bathymetry is referred to as supercritical. A further assumption made in the derivation of \( C_n \) in (8) is the smallness of the tidal excursion, given by \( U_0/\omega \), compared to the horizontal scale of the topography.

The integral in (8) is a convolution integral, demonstrating the nonlocality of the internal tide generation, so that it depends not only on the local bathymetric slope but also on that associated with the surrounding points. An alternate form of (8) can be obtained after integrating by parts, yielding the following equation:

\[ P_n = -\frac{\rho_0}{8} |\nabla h|^2 \sqrt{1 - \frac{f^2}{\omega^2}} \left[ U_0^2 \frac{\partial h}{\partial x} \right] \int h(x',\kappa_n|x - x'|) \frac{x - x'}{|x - x'|} \, d^2x' + U_0^2 \frac{\partial h}{\partial y} \int h(x',\kappa_n|x - x'|) \frac{y - y'}{|x - x'|} \, d^2x'. \]  

(12)

where we used \( dJ_0(x)dx = -J_1 \), and \( J_1 \) is the Bessel function of the first order. Only one of the factors, that is, \( \partial h/\partial x \), in (12) needs to be differentiated numerically. In (8), on the other hand, both factors, that is, \( \partial h/\partial x \) and \( \partial h/\partial x' \), must be differentiated numerically. Therefore, a more accurate result is obtained using (12). Thus, the results presented in this study are obtained by (12).

c. Formalism of Nycander (2005)

Nycander (2005) utilized an expression for the energy conversion rate given by Bell (1975a,b) for an ocean of infinite depth and constant stratification and modified it to take into account the main effect of finite depth and the vertically varying \( N(z) \) in an approximate way. The final expression in the tidal–ellipse coordinate, henceforth denoted \( C_{NYC} \), is given by

\[ C_{NYC} = \int P_+ \, dx \, dy + \int P_- \, dx \, dy, \]  

(13)

where

\[ P_+ = -\frac{\rho_0 N_B U_0^2}{4\pi} \sqrt{1 - \frac{f^2}{\omega^2}} \frac{\partial h}{\partial x} \int \frac{\partial g_a(x - x')}{\partial x'} h(x') \, dx' \, dx', \]  

(14)

and

\[ P_- = \frac{\rho_0 N_B U_0^2}{4\pi} \sqrt{1 - \frac{f^2}{\omega^2}} \frac{\partial h}{\partial y} \int \frac{\partial g_a(x - x')}{\partial y'} h(x') \, dx' \, dy'. \]  

(15)

Here, \( g_a \) is a filtered Green function introduced to take into account the finite depth of the ocean. It suppresses the influence of the topographic features beyond the horizontal wavelength of the mode-1 internal tide, and \( a \) is the cutoff length defined as

\[ a = \frac{1.455}{\pi \sqrt{\omega^2 - f^2}} \int_{-H}^0 N(z) \, dz. \]  

(16)

The numerical scheme for the computation is discussed in detail by Green and Nycander (2013).

From (14) and (15), it is seen that \( C_{NYC} \) is proportional to \( N_B \). Note that the expression (13) with the finite depth correction via \( g_a \) avoids expanding in the eigenmodes. As a consequence, it obviates the need for solving (1) numerically, as well as summing over the entire internal tide modal spectrum to calculate the total conversion rate. Furthermore, Green’s function \( g_a \) decreases rapidly, as \( r^{-3} \), at infinity. This leads to a significant reduction in the computational expense for calculating \( C_{NYC} \) in comparison to that associated with the calculation of \( C \) from (10).

In this work, we will compare the energy conversion obtained from (13) with the modal calculation using (7).
3. Test case with Witch of Agnesi

Our goal is to use (12) and calculate the modal energy flux \( P_n \) and the corresponding \( C_n \) for the global ocean. Before doing so, we want to test the numerical convergence of the modal conversion rate and find a suitable cutoff length for truncating the modal Green’s function \( J_1 \) in (12). To do so, we simplify the generation problem to the case of a unidirectional tide impinging on an idealized one-dimensional topography \( h(x) \). We use the Witch of Agnesi profile:

\[
h(x) = \frac{h_0}{1 + \left(\frac{x}{\Lambda}\right)^2},
\]

where \( h_0 \) is the maximum ridge height, and \( \Lambda \) is the half-width of the ridge.

When applied to the case of one-dimensional topography \( h(x) \), (7) yields the energy conversion rate per unit length in the horizontal \( y \) direction (W m\(^{-1}\)). A mathematically equivalent expression to (7) in Fourier space is given by LY02. This was utilized by Zarroug et al. (2010) for the case of one-dimensional topography, and the result is

\[
C_n = \frac{1}{4} \rho_0 f k_n^2 S_n^2 \left[ 1 - \frac{f^2}{\omega^2} U_0^2 |\tilde{h}(\kappa_n)|^2 \right],
\]

where \( \tilde{h} \) is the Fourier transform of \( h(x) \). For the ridge given by (17), we have

\[
\tilde{h}(\kappa_n) = h_0 \Lambda \pi \exp(-|\kappa_n|/\Lambda).
\]

Using (18) together with (19) enables us to validate the results obtained through solving (12) numerically for the case of the Witch of Agnesi profile. We consider both a uniform stratification \( N = \text{const} \) and a non-uniform \( N(z) \).

The maximum ridge height \( h_0 \) is set to 100 m, the ocean depth \( H \) is 4 km, unless otherwise stated, the Coriolis parameter \( f \) is taken to be \( 2 \times 10^{-5} \text{ s}^{-1} \), the mean density of seawater \( \rho_0 = 1040 \text{ kg m}^{-3} \), the tidal frequency corresponds to that associated with the M\(_2\) tidal constituent (i.e., \( \omega = 1.40 \times 10^{-2} \text{ s}^{-1} \)), and the tidal amplitude \( U_0 \) is assumed to be \( 4 \times 10^{-2} \text{ m s}^{-1} \). Furthermore, the ridge is placed in the middle of the domain that has the horizontal extent 4000 km.

For the Witch of Agnesi ridge, the steepness parameter \( \epsilon \) is

\[
\epsilon_{\text{max}} = \frac{3^{3/2} h_0}{8 \Lambda} \left( \frac{N_0^2 - \omega^2}{\omega^2 - f^2} \right)^{1/2},
\]

and the tidal excursion parameter is expressed as \( U_0/\Lambda \omega \).

The bottom boundary is discretized into 4000 intervals of equal size, yielding the uniform grid spacing \( \Delta x = 1 \text{ km} \). The numerical solution for the modal energy conversion rate is then compared to the exact solution (18).

a. Sensitivity to vertical profile of buoyancy frequency

Here, we present the results for both constant and realistic nonuniform stratification (Fig. 1). For the former case, \( N = 8 \times 10^{-4} \text{ s}^{-1} \), which is representative of the deep-ocean stratification. For the latter case, two \( N \) profiles are obtained from the World Ocean Circulation Experiment (WOCE) Global Hydrographic Climatology (WGHC; Gouretski and Koltermann 2004): one from a position in the South Atlantic (21°S, 18.5°W; hereinafter SNA), with a bottom depth of 4375 m and bottom buoyancy frequency \( N_B = 9.0 \times 10^{-4} \text{ s}^{-1} \), and one from a position in the North Atlantic (50°N, 30°W; hereinafter SSA), with a bottom depth of 2875 m and \( N_B = 9.4 \times 10^{-4} \text{ s}^{-1} \).

We now investigate the sensitivity of \( C_n \) to the topography width. To do so, (12) is truncated at the thirteenth and fourteenth zeros of \( J_1 \), based on the findings as discussed in detail in the appendix. The final result is the average of these two solutions. Four values of \( \Lambda \), that is, \( \Lambda = 2.5, 5, 10, \) and 20 km, are considered. For all three buoyancy frequency profiles, the steepness parameter and the tidal excursion parameter associated with \( \Lambda \) are small. For instance, for the case of constant \( N \), the \( \epsilon_{\text{max}} \) associated with the values of \( \Lambda \) are 0.178, 0.089, 0.044, and 0.022, respectively. The corresponding tidal excursion parameters are 0.114, 0.057, 0.028, and 0.014. Also, a total of 100 vertical normal modes are taken into account for the following sensitivity test to the topographic scale. These 100 modes capture almost all the energy conversion, as can be verified using (18). Thus, \( C_{\text{exact}} = \sum_{n=1}^{100} C_{n}^{\text{exact}} \).
For the case of constant stratification, the ratio $C_n^{\text{exact}}/C_n^{\text{NYC}}$ indicates that for all widths but $\Lambda = 2.5\text{ km}$, mode 1 is the most energetic one (Fig. 2a). For SSA and SNA, mode 1 is the most energetic mode only for $\Lambda = 20\text{ km}$ (Figs. 2b,c). As in the case of constant $N$, the bulk of the total energy goes to the first five modes for the four widths considered. Also, as the width decreases, modes 6-20 become more energetic.

To measure the difference between the numerical solution $C_n$ and the exact solution $C_n^{\text{exact}}$, we make use of the relative error $\gamma_n$, given by

$$\gamma_n = \frac{|C_n - C_n^{\text{exact}}|}{C_n^{\text{exact}}}. \quad (21)$$

We regard the error as being significant when $\gamma_n > 10\%$, and in this case $C_n$ will be discarded in the subsequent analyses. Along these lines, we introduce the $\Lambda$-dependent integer function $n_{\text{max}}(\Lambda)$ representing the maximum mode number for a given $\Lambda$ such that $\gamma_{n_{\text{max}}} \leq 10\%$.

Table 1 summarizes the results of the sensitivity test for the constant stratification, SSA and SNA, and for the four topographic scales considered.

For each of the three buoyancy frequency profiles, it is seen that $n_{\text{max}}$ decreases with increasing topographic scale. Also, the mode-$n_{\text{max}}$ half-wavelength $\pi k_{n_{\text{max}}}^{-1}$ obtained from (5) is always somewhat greater than the ridge half-width. Thus, modes with a shorter half-wavelength than the topographic length scale cannot be calculated accurately. This is because the energy conversion to such modes is very small. These modes therefore also account for only a small fraction of the total energy conversion. This can be readily verified by evaluating the ratio $C_{\text{sum}}/C_n$, where $C_{\text{sum}} = \sum_{n=1}^{n_{\text{max}}} C_n$. For each of the three buoyancy frequency profiles, this ratio is very close to 1, implying that the first $n_{\text{max}}$ modes contain almost all of the total energy conversion.

In Table 1, we see that for each $\Lambda$, the value of $n_{\text{max}}$ for the case of nonuniform stratification is larger than that for the case of constant $N$. This is because the modes have a larger wavelength with the nonuniform stratification, and hence more modes have a larger half-wavelength than the topographic scale.

b. Comparison with Nycander’s (2005) formalism

One motivation of this study is to improve the previous simplified method of Nycander (2005). For the Witch of Agnesi, we employ (13) to calculate $C_{\text{NYC}}$ for both constant $N$ and the depth-varying stratification $N(z)$. To do so, we set $U_+ = U_0$ and $U_- = 0$ in (14) and (15), respectively.

Figures 3a, 3b, and 3c show $C_{\text{sum}}$, $C_n^{\text{exact}}$, and $C_n^{\text{NYC}}$ versus the inverse of the topographic-scale $\Lambda^{-1}$ for the cases of constant $N$, SNA, and SSA, respectively. It is seen that for each of the three buoyancy frequency profiles, and for all widths considered, the modal calculations $C_{\text{sum}}$ agree well with the exact solutions. As discussed in the preceding subsection, this is because the energy contained in $n_{\text{max}}$ modes captures the total energy very well. For the case of constant $N$, and for all widths except the largest one, $\Lambda^{-1} = 0.05\text{ km}^{-1}$, there is also a good agreement between $C_n^{\text{NYC}}$ and the exact solution. In the case of SSA, the results based on $C_n^{\text{NYC}}$ are not in good agreement with the exact solution for the widths shown. However, $C_n^{\text{NYC}}$ for a narrower width $\Lambda = 1\text{ km}$ is in very good agreement with the exact solution (not shown). In the case of SNA, the values for $C_n^{\text{NYC}}$ are larger than the exact ones for all widths, except for the narrowest one.

4. Global calculation with realistic topography

In this section, we present the main results of this study, that is, the modal energy conversion rates for the global ocean. The results presented hitherto for the case of the Witch of Agnesi ridge indicate that the modal calculation of the energy conversion rate is more accurate than $C_n^{\text{NYC}}$. The objectives here are first to calculate $C_n$ using (12) and second to compare the results to $C_n^{\text{NYC}}$ on the global scale.

a. Data and method

The data required to perform the global calculation are the gridded bottom topography, the three-dimensional buoyancy frequency field, and barotropic tidal velocities. We used the 2-minute gridded global relief data (ETOPO2v2; available online at www.ngdc.noaa.gov/mgg/global/etopo2.html) topographic dataset with a resolution of $1/36^\circ$, approximately equal to 3 km. Horizontal barotropic tidal velocities are extracted from the Ocean Topography Experiment (TOPEX)/Poseidon Global Inverse Solution (TPXO6.2) model (Egbert and Erofeeva 2002) with a resolution of $1/4^\circ$. We only consider the most energetic tidal constituent $M_2$ in the modal calculation of the energy conversion rate. The global buoyancy frequency data originate from the WGHC, as utilized for the case of Witch of Agnesi in the previous section.

To do the global calculation, the conversion rate expressions (12), (14), and (15) are first transformed from the tidal–ellipse coordinate to the regular longitude–latitude coordinate associated with the bathymetry data. Then partial derivatives $\partial h/\partial x$ and $\partial h/\partial y$ are evaluated by the centered, finite-difference scheme in the midpoints between each pair of neighboring topographic grid points. The barotropic velocities extracted from the TPXO6.2 model are also interpolated onto these midpoints between the topographic grid points.
Fig. 2. The ratio of the exact modal energy conversion rate $C_{\text{exact}}^n$ to the total energy conversion rate $C_{\text{exact}} = \sum_{n=1}^{20} C_{\text{exact}}^n$ for the first 20 modes. (a) The ocean has constant stratification. (b) An ocean with the subtropical South Atlantic $N(z)$ shown as the red line in Fig. 1. (c) An ocean with the subpolar North Atlantic $N(z)$ shown as the blue line in Fig. 1.
The convolution integrals in the expressions (12), (14), and (15) are numerically evaluated by a simple sum, with $\mathbf{x}$ located at the evaluation points for $\partial h/\partial x$ and $\partial h/\partial y$ and $\mathbf{x}'$ located at the topographic points. Thus, the energy flux is obtained on a staggered grid, between the topographic grid points. The global ocean is split into 72 patches with overlapping buffer zones of approximately 16 km width outside of each patch. These buffer zones are needed when computing the convolution integral close to the boundary of each patch. In the calculation of the convolution integrals, all the topographic heights from the surface to the ocean bottom located on $\mathbf{x}'$ are used. Note, however, that the final results are found on the $\mathbf{x}$ points. As discussed below, we only show the results for depths larger than 400 m, which means we show the results for $\mathbf{x}$ points having a depth larger than 400 m.

The results for the Witch of Agnesi ridge indicated that the wavelength of the mode $n$, $2\pi k_n^{-1}$, needs to be comparable to or larger than the topographic scale in order to numerically predict the energy conversion rate associated with this mode with a good accuracy. However, in contrast to the Witch of Agnesi ridge, the gridded global topographic data used here do not have a uniform topographic scale. It is obtained using satellite gravity data [with the resolution of $O(10)\text{ km}$] merged with shipborne depth soundings, resulting in a very nonuniform topographic resolution. In some areas, such as coastal regions with plenty of multibeam data available, the effective topographic resolution is the same as the grid resolution of 3 km. On the other hand, in the deep ocean the multibeam coverage is poor, and hence the effective topographic resolution is close to that associated with the satellite gravity data. Since the focus of this study is directed toward the deep ocean, we thus regard the 10-km topographic scale as the smallest resolvable one.

Thus, we find the following criterion in order to utilize (12) for the calculation of $C_n$:

$$2\pi k_n^{-1} > 10 \text{ km}.$$ 

Based on the above criterion, we restrict our focus to the internal tide generated by the large topographic features, such as canyons and fracture zones conspicuous in Brazil basin, with a scale $O(10)$ km. At scales smaller than 10 km, abyssal hills are the dominant topographic features (Goff 1991) and their contribution to both the global and the regional internal tide energetics has recently been investigated by Melet et al. (2013).

The zonally averaged wavelength of modes 10, 11, 13, and 15, obtained by solving (1) numerically, shows that the mode 10 is the only one among the four modes considered to fully satisfy the above-mentioned criterion at every latitude band (Fig. 4). For modes 11, 13, and 15, it is marginally satisfied. Modes 1–10 consequently fulfill the aforementioned criterion, so that the global calculation will be conducted only for these modes. Note that very large wavelengths for each mode in Fig. 4 occur near the critical latitude, at which the frequency $\omega$ matches the Coriolis frequency $f$. For the $M_2$ tidal component, the critical latitude is $74.46^\circ$.

We calculate the energy conversion rate $C_n$ into the first 10 modes employing (12) with truncation at the thirteenth and fourteenth zeros of $J_1$, based on our findings for the case of the Witch of Agnesi. The final result is the average of these two solutions. Table 2 summarizes the results for the global ocean and its three major basins, the Pacific Ocean, Atlantic Ocean, and Indian Ocean shown in Fig. 5. These values are for depth greater than 400 m. For a depth shallower than 400 m, unreliable results are found for the energy conversion rate of the first 10 vertical modes $C_n$ and $C_{NYC}$. For example, $C_1$ up to the zero depth is found to be $-34.0 \text{ TW}$. For depths shallower than 100, 200, and 300 m, $C_1$ is found to be $-30.26$, $-30.0$, and 12.0 TW, respectively. These values are not realistic since their respective absolute values are larger than the 3.5 TW of the power input to the ocean by the astronomical tidal forcing.

There are two main reasons why the results are not reliable for depths shallower than 400 m. First, because of the generally strong stratification within 400 m of the ocean surface, the characteristic slope of the internal tides is smaller, causing the bathymetry to be more supercritical, in which case the semianalytical model used here is inappropriate. Second, as a consequence of mass continuity, tidal currents over shallow depths become so strong.

<table>
<thead>
<tr>
<th>$\lambda$ (km)</th>
<th>$n_{\text{max}}$</th>
<th>$\pi k_n^{-1} (\text{km})$</th>
<th>$C_{\text{sum}}/C_{\text{exact}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant stratification</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>11</td>
<td>2.53</td>
<td>0.98</td>
</tr>
<tr>
<td>5.0</td>
<td>5</td>
<td>5.57</td>
<td>0.99</td>
</tr>
<tr>
<td>10.0</td>
<td>2</td>
<td>13.93</td>
<td>0.97</td>
</tr>
<tr>
<td>20.0</td>
<td>1</td>
<td>27.85</td>
<td>0.99</td>
</tr>
<tr>
<td>Subtropical South Atlantic $N(z)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>33</td>
<td>2.45</td>
<td>0.98</td>
</tr>
<tr>
<td>5.0</td>
<td>18</td>
<td>4.51</td>
<td>1.0</td>
</tr>
<tr>
<td>10.0</td>
<td>8</td>
<td>10.21</td>
<td>0.99</td>
</tr>
<tr>
<td>20.0</td>
<td>4</td>
<td>20.80</td>
<td>0.99</td>
</tr>
<tr>
<td>Subpolar North Atlantic $N(z)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.5</td>
<td>16</td>
<td>2.81</td>
<td>0.97</td>
</tr>
<tr>
<td>5.0</td>
<td>9</td>
<td>5.00</td>
<td>0.99</td>
</tr>
<tr>
<td>10.0</td>
<td>4</td>
<td>11.46</td>
<td>0.98</td>
</tr>
<tr>
<td>20.0</td>
<td>2</td>
<td>19.84</td>
<td>0.99</td>
</tr>
</tbody>
</table>

TABLE 1. The maximum mode number $n_{\text{max}}$ for four different half-widths $\lambda$, the half wave-length associated with that $\pi k_n^{-1}$, and the ratio $C_{\text{sum}}/C_{\text{exact}}$, where $C_{\text{sum}} = \sum_{n=1}^{n_{\text{max}}} C_n$ and $C_{\text{exact}} = \sum_{n=1}^{\infty} C_n$. The ocean is characterized by a constant stratification, the subtropical South Atlantic $N(z)$, and subpolar North Atlantic $N(z)$, as shown in Fig. 1.
FIG. 3. The comparison between $C_{\text{sum}} = \sum_{n=1}^{n_{\text{max}}} C_n$ (blue triangle–solid line), $C_{\text{exact}} = \sum_{n=1}^{100} C_{n_{\text{exact}}}$ (red dot–solid line), and $C_{\text{NYC}}$ (black square–solid line). The horizontal axis shows the inverse of the half-width ridge in units of $\text{km}^{-1}$, and $n_{\text{max}}$ for each of these four widths is listed in Table 1. (a) The ocean has constant $N$, (b) an ocean with the subtropical South Atlantic $N(z)$ shown as the red line in Fig. 1, and (c) an ocean with the subpolar North Atlantic $N(z)$ shown as the blue line in Fig. 1.
that the tidal excursion parameter can exceed unity. For depths greater than 400 m, the tidal excursion, given by $U_1/v$, where $U_1$ is obtained from (9), rarely exceeds 1 km (not shown). This latter value is less than half the effective resolution of the topographic data used here (i.e., 5 km). Hence, for depths greater than 400 m, the assumption of the smallness of the tidal excursion parameter is justified. The choice of 400 m is motivated by the fact that most continental shelves are about 200–300-m deep, allowing their exclusion from our estimates. Therefore, in what follows, results for depths shallower than 400 m will not be presented.

b. Global distribution of energy conversion rate

From Table 2, it is seen that the global conversion rate into mode 1, which is characterized by a large horizontal wavelength and a fast propagation speed, is 0.237 TW. The conversion rate decreases with the increasing mode number, and the sum up to mode 10 is 0.707 TW (Table 2). The latter value is consistent with that estimated by Egbert and Ray (2000), who found that the abyssal dissipation of the M2 tidal constituent falls in the range 0.57–0.83 TW. For all three basins considered, mode 1 is the

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**TABLE 2.** The energy conversion rate $C_n$ for the first 10 internal tide modes and the sum of them $C_{\text{sum}} = \sum_{n=1}^{10} C_n$ for the global ocean and the Pacific, Atlantic, and Indian basins shown in Fig. 5. The listed numbers in percentage are the fraction of the sum of the energy conversion rate contained in each vertical normal mode. $C_{NYC}$ is the energy conversion rate obtained using the formalism of Nycander (2005). $C_{\text{cra}}$ is the global energy conversion rate into the first 10 modes corrected for supercritical slopes using Melet et al.'s (2013) method, and the correction is done for all the first 10 modes. $C_{\text{sub}}$ is the global energy conversion rate into the first 10 modes and only for subcritical regions. $C_{\text{crh}}$ is the global energy conversion rate into the first 10 modes corrected for supercritical slopes when the correction is only done for high modes (i.e., modes 3–10).

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Global Ocean</th>
<th>Pacific Ocean</th>
<th>Atlantic Ocean</th>
<th>Indian Ocean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.237 (34%)</td>
<td>0.140 (37%)</td>
<td>0.042 (36%)</td>
<td>0.040 (33%)</td>
</tr>
<tr>
<td>2</td>
<td>0.178 (25%)</td>
<td>0.091 (24%)</td>
<td>0.023 (20%)</td>
<td>0.034 (28%)</td>
</tr>
<tr>
<td>3</td>
<td>0.107 (15%)</td>
<td>0.053 (14%)</td>
<td>0.020 (17%)</td>
<td>0.020 (16%)</td>
</tr>
<tr>
<td>4</td>
<td>0.066 (9%)</td>
<td>0.033 (9%)</td>
<td>0.012 (10%)</td>
<td>0.010 (8%)</td>
</tr>
<tr>
<td>5</td>
<td>0.041 (6%)</td>
<td>0.020 (5%)</td>
<td>0.007 (6%)</td>
<td>0.008 (7%)</td>
</tr>
<tr>
<td>6</td>
<td>0.025 (4%)</td>
<td>0.012 (3%)</td>
<td>0.005 (4%)</td>
<td>0.005 (4%)</td>
</tr>
<tr>
<td>7</td>
<td>0.015 (2%)</td>
<td>0.008 (2%)</td>
<td>0.003 (3%)</td>
<td>0.003 (2%)</td>
</tr>
<tr>
<td>8</td>
<td>0.015 (2%)</td>
<td>0.007 (2%)</td>
<td>0.003 (3%)</td>
<td>0.002 (2%)</td>
</tr>
<tr>
<td>9</td>
<td>0.012 (2%)</td>
<td>0.006 (2%)</td>
<td>0.003 (3%)</td>
<td>0.001 (1%)</td>
</tr>
<tr>
<td>10</td>
<td>0.011 (2%)</td>
<td>0.006 (2%)</td>
<td>0.001 (1%)</td>
<td>0.002 (2%)</td>
</tr>
</tbody>
</table>

$C_{\text{sum}}$ 0.707 0.376 0.117 0.122

$C_{NYC}$ 0.820 0.411 0.143 0.147

$C_{\text{cra}}$ 0.422

$C_{\text{sub}}$ 0.321

$C_{\text{crh}}$ 0.619

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**FIG. 4.** The zonal average of horizontal wavelength $2\pi k_n^{-1}$ (km) for modes 10, 11, 13, and 15 internal tide.

**FIG. 5.** (a) The three ocean basins, Pacific, Atlantic, and Indian, considered in Table 2. All these three basins contain the Southern Ocean.
most energetic mode, and the energy conversion rate decreases with increasing mode number. On both the global and basin scales, the mode 1 contains between 33% and 37% of the total energy conversion rate into the first 10 modes. Also shown in this table is $C_{NYC}$, equal to $0.820$ TW on the global scale. The formalism of Nycander (2005) yields a somewhat larger conversion rate (16% larger) over the global scale than the modal calculation. This also holds true for each of the three basins, that is, the Pacific, Atlantic, and Indian Oceans.

Figure 6 shows the geographical distribution of the energy flux into the first-mode $P_1$ and the sum of that into the first 10 vertical modes: $P_{\text{sum}} = \sum_{n=1}^{10} P_n$. These are area-weighted averages of the energy flux over approximately $1^\circ \times 1^\circ$ squares. The energy flux is due to the $M_2$ tidal constituent. The color scale is logarithmic. For example, $-3$ means $10^{-3}$ W m$^{-2}$. The white shaded regions represent continents, regions poleward of the critical latitude 74.46º, and excluded regions with depths shallower than 400 m. The gray shaded regions represent the geographical points having the energy flux of either less than $10^{-6}$ W m$^{-2}$ or negative; (b) as in (a), but for the first 10 vertical modes $P_{\text{sum}}$. 

Fig. 6. (a) The global distribution of the total energy flux (W m$^{-2}$) into the first vertical mode. It is an area-weighted average of the calculated field, obtained on a staggered grid between topographic points, over approximately $1^\circ \times 1^\circ$ squares. The energy flux is due to the $M_2$ tidal constituent. The color scale is logarithmic. For example, $-3$ means $10^{-3}$ W m$^{-2}$. The white shaded regions represent continents, regions poleward of the critical latitude 74.46º, and excluded regions with depths shallower than 400 m. The gray shaded regions represent the geographical points having the energy flux of either less than $10^{-6}$ W m$^{-2}$ or negative; (b) as in (a), but for the first 10 vertical modes $P_{\text{sum}}$. 

The results in Table 2 show that on the global scale, the mode-1 internal tide is the most energetic one. This is not always the case on a regional scale, as shown by St. Laurent and Garrett (2002). Figures 7a–b show the
FIG. 7. (a), (b) As in Figs. 6a and 6b, respectively, but averaging over approximately $10^\circ \times 10^\circ$ squares. (c) The ratio of the energy flux shown in (a) to that in (b). The gray shaded regions here represent the areas in which either one of $P_1$ and $P_{sum}$ or both of them are negative.
global distribution of $P_1$ and $P_{\text{sum}}$, respectively, and the ratio between them is shown in Fig. 7c. Here, $P_n$ is obtained by an area-weighted averaging of $P_n$ over approximately $10^8 \times 10^8$ squares. This was done to get a less noisy plot. From this figure, it is recognized that in most parts of the ocean the ratio is less than 0.3, that is, the energy contained in the first mode accounts for less than 30% of the total energy. In many regions, the ratio is less than 0.1, which means that mode 1 is not the most energetic mode there.

Figure 8 shows the distribution of the energy conversion rates (GW) versus latitude for modes 1, 2, 3, $C_{\text{sum}}$, and $C_{\text{NYC}}$. For most latitudes, the mode 1 stands out as the most energetic mode. It peaks at a latitude very close to 20°N. This is very likely because of the strong mode-1 internal tide generation in the western and the central Pacific, for instance, the Hawaiian Ridge. The latitudinal distribution of the energy conversion rate obtained using $C_{\text{NYC}}$ closely follows $C_{\text{sum}}$. Aside from that, it is seen that in most latitudes, $C_{\text{NYC}}$ yields a higher value than $C_{\text{sum}}$.

The Hawaiian Ridge (located in the latitude range 16°–32°N and in the longitude range 154°–180°W) is regarded as a significant area of the internal tide generation and a large number of the theoretical, numerical, and observational studies have been devoted to this area. In our calculation, the mode-1 energy conversion rate and the sum of the conversion rate up to the mode 10 for this ridge are 10.83 and 14.38 GW, respectively. For a comparison, Merrifield and Holloway (2002) estimated 6 and 10 GW for the mode-1 $M_2$ internal tide and the total energy conversion rate, respectively, over the Hawaiian Ridge using a numerical model. Using a two-layer model, Kang et al. (2000) estimate 5.4 GW of the tidal conversion into the first vertical mode of the internal tide. Finally, using altimetry data, Ray and Mitchum (1997) estimated 15 GW of the mode-1 $M_2$ internal tide generation over the Hawaiian Ridge. Ray and Cartwright (2001) later refined the latter value of the mode-1 internal tide over the Hawaiian Ridge to 6 GW.

We make a comparison between the vertical distribution of the energy conversion rate into the first five modes (i.e., $C_1$–$C_5$, $C_{\text{sum}}$, and $C_{\text{NYC}}$). For most depths in the range 400–5500 m, $C_{\text{NYC}}$ yields larger values than the sum of the modal calculation, and the difference between them decreases as the depth increases (Fig. 9a). Also, the difference between the five globally most energetic modes (i.e., modes 1–5) decreases with increasing depth. This can be understood by using the WKB approximation.

In the deep ocean, it is appropriate to use the WKB method for calculating the horizontal wavenumber $\kappa_n$. In this case, the difference between two successive horizontal wavenumbers, $\delta \kappa = \kappa_{n+1} - \kappa_n$, is given by

$$
\delta \kappa = \sqrt{\omega^2 - f^2 \frac{\pi}{NH}},
$$

where $N$ is the vertical average of the buoyancy frequency, which is strongly affected by the strong stratification in vicinity of the thermocline. From the above equation, it is clear that for the very deep ocean (very large $H$), $\delta \kappa$ becomes so small that the discrete modal spectrum approaches a continuous one. Thus, the wavenumbers of the first three modes are closer to one another, and consequently so are their respective energy conversion rates.

The energy conversion rate contained between two successive depths and normalized by the 50-m distance between these two depths is higher for the shallow depth ranges than in the deep ocean (Fig. 9b). Also, the difference between $C_{\text{sum}}$ and $C_{\text{NYC}}$ is smaller in the deep ocean.

The difference between the ratio of the energy conversion rate for mode 1 to $C_{\text{sum}}$ and that for modes 1–5 to $C_{\text{sum}}$ is larger for the shallow depth ranges than in the deep ocean, and mode 1 contains the largest fraction of the total energy in the shallow depths (Fig. 9c). For the deep ocean (depths below 4000 m), the ratios associated with the first five modes become more close to one another. As noted above, this is a consequence of the fact that as the ocean depth becomes very large the difference between the energy conversion rate of the successive modes is getting smaller. In contrast to the shallow depth ranges, mode 1 does not stand out as the most energetic mode, which is instead mode 3.

The energetic, first-mode internal tide plays an important role in the remote generation of the internal tide and the subsequent mixing. This is because this mode
can propagate a long distance from the generation site without being subject to substantial dissipation. To a slightly lesser extent, the same holds true for mode 2, which is the second most energetic internal tide mode globally. To elaborate more on this, we consider the criterion by Klymak et al. (2010) stating that the vertical mode \( n \) with an eigenspeed \( c_n \) smaller than the maximum amplitude of the barotropic tidal velocity \( U_1 \) is dissipated locally near the generation site. This criterion was proposed and verified only in the case of a single ridge. Nevertheless, it is interesting to apply it globally. For the first 10 vertical modes whose energy conversion rates are calculated above, we check the mentioned criterion for all the geographic points of the ocean below the depth 400 m. The global energy conversion rate associated with points satisfying the criterion of Klymak et al. (2010) is then calculated. For mode 1, the energy conversion rate associated with these points is \(-0.027\) TW. For modes 2–7, the energy conversion rate is less than 5% of the energy conversion rate listed in Table 2. For modes 8–10, the energy conversion rates are 19%, 11%, and 16% of their corresponding energy conversion rate in Table 2, respectively. Hence, the modes 8–10 can be more dissipated locally compared to modes 1–7. Since we do not have the energy flux of the modes higher than 10 in this study, we cannot investigate what fraction of the energy conversion rate of these modes can be dissipated locally based on the above-mentioned criterion. It is worthwhile, however, noting that as the mode number increases more geographic points meet the criterion. This is because the eigenspeed decreases with increasing mode number.

We caution that the criterion by Klymak et al. (2010) may be too simple and does not capture some effects, such as wave–wave interactions, that may contribute to the local dissipation and were not included in their study. Also, this criterion is based on the formation of vigorous lee waves by an oscillatory flow such as the barotropic tide over the bottom topography. Tidally generated lee waves have a frequency at least twice the tidal frequency \( \omega \) (Bell 1975a). As discussed earlier, however, in the deep ocean, the tidal excursion parameter is mostly much less than unity based on the resolution of the ETOPO2v2 topography dataset, giving rise to the weak generation of lee waves by the oscillatory tidal flow. In this case, the internal tide energy predominantly stays at the fundamental tidal frequency \( \omega \). Regardless, it is worthwhile to use the criterion of Klymak et al. (2010) over the global ocean.

c. Energy conversion rate over supercritical slopes

The assumptions made in the semianalytical model used here to calculate \( C_n \) are not valid for a tall and
supercritical topography such as the Hawaiian Ridge, with a typical steepness parameter $\epsilon = 4$ and a large topographic height relative to the ocean depth with a typical height ratio of 0.8 (Legg and Klymak 2008). Hence, our results for the supercritical topography such as the Hawaiian Ridge are highly susceptible to uncertainties. In essence, for the case of the supercritical topography, because of the nonlinear bottom boundary condition for the vertical velocity, the eigenmodes have a horizontal variation, and hence they cannot only be a function of $z$, as assumed in this study. This gives rise to a modal coupling and intermodal energy conversion rate so that there is an exchange of the energy between the vertical normal modes (Kelly et al. 2012; Griffiths and Grimshaw 2007). A mathematical framework to derive a formulation for the modal energy conversion rate in the case of the modal coupling is discussed in detail by Griffiths and Grimshaw (2007). This is also of interest for future work to extend the semianalytical model presented here for calculating the modal energy conversion rate to the case of the supercritical topography, taking into account the modal coupling and the intermodal energy conversion rate.

The generation of the internal tides over the supercritical topography is still an open problem and much more work needs to be done to solve this problem (Llewellyn Smith and Young 2003; Nycander 2006; Balmforth and Peacock 2009). The semianalytical model valid only for the subcritical topography predicts the energy conversion rate proportional to the steepness parameter squared $\epsilon^2$, and hence it most likely overestimates the energy conversion rate for the supercritical topography (Melet et al. 2013).

Previous studies, on the other hand, showed that the energy conversion rate saturates when the topography is supercritical (Khatiwala 2003; Nycander 2006; Balmforth and Peacock 2009).

To improve predictions of the internal tide energy flux using a semianalytical model valid only for the subcritical topography, Melet et al. (2013) proposed a method where the internal tide energy flux is divided by the squared steepness parameter $\epsilon^2$ when the bottom topography is supercritical and is kept unaltered otherwise. We apply this method to the sum of the energy flux into the first 10 modes $P_{\text{sum}}$ over the global ocean, and the resulting globally integrated energy flux for depths greater than 400 m, denoted $C_{\text{crh}}$, amounts to 0.422 TW (Table 2). Also shown in this table is $C_{\text{sub}}$, which is the globally area-integrated $P_{\text{sum}}$ only for the subcritical slopes and is equal to 0.321 TW. It therefore implies that 55% of the total energy conversion rate, $C_{\text{sum}} = 0.707$ TW, is due to the supercritical slopes, consistent with Nycander (2005), who found that about half of the $M_2$ internal tide energy conversion rate comes from the supercritical slopes.

As noted above, the semianalytical model used in this study cannot take into account the intermodal energy conversion rate. That is, the correction for the supercritical topography in the calculation of $C_{\text{crh}}$ was equally done for all the first 10 modes. As mentioned by Garrett and Kunze (2007), for the steep isolated topographic features responsible for a large fraction of the tidal energy conversion into the internal tides, it is very likely that most of the internal tide energy flux is in low modes. Some support for this is given by Llewellyn Smith and Young (2003), who offered an analytical solution of the energy conversion rate for the knife edge topography (where $\epsilon \to \infty$) in an ocean of finite depth. Llewellyn Smith and Young (2003) used their knife edge model to calculate the internal tide energy conversion rate using values of the buoyancy frequency and the tidal velocities representative of the Hawaiian Ridge and found that most of the energy conversion rate is in the first-mode internal tide. A similar numerical result for the Hawaiian Ridge was also found by St. Laurent et al. (2003) using the knife edge model. These results are in accord with the fraction of the internal tide energy flux in mode 1 observed at Hawaii (Ray and Mitchum 1996). Nash et al. (2006) investigated in detail the internal tide energy flux at the steep Kaena Ridge, Hawaii, and found the predominance of the energy flux in modes 1 and 2. Klymak et al. (2010) stated that the knife edge model (Llewellyn Smith and Young 2003; St. Laurent et al. 2003) predicts the energy flux of low modes quite well, while overestimating that of high modes.

In view of the above-mentioned points, we regard modes 1 and 2 as low modes and apply Melet et al.’s (2013) method only to high modes, that is, modes 3–10 in this case, in the calculation of $P_{\text{sum}}$ over the global ocean. By doing so, we assume that the semianalytical method overestimates the energy flux of the high modes for isolated supercritical topographic features. In this case, the globally area-integrated energy flux of $P_{\text{sum}}$, denoted $C_{\text{crh}}$, is equal to 0.619 TW (Table 2). The latter value is close to $C_{\text{sum}}$, again demonstrating the dominance of the first two modes in the global internal tide energy conversion rate.

Figures 10a and 10b show the depth distribution of $C_{\text{sub}}$, $C_{\text{crh}}$, $C_{\text{sum}}$, and $C_{\text{sum}}$. From these figures, it is seen that the difference between these four energy conversion rates is significant mostly at shallow depths, and it becomes smaller with increasing depth, so that for depths greater than 3500 m, there is almost no difference because the bottom topography is mostly subcritical for greater depths. The comparison between these four solutions provides some indication on how sensitive the semianalytical model might be to the subcritical slope assumption. It is likely that true conversion rates might
stand somewhere between $C_{crh}$ and $C_{cra}$, implying a total conversion 20%–30% smaller because of the reduced maxima in supercritical spots.

d. Negative energy conversion rate

The energy flux field obtained using the modal calculation can have negative values. This is the case for both the global calculation presented here and the test case problem examined in the preceding section. For the latter case, it was shown that the line-integrated modal energy flux $C_n = \int P_n \, dx$ is positive for all the topography widths considered, and the agreement between the analytical solution $C_n^\text{exact}$ and the numerical solution $C_n$ up to mode $n_{\text{max}}$ is remarkable. Thus, the negative values are an intrinsic property of the exact solution, and not caused by numerical errors. Nevertheless, they may degrade the numerical accuracy.

For the case of the global calculation, we note in Fig. 6 that the gray shaded regions have an energy flux either less than $10^{-6}$ W m$^{-2}$ or negative. Globally, for the energy flux field shown in Fig. 6b, the total energy conversion rate associated with these negative values is $-0.034$ TW, which is less than 5% of the sum of the energy conversion rate into the first 10 modes. Hence, these negative values do not raise a significant issue and cannot severely affect our results for the case of the energy sum. For the case of mode 1, the total negative energy conversion is $-0.049$ TW, which is less than 21% of the energy conversion rate into the mode 1. As is apparent from Fig. 6a, for the case of mode 1, the gray shaded regions are more widespread than that for the case of the sum shown in Fig. 6b. This means that the geographic distribution of the mode-1 energy flux is noisier than the sum, since it embraces more negative and very small values of the energy flux. By area-weighted averaging over $10^6 \times 10^6$ squares, the total negative energy conversion rate is $-0.01$ TW, which is less than 5% of the mode-1 internal tide energy conversion rate. Note that the total energy conversion rate, which is the sum of the positive and negative energy conversion rates, is independent of the size of the squares over which the averaging procedure is done. In fact, by increasing the size of squares, we obtain a less negative energy conversion rate and a more positive energy conversion rate with their sum, that is, the total energy conversion rate is unaltered.

The global energy flux obtained using the formulation of Nycander (2005) also contains negative values, but the total energy conversion rate associated with them, after area-weighted averaging over $10^6 \times 10^6$ squares, is only $-0.87 \times 10^{-8}$ TW. This is utterly negligible relative to the total energy conversion rate $C_{NYC} = 0.820$ TW, as listed in Table 2.

The issue of the negative energy flux is also brought up by Zilberman et al. (2009) when studying the internal tide generation over a region in the Mid-Atlantic Ridge, utilizing both numerical and analytical approaches. Further, in the calculation of the energy flux of the $M_2$ internal tide done by Simmons et al. (2004) over the global ocean using a two-layer model, the areas having the negative energy flux are easily noticeable (their Fig. 12).

5. Summary and conclusions

We have examined the internal tide generation problem for the global ocean. This is quantified in terms
of the energy conversion rate into vertical modes whose structures are governed by the vertical profile of the buoyancy frequency $N(z)$.

One of our goals was to assess the advantage in accuracy of using vertical modes instead of Nycander's (2005) formulation $C_{NYC}$, which is based on the WKB approximation. This was done for a range of widths of an idealized ridge in both an ocean of constant stratification and for the nonuniform buoyancy frequency. In all cases, we found that the modal calculation of the conversion rate yields more accurate results than $C_{NYC}$ (Fig. 3).

Building on this test case, a global calculation of the energy conversion rate into the first 10 vertical modes was carried out. The insufficient resolution of the global topography data in the present study for the deep ocean did not allow us to take higher modes into account. However, on the global scale, the bulk of internal tide energetics is concentrated in modes 1–10. When topography data at a higher resolution become available, the global calculation can be readily extended to higher modes generated by topographic features of a scale below $O(10)$ km, that is, abyssal hills. As shown by Melet et al. (2013), taking into account the abyssal hills can result in a 10% increase in the global energy conversion rate of the internal tide, although on the regional scale the increase of the energy conversion rate can be up to 89%.

Besides the topographic resolution, since the energy conversion rate is quadratic in the barotropic tidal velocity, as evident, for example, from (18), the predicted energy conversion rate is subject to an uncertainty associated with the barotropic tidal velocity product. Because of this quadratic dependency, a very crude estimate of the uncertainty in the energy conversion rate is in the range $(2\beta - 1)\% - (2\beta + 1)\%$ when, for example, there is a globally uniform $\beta$% error in the barotropic tidal velocity and $\beta$ is a numerical value. This source of error is expected to decrease as barotropic tidal models are constantly improving.

The sum of the global energy conversion rate up to mode 10 ($C_{sum}$) for depth greater than 400 m was found to be 0.707 TW, which is in good agreement with the $M_2$ tide dissipation in the deep ocean obtained using inverse methods (Egbert and Ray 2000). Of the total, $0.237$ TW is accounted for by the mode-1 internal tide, which was found to be the most energetic mode on the global and basin scales. The global calculation of the energy conversion rate using $C_{NYC}$ returned the value of $0.820$ TW for the same depth range. Although being the most energetic internal wave mode globally, on a regional scale it is found that other modes than mode 1 can be the most energetic one (Fig. 7).

The latitudinal distributions of $C_{sum}$ and $C_{NYC}$ show that the latter follows closely the former for most latitude bands (Fig. 8). Also, with this distribution, mode 1 stands out as the most energetic internal wave mode.

The vertical distribution of the energy conversion rate indicates that the difference between $C_{NYC}$ and $C_{sum}$ decreases with increasing depth (Fig. 9). In the shallow depth ranges, the energy conversion is largest for mode 1 as compared to modes 2–4. In the case of the deep ocean, the difference between energy conversion rates of these modes is less significant than the shallower depth, and for depths below 4000 m, mode 3 is the most energetic internal tide mode.

The main purpose of considering the generation problem for the case of unidirectional tide flowing over the one-dimensional idealized ridge was to test the accuracy and convergence of the numerical method and to find an optimal cutoff length to truncate the modal Green’s function in the numerical calculation. For each of the three buoyancy profiles considered (Fig. 1), the convergence of the numerical solution $C_n$ to $C_n^{exact}$ as the truncation of the integral in (12) increases was examined. Using the thirteenth and fourteenth zeros of the Bessel function $J_1$ was found to be a good compromise between computational expense and numerical accuracy (e.g., Fig. 11). The final result is the average of those obtained for these two truncation numbers.

The sensitivity of $C_n$ to the topographic scale was also examined. It turned out that if the conversion rate into a given mode $n$ needs to be predicted numerically with a good degree of accuracy, its wavelength should be not shorter than the topography width (Table 1).

The use of (12) to calculate the energy conversion rate gives two advantages over $C_{NYC}$. First, the energy conversion rate into each vertical mode can be obtained using (12) in contrast to $C_{NYC}$. The lowest modes can propagate over a considerable distance, and hence they can contribute to the remote generation and subsequent mixing, while high modes are more likely to dissipate locally, near the generation site. Thus, the energy distribution among the vertical modes needs to be determined prior to investigating the energy pathway for the generation to the dissipation. Second, issues related to WKB approximation on which $C_{NYC}$ is based are resolved by considering the decomposition of internal tide into vertical modes. On the other hand, the Green’s function of the convolution integral decreases much more slowly at infinity when using vertical modes. Because of this, the convolution integral must be calculated over a much larger area, and the numerical convergence is slower than when using the method of Nycander (2005). It is therefore reassuring that the global value differs only by 16% when using the two methods.

We used Melet et al.’s (2013) method to mitigate the issue of most likely overestimating the energy flux by the semianalytical model over the supercritical slopes. Applying this method equally to all the first 10 modes, for
depths greater than 400 m, we found the sum of the global energy conversion rate up to the mode 10 (\(C_{\text{cra}}\)) to be 0.422 TW (Table 2). Regarding modes 1 and 2 as low modes and modes 3–10 as high modes and applying Melet et al.’s (2013) method only to high modes, for depths greater than 400 m, we found the sum of the global energy conversion rate up to the mode 10 (\(C_{\text{crh}}\)) to be 0.619 TW (Table 2). For depths greater than 400 m, the sum of the global energy conversion rate up to mode 10 over the subcritical slopes \(C_{\text{sub}}\) was found to be 0.321 TW (Table 2). The depth distribution of \(C_{\text{cra}}, C_{\text{crh}}, C_{\text{sub}},\) and \(C_{\text{sum}}\) showed that the difference between these four energy conversion rates is significant at shallow depths, and, for depths greater than 3500 m, there is almost no difference between them (Fig. 10). Admittedly, the results of the modal energy conversion rate are subject to high uncertainty at shallow depths where more supercritical slopes exist as compared to the deep ocean.

The semianalytical model used in this study to calculate the modal energy conversion rate is not applicable when the topography is shallow or supercritical. To fully account for these limitations, the mathematical framework given by Griffiths and Grimshaw (2007) can be used to derive a formulation for the modal energy conversion rate, assuming a linear wave field and taking into the account the modal coupling. In any case, the most comprehensive way of calculating the energy conversion rate is to use numerical models based on 3D primitive equations that do not require the assumption of the subcritical slope nor that of the linearity of the wave field (e.g., Zilberman et al. 2009; Niwa and Hibiya 2011). Also, in the 3D numerical simulations, the effect of the propagating internal tides is taken into account, which can have an influence on a distant internal tide generation (e.g., Kelly et al. 2010; Zilberman et al. 2011; Kerry et al. 2013). Nevertheless, these models are computationally intensive, in particular, when they are run over a large domain with fine horizontal grid spacing. In contrast, the semianalytical models can be used to calculate the energy conversion rate over a large domain with a high topography resolution. It is suggested that the semianalytical models could also be used as a complement to forward 3D models to parameterize unresolved high-mode internal tide generation.
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APPENDIX

Sensitivity of the Energy Conversion Rate to the Truncation Number

When evaluating $C_n$ in (12) numerically, $J_1$ has to be truncated at an appropriate distance, $r_c = |x - x'|$, from the generation site at $x$. We choose $r_c$ so that $J_1(\kappa, r_c) = 0$. The numerical results to be presented in the following are the average of two solutions obtained for two successive zeros of $J_1$.

Here, we examine the sensitivity of $C_n$ to the truncation number for the case of Witch of Agnesi presented in section 3. When doing this, we keep the half-width $\Lambda$ fixed at 5 km, for which only the first four modes carry a significant amount of the energy for each of the three buoyancy frequency profiles shown in Fig. 1. For the chosen values of $\Lambda$ and $N_p$ corresponding to each of three buoyancy frequency profiles, $\epsilon_{\text{max}}$ and $U_0/\Delta\omega$ are small. For instance, for the case of constant stratification, $\epsilon_{\text{max}} = 0.089$ and $U_0/\Delta\omega = 0.057$. Hence, the use of (12) for calculating $C_n$ is justified.

We consider the first 20 zeros of the Bessel function with $\kappa r_1$ in the range from 3.83 to 63.61. For the case of constant stratification, Fig. 11 shows the values of the energy conversion rate $C_n$ and their corresponding exact solutions $C_n^{\text{exact}}$ versus the number of zeros for the first four modes. As noted above, $C_n$ is the average of the results obtained from two successive zero numbers. For instance, the notation two on the horizontal axis of Fig. 11 means that $C_n$ is the average of those obtained for the first and the second zeros of $J_1$. As evident, the departure of $C_n$ from $C_n^{\text{exact}}$ diminishes as the zero number increases. The same sensitivity test was carried out for $\Lambda = 10$ km and for $\Lambda = 2.5$ km. In the case of $\Lambda = 10$ km, a similar convergence was obtained for the modes 1 and 2. In contrast, for the mode 3, the convergence rate was so slow that the numerical solution was accurate only to 5% for the twentieth zero. The reason for this is the smallness of the wavelength of modes 3 and 4 as compared to the topographic scale 2A. This will be further discussed below. In the case of $\Lambda = 2.5$ km, for all four modes, the difference between $C_n$ and $C_n^{\text{exact}}$ becomes small with an increasing truncation number.

For the case of SSA, the solutions converge to the exact solution for zero numbers greater than 10. The sensitivity test also was done for $\Lambda = 2.5$ km and $\Lambda = 10$ km. The results (not shown) for the first four modes revealed that thirteenth and fourteenth zeros are sufficient.

For the case of SNA, for all the modes considered, the numerical solutions converge to the exact ones as the number of zeros increases. For the widths $\Lambda = 2.5$ km and $\Lambda = 10$ km, the numerical solutions for modes 1–4 approached the exact solutions as the zero number increased.

Notice that the convergence rate for both cases SSA and SNA is faster than for the case of constant stratification. This is because the low-mode wavelength in the case of the vertically nonuniform stratification is larger than the constant counterpart, since in the real ocean the strong stratification in the thermocline exerts a profound influence on the wavelength. From (12), it follows that the nonlocality of the internal tide generation problem is proportional to the wavelength. Consequently, with the realistic profile, more topographic points are used in the calculation of $C_n$.

Similar sensitivity tests with the number of zeros of $J_0$ were done using (8) instead of (12). For each of three buoyancy frequency profiles considered, this demonstrated that (12) yields more accurate results and a faster convergence rate to the exact solution than those (not shown) obtained using (8).

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