An Improved Second-Moment Closure Model of Langmuir Turbulence

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ABSTRACT

A prior second-moment closure (SMC) model of Langmuir turbulence in the upper ocean is modified by introduction of inhomogeneous pressure–strain rate and pressure–scalar gradient closures that are similar to the high Reynolds number, near-wall treatments for solid wall boundaries. This repairs several near-surface defects in the algebraic Reynolds stress model (ARSM) of the prior SMC by redirecting Craik–Leibovich (CL) vortex force production of turbulent kinetic energy out of the surface-normal vertical component and into a horizontal one, with an associated reduction in near-surface CL production of vertical momentum flux. A surface-proximity function introduces a new closure parameter that is tuned to previous results from large-eddy simulations (LES), and a numerical SMC model based on stability functions from the new ARSM produces improved comparisons with mean profiles of momentum and TKE components from steady-state LES results forced by aligned wind and waves. An examination of higher-order quasi-homogeneous closures and a numerical simulation of Langmuir turbulence away from the boundaries both show the near-surface inhomogeneous closure to be both necessary for consistency and preferable for simplicity.

1. Introduction

A significant source of uncertainty in parameterizations of upper-ocean mixing stems from limitations in our ability to describe and predict the role of surface waves in boundary layer turbulence. The now classical generation of upper-ocean mixing parameterizations [e.g., Mellor and Yamada 1982, hereinafter MY2.5; Price et al. 1986; Kantha and Clayson 1994, hereinafter KC94; K-profile parameterization (KPP), Large et al. 1994] uses surface stress, heat, and salinity fluxes and the subsurface profiles of shear \( \partial_z \mathbf{u} \) and stability \( N^2 \) to predict turbulent vertical eddy fluxes. A notable independence from surface wave forcing stems from their introduction to oceanographic applications from rigid-wall atmospheric, engineering, and laboratory boundary layers. Observations (obs) show that open-ocean and coastal free-surface boundary layers differ significantly from wall-bounded layers in several ways attributable to surface waves, including (obs 1) near-surface measurements of turbulent microscale dissipation \( \varepsilon \) are elevated by \( O(10–100) \) near surfaces with breaking waves (e.g., Agrawal et al. 1992; Terray et al. 1996); (obs 2) downwind upper-ocean mean shear \( \partial_z \mathbf{u} \) is much more well-mixed than in smooth, rigid-wall atmospheric or laboratory boundary layer flows in both open-ocean (e.g., Schudlich and Price 1998; Price and Sunderland 1999) and coastal (Gargett et al. 2004) environments; (obs 3) the near-surface crosswind/wave turbulent kinetic energy (TKE) component, normalized by friction velocity, \( (\mathbf{w}_{\text{rms}}/u^*)^2 \) is significantly elevated in the presence of waves over the wall boundary layer scaling (e.g., Smith 1992; Plueddemann et al. 1996; Gargett et al. 2004); (obs 4) the mixed layer vertical TKE scaling \( (w_{\text{rms}}/u^*)^2 \) is similarly elevated, nearly doubling under strong wave forcing (e.g., D’Asaro and Dairiki 1997; D’Asaro 2001; Gargett and Wells 2007); and (obs 5) crosswind length scales of the largest mixed layer turbulent eddies are significantly increased over wall-bounded layers (e.g., Weller et al. 1985; Zedel and Farmer 1991; Thorpe et al. 1994). The latter three features are associated by the field studies (obs 3–5) with the surface streak and downwelling signatures of Langmuir circulations (Langmuir 1938), upper-ocean structures composed of counterrotating surface vortex pairs below surface convergence zones, aligned to the wind and waves. Interactions between the surface waves’ Stokes drift \( \mathbf{u}^S \) and the wave phase–averaged Eulerian momentum \( \mathbf{u} \) was identified by Craik and Leibovich (1976) as responsible for generating Langmuir structures and approximated...
as a “vortex force” $\mathbf{u}^5 \times (\nabla \times \mathbf{u})$ acting on the Eulerian current. Recent advances in predicting the structure and scaling of upper-ocean mixing have benefited from large-eddy simulation (LES) techniques to study wind- and wave-forced Langmuir turbulence—the less coherent mix of Langmuir circulations and turbulence (Skillicorn and Denbo 1995; McWilliams et al. 1997; Li et al. 2005; Harcourt and D’Asaro 2008, hereinafter HD08; Grant and Belcher 2009). With wind and waves aligned, Langmuir structures develop in LES-resolved turbulent momentum fluctuations $\mathbf{u}$ when model momentum equations include the Craik–Leibovich (CL) vortex force. LES model-to-data comparisons have provided further quantitative validations (HD08; D’Asaro et al. 2014; Kukulka et al. 2009; Tejada-Martinez and Grosch 2007) that the CL vortex force explains well-mixed mean momentum profiles (obs 2), increased anisotropy of TKE into vertical and crosswind components (obs 3–4), and the crosswind length scales of Langmuir structures (obs 5). However, additional CL vortex force TKE production only increases dissipation by $O(2)$ in the above LES studies or in second-moment closure (SMC) models (e.g., Kantha and Clayson 2004) and does not account for the observed near-surface increase in $\varepsilon$ (obs 1), attributed instead to wave breaking or other wave dissipation processes.

The combination of wave breaking and Langmuir turbulence has also been studied using LES techniques, combining CL vortex force with TKE injection into sundry combinations of the resolved and unresolved (subgrid) model scales (Noh et al. 2004, Sullivan et al. 2004, 2007; McWilliams et al. 2012), all to various effects depending on the relative scales of TKE injection by breaker forcing. The partition into resolved [typically $> O(1)$ m] and unresolved components, and the distribution of resolved forcing scales, can have large impacts on mixed layer turbulent dynamics. Sullivan et al. (2007) shows that for young seas CL breaker–Stokes drift interactions can strongly impact mixed layer entrainment rates and increase the skewness of $w$. Formulations for mature seas or deep mixed layers may have different nondimensional forcing scales, and for one such case, McWilliams et al. (2012) report breaker effects to be secondary to CL forcing and mostly localized within $O(1)$ wave height of the surface.

As turbulence-resolving LES methods are prohibitively expensive or slow for predicting larger-scale oceanographic flows, efforts to modify models for wave effects have also included those that instead parameterize mixing using vertical diffusivity $K_{\theta}$ and viscosity $K_{H}$ to predict vertical fluxes of momentum and temperature $\theta$, that is,

$$
\mathbf{u}'\mathbf{w}' = -K_M \partial_z \mathbf{u}', \quad \mathbf{v}'\mathbf{w}' = -K_M \partial_z \mathbf{v}', \quad \theta'\mathbf{w}' = -K_H \partial_z \theta'.
$$

Two-equation $q^2 - q^2 l$ SMC models (MY2.5 and its KC94 variant) combine prognostic equations for $q^2 = 2k = u'w'$ and $q^2 l$ (for dissipation length $l \approx q^2/\varepsilon$) with “stability functions” $S_M$, $S_H$ from an algebraic Reynolds stress model (ARSM) to determine second moments and

$$
K_M = S_M q l, \quad K_H = S_H q l.
$$

As the dissipation of surface wave energy typically exceeds other upper-ocean TKE sources, initial modifications of SMC models after Craig and Banner (1994) and Craig (1996) addressed obs 1, representing energy lost from breaking waves as a surface TKE flux proportional to $u^3$ and an associated boundary condition on $l$ in $q^2 - q^2 l$ models (Mellor and Blumberg 2004; Kantha and Clayson 2004, hereinafter KC04). These and similar modifications implemented in a broader class of SMC models obtain improvements in near-surface homogeneity (i.e., obs 2), but restricted to a thin near-surface layer. Qiao et al. (2004), Babanin (2006), Qiao et al. (2010), and Huang and Qiao (2010) argue that additional nonbreaking wave dissipation processes driven by orbital wave motions redistribute the energy from wave attenuation into the boundary layer interior and propose modifications of SMC models or of KPP with additional interior TKE source terms or supplementary $K_M$, $K_H$ profiles. Using rapid distortion methods, Teixeira and Belcher (2002) describe pathways for wave attenuation that include a down-Stokes-gradient turbulent covariance stress. The subsequent study of Teixeira (2012) is similar to the analysis of Babanin in suggesting nonbreaking wave attenuation could account for much of observed near-surface dissipation.

Separately, several Langmuir turbulence modifications of KPP have followed LES-motivated suggestions (McWilliams and Sullivan 2000; Smyth et al. 2002) to adjust the empirical constants governing parameterized profiles of $K_H$, $K_M$ and to introduce a momentum flux component directed down the Stokes gradient $\partial_z \mathbf{w}'$. McWilliams et al. (2012) further examines this idea to account for LES Langmuir turbulence results, with and without wave-breaking contributions, by taking the momentum flux down the gradient of the mean Lagrangian momentum $\mathbf{u}' = \mathbf{u} + \mathbf{u}'$, that is, $\mathbf{u}'\mathbf{w}' = -K_M (\partial_z \mathbf{u}' + \partial_z \mathbf{u})$. A down-Stokes-gradient momentum flux component is effectively similar to the earlier conjecture of Jenkins (1986, 1987) that the transfer of momentum out of the wave field $\mathbf{u}'$ and into the Eulerian current $\mathbf{u}$ should be distributed with depth over the profile of $\mathbf{u}'$. Empirical support for down-Stokes-gradient flux lies in the small
levels of downwind upper-ocean shear (obs 2), coupled with LES results demonstrating that the inclusion of CL vortex forcing also yields improved agreement with TKE component asymmetries (obs 3–4), mixed layer structures with large crosswind length scales (obs 5), and downwind shear profiles more well mixed than in wall-bounded layers. Interior downwind LES mean shear profiles are often slightly retrograde (i.e., $\partial_x u$ has a sign of $\bar{u}(w^2)$), a feature problematic in mixing parameterizations using conventional down-Eulerian-gradient momentum flux closure.

SMC modifications for Langmuir turbulence in D’Alessio et al. (1998) and KC04 included the additional CL vortex force TKE production $P^s = -(\bar{u} w \partial_x u^S + \bar{v} w \partial_x v^S) + \kappa \nabla^2 \theta$ in the TKE and $q^2 l$ equations. These studies found impacts on mixed layer homogeneity and deepening, both from the internal TKE source $P^s$ alone and in conjunction with a surface TKE flux from wave breaking. However, these studies used the conventional down-Eulerian-gradient momentum flux closure [Eq. (1)] and stability functions from the MY2.5 or KC94 ARSM, without CL vortex forcing. As SMC models are built upon an ARSM consisting of a set of linear equations without CL vortex forcing. As SMC models are built upon an ARSM consisting of a set of linear equations that specifies all of the second moments $\bar{u}^2 \bar{v}_y$, $\bar{u} \bar{v}_y \partial_x \theta$, and $\partial_x \theta \partial_y \theta$ and that is solved to determine $\bar{u} w$, $\bar{v} w$, $\theta^S$ from $q^2$ and $l$ via $S_M$ and $S_H$, stability functions that encapsulate the effect of Reynolds stress anisotropy $(\bar{u}^2 \bar{v}_y^2/q^2 - \delta_{ij} l^2/3)$ on $K_M$, $K_H$; for example, since in the ARSM $w \theta^S \sim l q^{-1} \bar{w} \bar{w} \partial_x \theta$, then $S_H \sim \bar{w} \bar{w} / q^2$. SMC changes that affect only $q^2$ and $l$ but not $S_M$, $S_H$ therefore apply wall boundary predictions of Reynolds stress anisotropies to wave boundary layers, where these clearly differ (i.e., obs 3–4). Further adjusting SMC models as in Paskyabi et al. (2012) by adding the Jenkins (1986, 1987) surface momentum flux redistribution improves predictions of mean shear profiles, but $K_M$, $K_H$ are still too small if expressions for the stability functions are based on an ARSM with lower predictions for $\bar{w} \bar{w} / q^2$.

Harcourt (2013, hereinafter H13) rederives the Reynolds transport equations for $\bar{u} \bar{u}_i$ and $\bar{u} \bar{v}_j$ [Eqs. (5)–(6) in H13] starting from the Boussinesq Navier–Stokes equations that include the CL vortex force $\bar{u} \times (\bar{v} \times u) = e_{ijk} e_{lmn} u^S_i \partial_m u_n$ in both the momentum equation

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial x_j} - g \partial_x \theta - e_{ijk} f_k (u_t + u^S_t) + e_{ijk} e_{lmn} u^S_i \partial_m u_n + \nu \nabla^2 u_i, \hspace{1cm} (3)$$

where $g_j = (0, 0, -g)$ is gravitational acceleration, $f_k$ are Coriolis coefficients, and $\nu$ is viscosity, and in the governing equations for thermodynamically active scalars such as potential temperature, with buoyancy $b = \gamma g \theta$, expansion coefficient $\alpha$ and diffusivity $\kappa$ contain additional Stokes drift advection:

$$\frac{D\theta}{Dt} = -\frac{\partial u_i^S \theta}{\partial x_k} + \kappa \nabla^2 \theta. \hspace{1cm} (4)$$

The interaction with surface waves in Eq. (3) includes the Stokes–Coriolis force $-e_{ijk} f_k u^S_j$ (Ursell and Deacon 1950; Holm 1996; McWilliams et al. 1997), and incompressibility $\partial_t \rho / \partial x_i = \partial_x u_i / \partial x_i = 0$ entails additional Bernoulli terms stemming from the CL vortex force that are incorporated into $P^s = \rho + (|\bar{u} + \bar{w}|^2 - |\bar{u}|^2)/2 = \rho + u_t^S (u_t + u_i^S)/2$ to modify nonhydrostatic pressure $P$, which is scaled to $P = \rho l / 9$ on reference density $\rho_0$.

The reader is directed to Eqs. (7)–(17) of H13 for details of the algebraic closure for Langmuir turbulence, except to note here by way of erratum that the statement of the governing equations for thermodynamically active scalars is viscosity, and in the governing equations for thermodynamically active scalars such as potential temperature, with buoyancy $b = \alpha g \theta$, expansion coefficient $\alpha$ and diffusivity $\kappa$ contain additional Stokes drift advection:
The SMC column model implemented in H13 can be tuned by dimensional Stokes forcing functions that, through the stability functions, depend upon nondimensional covariances decomposed into slow and rapid components, the pressure–strain rate and pressure–scalar gradients. Following standard treatments, the pressure–scalar correlations modeled in KC94 and extended in H13, introducing a Stokes drift strain term and a new constant $C_1^S$: 

$$
\Pi_{ij}^{(1)} = -\frac{q}{3A_1^2} \left( \overline{u'w'} - \delta_{ij} \frac{q^2}{3} \right), \quad \Pi_{ij}^{(1)} = -\frac{q}{3A_2^2} \overline{u'w'} ,
$$

are closed in H13 by using the same return-to-isotropy formulation of Rotta (1951) and the scalar variance term of Moeng and Wyngaard (1986) as in KC94 and KC04, introducing model coefficients $A_1$, $A_2$. For the rapid components, the Eulerian form was generalized to introduce a Stokes drift strain term and a new constant $C_2^S$:

$$
\Pi_{ij}^{(2)} = C_1 q^2 \left( \frac{\partial \overline{u}}{\partial x_j} + \frac{\partial \overline{u}}{\partial x_i} \right) + C_2 q^2 \left( \frac{\partial \overline{S}}{\partial x_j} + \frac{\partial \overline{S}}{\partial x_i} \right),
$$

The pressure–scalar correlations modeled in KC94 and KC04 after Moeng and Wyngaard (1986) were similarly extended in H13, introducing $C_3^S$:

$$
\Pi_{ij}^{(2)} = C_3 \alpha \overline{g' \theta' \theta'} + C_3 \overline{u'k'} \overline{\theta' \theta'} + C_3 \overline{u'k'} \overline{\theta' \theta'},
$$

As a starting point, for purposes of modification, the linearized H13 ARSM and flux equations are reiterated here, retaining the slow terms and the rapid Eulerian strain terms of [i.e., Eqs. (15)–(16) with $C_1^S = C_2^S = 0$]

$$
\Pi_{ij}^{(2)} = C_1 q^2 \left( \frac{\partial \overline{u}}{\partial x_j} + \frac{\partial \overline{u}}{\partial x_i} \right) + \Pi_{ij}^{(LC)} \quad (17a)
$$

$$
\Pi_{ij}^{(2)} = C_2 \overline{u'k'} \overline{\theta' \theta'} + C_3 \alpha \overline{g' \theta' \theta'} + \Pi_{ij}^{(LC)}, \quad (17b)
$$

while leaving the rapid correlation terms $\Pi_{ij}^{(LC)}$ and $\Pi_{ij}^{(LC)}$ associated with Langmuir circulations and Stokes shear to be specified. With this modification in place, the ARSM template for study is

$$
\overline{u'^2} = \frac{q^2}{3} \left( 1 - \frac{6A_1}{B_1} \right) - 6A_1 lq^{-1} \left[ \overline{u'w'} - \frac{\Pi_{ij}^{(LC)}}{2} \right],
$$

$$
\overline{v'^2} = \frac{q^2}{3} \left( 1 - \frac{6A_1}{B_1} \right) - 6A_1 lq^{-1} \left[ \overline{v'w'} - \frac{\Pi_{ij}^{(LC)}}{2} \right].
$$
Inhomogeneous closures developed for near-wall treatments depend upon the orientation and distance to the nearest boundary (e.g., Pope 2000, section 11.7). While they have been generally avoided (e.g., Cheng et al. 2002) or little used for atmospheric and oceanic boundary layer turbulence in favor of higher-order quasi-homogeneous closures, adoption in SMC models outside these fields, as reviewed in Speziale (1991), Hanjalic (1994), and Jaw and Chen (1998), has grown steadily since the 1980s to a strong current level of interest and application (e.g., Gerolymos et al. 2012). Several inhomogeneous closures represent near-wall pressure–strain correlations using variations on an expression of Daly and Harlow (1970) that is traditionally written

\[
\Pi_{ij}^{(W)} \approx -\frac{C_1}{k} \left[ u_n u_m n_m n_i \delta_{ij} - \frac{3}{2} (w \omega_k^i n_k n_j + \omega \omega_k^i n_k n_j) f(z),
\right]
\]

where \( \mathbf{n} = (n_l) \) is a unit vector normal to the (nonlocal) boundary. Anisotropy is introduced by both the wall-proximity function \( f(z) \propto -1/lz \) and the surface-normal direction, \( \mathbf{n} = -\mathbf{x}_z \) and \( n_m = -\delta_{m3} \) for the phase-averaged ocean surface boundary. This form has been used to model both low Reynolds number wall boundaries where turbulent momentum fluxes transition to molecular ones toward a wall (Shima 1988; Lai and So 1990) and high Reynolds number flows where the wall-normal TKE component is limited by proximity to the surface, using Eq. (19) coupled with an analogous term where the traceless shear production of \( \tilde{\omega}^i \tilde{\omega}^j \) anisotropy substitutes for \( u_n u_m n_m n_i \) (Lauder and Shima 1989). Stated in Eq. (19) as a slow pressure–strain term (i.e., independent of mean gradients), it represents the “echo” effect of the wall on pressure fluctuations, inhibiting the rotation of either horizontal TKE component into the wall-normal one, and damping the associated turbulent momentum fluxes in the wall-normal direction. This near-wall case is different to LC where the near-surface pressure–strain correlations “block” the near-surface vertical TKE production along the downwind jet and redirect it anisotropically into a single transverse horizontal component as fluid is pulled into the counterrotating vortices.

To recast the near-wall expression of Eq. (19) for LC, it is applied to only block and redirect the CL vortex force production by substituting

\[
P_{ij}^S = \tilde{u}_m \tilde{\omega}_k \frac{\partial u_i^S}{\partial x_k} + \tilde{u}_m \omega_k \frac{\partial u_i^S}{\partial x_k}
\]

in place of \( u_n u_m n_m n_i \) in \( \Pi_{ij}^{(W)} \). The CL vortex force vertical TKE production

\[
P_{ij}^S \frac{1}{2} P_{ij}^S = \frac{1}{2} P_{ij}^S \left( \tilde{u}_m \tilde{\omega}_k \frac{\partial u_i^S}{\partial x_k} + \tilde{u}_m \omega_k \frac{\partial u_i^S}{\partial x_k} \right)
\]

2. Near-surface closure for Langmuir turbulence in quasi-equilibrium ARSM

Section 2 introduces inhomogeneous near-surface pressure–strain \( \Pi_{ij}^{(LC)} \) and pressure–scalar \( \Pi_{ij}^{(LC)} \) closures that better represent their anisotropic effects in Langmuir circulations and tunes them for consistency between the ARSM and LES predictions. In section 3, a model using this closure is compared with LES results. Sections 4 and 5 were added after the initial submission. Section 4 examines alternatives among more complex quasi-homogeneous closure forms, and section 5 presents a simulation of Langmuir turbulence away from boundaries to demonstrate the necessity of the near-surface inhomogeneous closure.
is then redirected into just one horizontal TKE component along \( \mathbf{\hat{s}} = \mathbf{\hat{s}} \times \mathbf{x}_3 \), perpendicular to \( \mathbf{s} = \mathbf{\hat{x}}_1 \sin \phi + \mathbf{x}_2 \cos \phi \), as illustrated in Fig. 1. This was arrived at from Eq. (19) by first redundantly substituting \( \mathbf{\hat{x}}_i \cdot \mathbf{\hat{x}}_i / 2 \) for the \( \frac{3}{2} \) factor on the second and third right terms, rewriting \( \delta_{ij} = \mathbf{\hat{x}}_i \cdot \mathbf{\hat{x}}_j \) in the first term, and then replacing \( \mathbf{\hat{x}}_i \to \mathbf{\hat{s}} \times \mathbf{\hat{x}}_i \) (22) for each unit directional vector \( \mathbf{\hat{x}}_i \) and using vector identities to obtain

\[
\Pi_j^{(LCW)}(\mathbf{\hat{s}}) = \left( P_{33}^S \delta_{ij} - (\mathbf{\hat{s}} \cdot \mathbf{\hat{x}}_j)(\mathbf{\hat{s}} \cdot \mathbf{\hat{x}}_i) \right) - P_{13}^S \delta_{ij} - P_{13}^S \delta_{ij} \tanh(0.25 \zeta^l) \\
= \begin{cases} 
  P_{33}^S \cos^2 \phi & -P_{33}^S \sin \phi \cos \phi & -P_{13}^S \\
  -P_{33}^S \sin \phi \cos \phi & P_{33}^S \sin^2 \phi & -P_{23}^S \\
  -P_{13}^S & -P_{23}^S & -P_{33}^S 
\end{cases} f_z^S. 
\] (23)

Because Eq. (22) systematically replaces one “objective” vector with another, the invariance of the Eq. (19) tensor to rotation about the \( z \) axis is preserved in Eq. (23). In addition to the horizontal anisotropy of redirected TKE production, several other features of Eq. (23) distinguish it from the near-wall form of Eq. (19). Redirection of \( P_{33}^S \) into a horizontal component intermediate to \( \mathbf{\hat{x}}_1 \) and \( \mathbf{\hat{x}}_2 \) generates the nonzero \( \Pi_1^{(LCW)} \) contribution evident in LES results (Figs. 15g–i of H13). Also, reductions in \( P_{13}^S \) and \( P_{23}^S \) are commensurate with those in \( P_{33}^S \), while for Eq. (19) the [1, 3] and [2, 3] components are only reduced by \( \frac{3}{2} \) when vertical TKE, or its production, is fully blocked. For the \( \mathbf{\hat{x}}_2 \)-aligned wind-wave forcing cases of H13 and HD08 LES solutions, a small deflection of \( \mathbf{\hat{s}} \) from the direction \( \phi^S = 0 \) of the Stokes shear is entirely due to Coriolis effects. When Stokes shear is not aligned with surface stress, variations in the direction \( \phi^S \) are generally expected to follow a near-surface average of \( \phi^S \), adjusted for the Coriolis effects, as discussed further below.

The surface-proximity function \( f_z^S \) decreases monotonically with depth, but apparently differs from the \( f(z) \propto -i/z \) form used for wall-bounded layers with Eq. (19). Equation (23) does offer an intuitively appealing representation of LC physics addressing several of the H13 shortcomings as it transfers near-surface TKE production from vertical to transverse components, while curtailing the production of downwind covariance \( \mathbf{\hat{u}} \cdot \mathbf{\hat{w}} \). However, substitution of this term alone for \( \Pi_j^{(LC)} \) does not address the H13 defect in cross-stream \( \mathbf{\hat{u}} \cdot \mathbf{\hat{w}} \) generated by the \( \mathbf{\hat{u}} \cdot \partial \mathbf{\hat{u}} \cdot \mathbf{\hat{w}} \) source term in Eq. (18e). More problematic is that on its own also requires a momentum closure assumption with slightly different vertical eddy diffusivities for \( \mathbf{\hat{u}} \cdot \mathbf{\hat{w}} \) and \( \mathbf{\hat{u}} \cdot \mathbf{\hat{w}} \) to solve the ARSM, as demonstrated below. To address both of these issues an additional “balancing” term \( \Pi_{ij}^{(LCB)} \) is added to the LC rapid pressure–strain closure

\[
\Pi_{ij}^{(LC)} = \Pi_{ij}^{(LCW)} + \Pi_{ij}^{(LCB)}, 
\] (24)
as further identified below in the course of solving the ARSM.

In the pressure–buoyancy closure, the Stokes drift terms in Eqs. (16) and (17b) are modified so that the vertical flux production by the CL vortex force is damped near the surface, in proportion to the degree Eq. (23) opposes near-surface CL momentum flux production:

\[
\Pi_{ij}^{(LC)} = -f_z^S P_{ij}^S = -\frac{f_z^S (u^S_i \partial_z u^S_j + \bar{w}^S_i \partial_z \bar{w}^S_j)}{2}.
\]  

(25)

Leaving \( \Pi_{ij}^{(LCB)} \) to be determined, substitution of these near-wall rapid pressure covariance closures \( \Pi_{ij}^{(LC)} \), \( \Pi_{ij}^{(LCB)} \) into Eq. (18) gives a new ARSM using the quasi-equilibrium approximation of Galperin et al. (1988):

\[
\begin{align*}
\bar{u}^2 &= \frac{q^2}{3} \left( 1 - \frac{6A_1}{B_1} \right) - 6A_1 lq^{-1} \left[ \bar{u} \bar{w} \partial_z \bar{n} + f_z^S \cos^2 \phi (\bar{u} \bar{w} \partial_z \bar{u}^S + \bar{w} \bar{w} \partial_z \bar{v}^S) - \frac{\Pi_{ij}^{(LCB)}}{2} \right], \\
\bar{v}^2 &= \frac{q^2}{3} \left( 1 - \frac{6A_1}{B_1} \right) - 6A_1 lq^{-1} \left[ \bar{v} \bar{w} \partial_z \bar{v} + f_z^S \sin^2 \phi (\bar{u} \bar{w} \partial_z \bar{u}^S + \bar{w} \bar{w} \partial_z \bar{v}^S) - \frac{\Pi_{ij}^{(LCB)}}{2} \right], \\
\bar{w}^2 &= \frac{q^2}{3} \left( 1 - \frac{6A_1}{B_1} \right) - 6A_1 lq^{-1} \left[ -\alpha g \bar{w} \bar{\theta} + (1 - f_z^S) (\bar{u} \bar{w} \partial_z \bar{u}^S + \bar{w} \bar{w} \partial_z \bar{v}^S) - \frac{\Pi_{ij}^{(LCB)}}{2} \right], \\
\bar{u} \bar{w} &= -3A_1 lq^{-1} \left[ (\bar{w}^2 - C_1 q^2) \partial_z \bar{u} - \alpha g \bar{w} \bar{\theta} + (1 - f_z^S) (\bar{u} \bar{w} \partial_z \bar{u}^S + \bar{w} \bar{w} \partial_z \bar{v}^S) - \Pi_{ij}^{(LCB)} \right], \\
\bar{v} \bar{w} &= -3A_1 lq^{-1} \left[ (\bar{w}^2 - C_1 q^2) \partial_z \bar{v} - \alpha g \bar{w} \bar{\theta} + (1 - f_z^S) (\bar{u} \bar{w} \partial_z \bar{u}^S + \bar{w} \bar{w} \partial_z \bar{v}^S) - \Pi_{ij}^{(LCB)} \right], \\
\bar{u} \bar{\theta} &= -3A_1 lq^{-1} \left[ (\bar{w}^2 \partial_z \bar{v} + \bar{v} \bar{w} \partial_z \bar{u}) - 2f_z^S \cos \phi \cos \phi (\bar{u} \bar{w} \partial_z \bar{u}^S + \bar{w} \bar{w} \partial_z \bar{v}^S) - \Pi_{ij}^{(LCB)} \right], \\
\bar{v} \bar{\theta} &= -3A_1 lq^{-1} \left[ (\bar{w}^2 \partial_z \bar{v} + \bar{v} \bar{w} \partial_z \bar{u}) - 2f_z^S \cos \phi \cos \phi (\bar{u} \bar{w} \partial_z \bar{u}^S + \bar{w} \bar{w} \partial_z \bar{v}^S) - \Pi_{ij}^{(LCB)} \right], \\
\bar{w} \bar{\theta} &= -3A_1 lq^{-1} \left[ (\bar{w}^2 \partial_z \bar{v} - (1 - C_3) \alpha g \bar{w} \bar{\theta} + (1 - f_z^S) (\bar{u} \bar{w} \partial_z \bar{u}^S + \bar{w} \bar{w} \partial_z \bar{v}^S) \right], \quad \text{and} \\
\bar{\theta}^2 &= -B_2 lq^{-1} \bar{w} \bar{w} \partial_z \bar{\theta}.
\end{align*}

The first step in solving the ARSM of Eqs. (26a)–(26j) for the covariance \( \bar{u} \bar{\theta} \) into the expressions for vertical momentum and scalar fluxes:

\[
\begin{align*}
\bar{u} \bar{w} &= -3A_1 lq^{-1} \left[ \left( \frac{q^2}{3} \left( 1 - \frac{6A_1}{B_1} - 3C_1 \right) + 6A_1 lq^{-1} \left( \alpha g \bar{w} \bar{\theta} + (1 - f_z^S) P^S - \frac{\Pi_{ij}^{(LCB)}}{2} \right) \right) \partial_z \bar{u} + 3\alpha g A_2 lq^{-1} \left( \bar{u} \bar{w} \partial_z \bar{\theta} + \bar{v} \bar{w} \partial_z \bar{u} \right) \right. \\
&\quad + (1 - C_3) \bar{w} \bar{\theta} \partial_z \bar{\theta} - \Pi_{ij}^{(LCB)} + (1 - f_z^S) \left[ \left( \frac{q^2}{3} \left( 1 - \frac{6A_1}{B_1} \right) - 6A_1 lq^{-1} \left( \bar{u} \bar{w} \partial_z \bar{u} - f_z^S \cos^2 \phi P^S - \frac{\Pi_{ij}^{(LCB)}}{2} \right) \right) \partial_z \bar{u} \\
&\quad - 3A_1 lq^{-1} \left( \bar{w} \bar{w} \partial_z \bar{v} + \bar{v} \bar{w} \partial_z \bar{u} \right) + 2f_z^S \cos \phi \cos \phi P^S - \Pi_{ij}^{(LCB)} \right] \}, \\
\bar{v} \bar{w} &= -3A_1 lq^{-1} \left[ \bar{u} \bar{w} \partial_z \bar{u} + \bar{v} \bar{w} \partial_z \bar{v} \right], \quad \text{and} \\
\bar{w} \bar{\theta} &= -3A_1 lq^{-1} \left( \bar{w} \bar{w} \partial_z \bar{v} - (1 - C_3) \alpha g \bar{w} \bar{\theta} + (1 - f_z^S) \bar{u} \bar{w} \partial_z \bar{u} \right), \\
\bar{\theta}^2 &= -B_2 lq^{-1} \bar{w} \bar{w} \partial_z \bar{\theta}.
\end{align*}
\]

(27a)
\[ v'w' = -3A_1 q^{-1} \left( \frac{q^2}{3} \left( 1 - \frac{6A_1}{B_1} - 3C_1 \right) + 6A_1 q^{-1} \left( \alpha g w' \bar{\theta} + (1 - f_2^S) \lambda S - \frac{\Pi_{33}^{(LCB)}}{2} \right) \right) \partial_z \bar{\nu} + 3 \alpha a A_1 q^{-1} \left[ v' w' \partial_z \bar{\theta} \right] \]

\[ + (1 - C_2) w' \theta \partial_z \bar{w} \Pi_{23}^{(LCB)} + (1 - f_2^S) \left\{ \left[ \frac{q^2}{3} \left( 1 - \frac{6A_1}{B_1} \right) - 6A_1 q^{-1} \left( w' \partial_z \bar{w} - f_2^S \sin^2 \phi \lambda S - \frac{\Pi_{22}^{(LCB)}}{2} \right) \right] \partial_z u^S \right\} \]

\[ - 3A_1 q^{-1} \left[ (w' w' \partial_z \bar{w} + w' \theta \partial_z \bar{w}) [2 f_2^S \cos \phi \sin \phi \lambda S - \Pi_{12}^{(LCB)}] \partial_z u^S \right] \}

and

\[ \Pi_{ij}^{(LCB)} = \frac{6A_1 f}{q} p f^S (1 - f_2^S) \left[ \begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \partial_z u^S \cos^2 \phi & \partial_z u^S \sin^2 \phi \\ - \partial_z u^S \sin \phi \cos \phi & - \partial_z u^S \sin \phi \cos \phi \end{array} \right] \].

Without further adjustment from \( \Pi_{ij}^{(LCB)} \), closing the ARSM with different dependencies on \( \phi \)—between the \( \cos^2 \phi \) coefficient of \( \partial_z u^S \) in Eq. (27a) and the \( \sin^2 \phi \) coefficient of \( \partial_z u^S \) in Eq. (27b)—would require the Stokes eddy coefficient \( K_M^S \) to be different for the two components of vertical momentum flux. Furthermore, the \( \cos \phi \sin \phi \) terms passing into these flux expressions from \( \Pi_{12}^{(LCB)} \) modifications of \( w'u' \) (correcting error shown in Figs. 14g of H13) are largely responsible for the erroneous ARSM predictions of near-surface \( w'u' \) (illustrated in Figs. 14a-c of H13). To address these issues, the balancing component of the rapid pressure–strain closure \( \Pi_{ij}^{(LCB)} \) is set to cancel both of these terms by taking

Note that if \( f = 0 \) and \( \partial_z u^S \) is uniformly oriented along \( \phi = \phi^S \), then \( \Pi_{ij}^{(LCB)} = 0 \).

While \( \Pi_{ij}^{(LCB)} \) preserves the critical features of \( \Pi_{ij}^{(LC)} \), addresses the near-surface H13 defect in \( w'u' \), and averts the introduction of second rank tensor viscosities and diffusivities of uncertain rotational invariance, the physical meaning of the higher-order (in \( l/q \)) cancellation from this closure-balancing term is admittedly obscure. It enforces a “sterilization” of sorts in the damping of vertical momentum flux by \( \Pi_{ij}^{(LCB)} \), preventing TKE production redirected from the vertical to horizontal components from in turn generating vertical momentum fluxes via their incompletely damped, \( (1 - f_2^S) \), CL production source terms in Eqs. (26d)–(26e).

Cancellations from \( \Pi_{12}^{(LCB)} \) eliminate dependence on \( \phi \) from the vertical eddy coefficients required by Eq. (27) and greatly simplify the stability functions, as the net effect of \( \Pi_{ij}^{(LC)} \), \( \Pi_{ij}^{(LCB)} \) can now be determined from the solution in Eqs. (25)–(27) of H13 by setting \( C_1^S = C_2^S = 0 \) and absorbing a factor of \( (1 - f_2^S) \) into the Stokes shear.

Substituting

\[ w'u' = - (K_M^S \partial_z \bar{\pi} + K_M^S \partial_z u^S) \]

\[ = - lq [S_M^\partial \partial_z \bar{\pi} + S_M(1 - f_2^S) \partial_z u^S], \]
FIG. 2. Evaluation of the equilibrium model from LES results for the three example forcing cases as in Fig. 13 of H13. Using steady-state LES profiles of Reynolds buoyancy flux \( \langle \overline{w} \overline{B} \rangle \) and stress tensor \( \langle \overline{u} \overline{u} \rangle \) and the dissipation length scale \( \lambda_{LES} = q_{LES} / B_1 \rho g_0 \), the equilibrium model predictions of TKE components (i.e., the diagonal Reynolds stress tensor elements) are evaluated using the new ARSM of Eqs. (26a)–(26c) (solid gray), the new ARSM with \( \Pi^{LC}_{ij} = 0 \) (wide gray dashed, equivalent to H13 with \( C_1 = C_2 = 0 \)), and excluding Stokes drift contributions (thin dashed) for self-consistency in the LES solution (dotted). Equations (26a)–(26c) are reiterated below each TKE component’s set of plots, underscored to match the corresponding part of the equation plotted.
\[ \nabla \vec{w} = -(K_M \partial_z \vec{v} + K_S \partial_z \psi) \]
\[ = -lq[S_M \partial_z \vec{v} + \tilde{S}_M (1 - f_z^S) \partial_z \psi], \quad (30) \]
\[ \nabla \vec{w} = -K_H \partial_z \vec{u} = -lq S_H \partial_z \vec{u}, \quad (31) \]
and
\[ \tilde{G}_V = (1 - f_z^S)^2 \hat{p} q^{-2} \partial_z \vec{u} \cdot \partial_z \vec{u}^S; \]
\[ \tilde{G}_S = (1 - f_z^S)^2 \hat{p} q^{-2} [\partial_z \vec{u}]^2, \quad (32) \]
in the solution for stability functions gives
\[
\begin{align*}
\text{den}(S_H) &= [1 - 9A_1(A_2G_H + 4A_1 \tilde{G}_V)] [1 - 3A_2G_H(6A_1 + B_z(1 - C_3))] - 9 \tilde{G}_V(A_2^2(1 - C_2) \\
&+ 18A_1^2(A_2[2A_1 + A_2(2 - C_2)]G_H - 2A_2^2(1 - C_2) \tilde{G}_V)],
\end{align*}
\]
\[
S_M = \frac{A_1(\gamma_1 - 3C_1) + 9 \left[ 2A_1 + A_2(1 - C_2) \right] G_H S_H + 3A_1 \tilde{G}_S S_M^S}{\left[ 1 - 9A_1(A_2G_H + 4A_1 \tilde{G}_V) \right]}, \quad (33c)
\]
where \( \gamma_1 = 1 - 6A_1/B_1 \).

A rough tuning of adjustable parameters in \( \Pi_y^{(LC)} \) is illustrated in Figs. 2–3, which reprise Figs. 13–14 of H13 using the new ARSM, but here the comparison is restricted to resolved variables, excluding subgrid contributions that were estimated in the H13 version. After considering several different expressions, many nearly equivalent, the surface-proximity function was set to
\[ f_z^S = 1 + \tanh(C_z^S \partial_z \vec{u}), \quad (34) \]
with \( C_z^S = 0.25 \), and where the length scale \( l^S \) is an average of the dissipation length scale weighted by positive CL vortex production,
\[ l^S = \langle |l|^S \rangle_{p^S > 0} = \int_{P^S}^{fS} \langle |l|^S \rangle dz \]
\[ = \int_{P^S}^{fS} \langle |l|^S \rangle dz \]
where \( P^S = \max(0, P^S) \). The model prediction of vertical fluxes is independent of the angle \( \phi \) of the horizontal unit vector \( \vec{s} \). Comparisons with the ARSM for the midlatitude LES with aligned wind and wave cases in HD08 found this angle to be much larger than the direction of total Lagrangian shear that is proposed in Van Roekel et al. (2012) to predict the mean direction of LC vortices. Better agreement gauged principally by the \( \vec{u} \vec{w} \) comparisons in Figs. 3g–i was found by taking \( \vec{s} \) to be in the direction \( \phi^S = 0 \) of the Stokes shear, but rotated by the difference \( \phi^S \) between the surface wind stress direction \( \phi^S \) and the \( P^S \)-weighted, resolved subsurface stress:
\[ S_M = S_M^S(1 - f_z^S) = \frac{A_1(1 - f_z^S) \gamma_1}{1 - 9A_1A_2G_H - 9A_1^2G_V}, \quad (33a) \]
\[ S_H = \text{num}(S_H)/\text{den}(S_H), \quad (33b) \]
where
\[ \text{num}(S_H) = A_2 \gamma_1 - 9A_1A_2[2A_2 \gamma_1 G_H - \gamma_1 (2A_1 + A_2) \tilde{G}_S] - [A_2(\gamma_1 - 3C_1) - 2A_1(\gamma_1 + 3C_1)] \tilde{G}_V, \]

Using these choices produced a good overall fit to the anisotropy of the forcing because of the pressure correlations in the ARSM. The agreement shown in Figs. 2–3 is representative of comparisons against a larger set of LES cases, including those presented and examined in HD08 and H13.

There are several remaining ARSM discrepancies and uncertainties in the near-surface closure. Errors in the shear layer adjacent the surface may call for additional inhomogeneous near-wall treatment using the form of Eq. (19) to represent the pressure “echo” effect, but an accurate parameterization of this would require higher LES resolution to reduce dependence on subgrid dynamics. Presumably, \( \phi^S \) in Eq. (36) should also be computed by a \( P^S \)-weighted average of \( \partial_z \vec{u}^S \) when the direction is not uniformly aligned with depth, so as to represent the mean reflection of this forcing direction through the surface pressure anomaly. However, the LES cases examined here and in H13 do not include forcing by waves not aligned to the wind, so the \( \phi - \Delta \phi^S = \phi^S \) implication in Eq. (36), namely, that it does not follow the Langmuir roll orientation approximated as \( \langle \phi^1 + \phi^2 \rangle/2 \) by Van Roekel et al. (2012), should be regarded at this point as conjecture with respect to any \( \phi^S \neq \phi^S \) forcing cases. The suggestion that \( \phi - \Delta \phi^S = \phi^S \) is based on inverting the ARSM for \( \vec{u} \vec{w} \) [Eq. (26)] to get \( \phi \) from a limited set of LES results for modest angles \( |\phi^S - \phi^S| > \pi/6 \) between wind and wave directions,
FIG. 3. Evaluation as in Fig. 2 of the new ARSM from LES results for the three example forcing cases, as in Fig. 14 of H13. Given steady-state LES profiles, the equilibrium model predictions of momentum flux profiles (i.e., the off-diagonal Reynolds stress tensor elements) are evaluated using the new ARSM of Eqs. (26d)–(26f) (solid gray), the new ARSM with $P_{ij}^{(LC)} = 0$ (wide gray dashed, equivalent to H13 with $C_1 = C_3 = 0$), and excluding Stokes drift contributions (thin dashed) for self-consistency in the LES solution (dotted). Equations (26d)–(26f) are reiterated below each TKE component’s set of plots, underscored to match the corresponding part of the equation plotted.
results that cannot be fully accommodated in this presentation. Profiles of $\nu$ from inverting Eqs. (26a)–(26b) for $\overline{u'u'}$ and $\overline{v'v'}$ are much less uniform, suggesting that for modest wind-wave angles, $\phi - \Delta \phi$ may range with depth between $-\phi^L$ and $-(\phi^F + \phi^S)/2$. The incongruity between these divinations of $\phi$ may be consistent with the omission from the ARSM [Eq. (26)] of any wall echo for shear turbulence production, which would inhibit the rotation of horizontal TKE into the vertical component [i.e., the form of Eq. (19)] as increasing the wind-wave angle results in larger near-surface Eulerian shear. The forms of $\Pi_{ij}^{(LC)}$ and $\Pi_{ij}^{(LC)}$ may therefore not be accurate for addressing all the same Reynolds stress anisotropies for misaligned wind and waves, nor, for that matter, have they been adequately evaluated for variations due to surface buoyancy fluxes.

3. SMC model improvements

A SMC model of Langmuir turbulence using the new stability functions [Eq. (33)] has been numerically implemented along the lines of that produced and demonstrated in H13 but with several additional differences. Length scale $q^2l$ equation constants $E_i$ detailed in H13 were adopted, except that the coefficient of CL production was reduced to $E_b = 6.0$, marginally limiting some excessively high midlayer maxima in $l$. The H13 dependence of some constants $E_i$ on the alignment of Eulerian and Stokes shear was dropped [replacing Eq. (32) in H13 by $r_V = r_E = 1$] as the relevance of this adjustment is not yet clear with the new ARSM. Also, a compromise between the KC94 and H13 stability functions was adopted with

$$S_q = S_l = [0.2^2 + (0.41S_H)^2]^{1/2}$$

(37)

to retain the higher entrainment rate stemming from the KC94 constant value at low $S_H$ while enhancing the vertical transports of turbulence when $S_H$ is augmented by CL vortex forcing. These additional SMC model adjustments from H13 are not presented as definitive, but only as sufficient to demonstrate the effects of the new pressure–strain closure in the model predictions of the profiles of downwind Eulerian momentum and vertical TKE.

Figure 4 shows the SMC based on the new ARSM and stability functions [Eq. (33)] replaces near-surface retrograde shear of H13 with a small prograde downwind shear layer in all three example LES cases, and for sufficiently strong CL forcing, the vertical TKE exhibits a subsurface maximum at a depth similar to LES predictions. Profile details in Fig. 4 occasionally correspond very closely, but because of the unaltered slow pressure–strain closure and the isotropic dissipation assumption, vertical TKE does not continue to decrease right next to the surface, limiting the SMC Eulerian shear to levels below LES predictions. The Figs. 2g–i vertical TKE comparison between LES and ARSM predictions at these depths suggests the need for a near-wall term for the slow pressure–strain closure. However, this model–model comparison is ambivalent over these top grid-layer depths, as LES predictions and their interpretation there depend strongly on the LES subgrid model that involves neither a momentum flux down the Stokes gradient nor any implications from the anisotropy of subgrid TKE (i.e., the LES subgrid is consistent with an SMC with constant $S_H$, $S_M$ and with $S_M = 0$). Moreover, including TKE injection from breaking waves might have impacts also comparable to these discrepancies.

Low SMC levels of midlayer vertical TKE in Fig. 4 correspond to an inadequate mixing of momentum below the depths of strong CL production. The quasi-equilibrium assumption of the ARSM, responsible for generating $\overline{u'u'} \neq q^2$ predictions under departures from equilibrium, plus the neglect of third-moment transport divergence in SMCs, may both contribute to this model behavior. Compensating for these limitations by overtuning available constants may result in unintended consequences.

Figures 5 and 6 reprise Figs. 5 and 6 of H13 to demonstrate the effect of changes to the SMC of Langmuir turbulence on bulk layer–averaged turbulence statistics for $l$, $q^2$, and $\overline{w'w'}$ (Fig. 5) and for two metrics of mixed layer entrainment. In general, the effect of the new ARSM on these metrics is small, and the notable but uneven improvement in entrainment predictions (Fig. 6) is due to the lower limit enforced on the stability functions for turbulence transport [Eq. (37)]. While near-surface profiles are clearly improved by the new ARSM, SMC defects in the middepth maximum $l$ values were exacerbated by the new ARSM, and vertical TKE levels were reduced below LES predictions.

4. Higher-order quasi-homogeneous closures and near-surface effects

Quasi-homogeneous pressure–strain and pressure–scalar closures more complex than those [Eq. (15)–(16)] generalized to Stokes strain terms in H13 may offer alternatives to the near-surface closure in sections 2–3. In the higher-order closures, terms in Eqs. (15)–(16) compose only the first in a series expansion of traceless symmetric orthogonal tensors that describe the anisotropy of $\Pi_{ij}^{(LC)}$
and Π[LC]_{ij} as a function of local second moments and mean strain rates (see Pope 2000, section 11.5.2; Speziale 1991; Smith 1971). Adopting the notation of Umlauf and Burchard [2005, see their Eq. (15)] with different constants $c_i$ ($\neq C_i$), the next-order quasi-homogeneous terms are phrased in terms of anisotropy $b_{ij} = (\bar{u}_i \bar{u}_j - \delta_{ij}/3)q^{-2}$, strain $S_{ij} = (\partial_i \bar{u}_i + \partial_j \bar{u}_j)/2$ vorticity $W_{ij} = (\partial_j \bar{u}_i - \partial_i \bar{u}_j)/2$ tensors, and in their various tensor inner products. A naive generalization of these second-order forms to terms where $\partial_i \bar{u}_i$ substitutes for $\partial_i \bar{u}_j$, each introducing new coefficients $c_S$, may be recast into the MY2.5 formalism of Eq. (18) to give $\Pi_{ij}^{(LC)}$ with diagonal elements

$$
\Pi_{ij}^{(LC)} = \begin{bmatrix}
\frac{c_s^3 + 3c_s^5}{6} \bar{u}' \bar{w}' \partial_z \bar{u}' - \frac{c_s^6}{3} \bar{u}' \bar{w}' \partial_z \bar{v}' \\
\frac{c_s^3 + 3c_s^5}{6} \bar{w}' \bar{u}' \partial_z \bar{v}' - \frac{c_s^6}{3} \bar{w}' \bar{u}' \partial_z \bar{u}' \\
\frac{c_s^3 - 3c_s^5}{6} (\bar{w}' \bar{w}' \partial_z \bar{v}' + \bar{u}' \bar{w}' \partial_z \bar{u}') \\
\end{bmatrix}
$$

(38a)

and off diagonal

$$
\begin{bmatrix}
\Pi_{12}^{(LC)} \\
\Pi_{13}^{(LC)} \\
\Pi_{23}^{(LC)}
\end{bmatrix} = \begin{bmatrix}
\frac{c_s^3 + 3c_s^5}{4} \bar{u}' \bar{w}' \partial_z \bar{u}' + \frac{c_s^5 + c_s^6}{4} \bar{u}' \bar{w}' \partial_z \bar{v}' \\
\frac{c_s^3 + 3c_s^5}{4} \bar{w}' \bar{u}' \partial_z \bar{v}' + \frac{c_s^5 + c_s^6}{4} \bar{w}' \bar{u}' \partial_z \bar{u}' \\
\frac{c_s^3 - 3c_s^5}{4} (\bar{w}' \bar{w}' \partial_z \bar{v}' + \bar{u}' \bar{w}' \partial_z \bar{u}') + \frac{c_s^5 + c_s^6}{4} \bar{w}' \bar{w}' \partial_z \bar{v}' + \frac{c_s^5 + c_s^6}{4} \bar{u}' \bar{u}' \partial_z \bar{u}'
\end{bmatrix}
$$

(38b)
Mixed layer turbulence properties from the new SMC with (a), (c), (e) $E_6 = 6$ and from the H13 SMC with (b), (d), (f) $E_6 = 7$ are compared against LES results for forcing case sets identified in HD08 as $S_1$, $S_2$, $S_{3a}$, $S_{3b}$, and $S_4$. Properties compared are (top) the maximum nondimensional dissipation length scale $l_{\text{max}}/H_{\text{ML}}$, where the LES estimates from mean profiles are corrected by 1/1.2 as in H13, (middle) the $u^*$-scaled layer-averaged turbulent energy scale $\langle q^2 \rangle/u^2$, and (bottom) the scaled, layer-averaged vertical kinetic energy $\langle w^2 \rangle/u^2$ implied by the ARSM [Eq. (18c)]. For each LES case, a gray dot gives a corresponding comparison to a SMC with $u^3 = 0$ (i.e., KC94).
where the first-order contribution in Eq. (15) is provided by $c_S^2 = 4C_1^2$. The closure terms to the second order in the anisotropy and Eulerian strain are recovered on the right of Eq. (38) by dropping all $S$ superscripts and where the first-order Eulerian term from KC94 in Eq. (15) would correspond to setting $c_2^2 = 4C_1$ with $c_3 = c_4 = 0$.

The structure of Eq. (38) may be compared with the defects of the H13 ARSM (Figs. 2–3), though details may differ in the context of other SMCs that use the second-order Eulerian counterparts to Eq. (38) in the pressure–strain closure or that do not make the quasi-equilibrium assumption. The diagonal terms [Eq. (38a)] can indeed rotate the CL vortex TKE production entirely from $w_0^2$ into just the transverse component with the selection of $c_3^2 = 6$ and $c_4^2 = -2$ to produce an ARSM to match LES predictions several grid levels below the surface where $\overline{\omega w} \approx q^2 (1 - 6A_1/B_1)/3$ (at the intersection of LES dots and the thin black dashed lines in Figs. 2g–i).

Consistency with profiles of $\overline{w w}$ below that, for example, at the $\overline{w w}$ maximum, would entail coefficients $c_3^2$ and $c_4^2$ that decay with depth as $f_2^2$, while retaining $c_4^2 \leq -3c_3^2$. Modeling the off-diagonal component $\Pi^{(2C)}_{23}$ for downstream momentum flux entails at least partial cancellation of the $\overline{w w}$ CL production terms at similar near-surface depths, but this generally entails a different choice of constants, with $c_3^2 \equiv -c_4^2$. On the other hand, there is no choice of Eq. (38) constants that will address the surface-intensified $\overline{w w}$ discrepancy in Figs. 3g–i, since $\overline{u w \partial z \overline{w}} + \overline{v w \partial z \overline{w}} \rightarrow 0$ as $z \rightarrow 0$ and is decreasing as $|z|$ within LES-resolved near-surface depths. Harcourt (2012) developed a generalized ARSM based on including both Eq. (38) and its Eulerian counterpart in a weak equilibrium closure, which produces rational but rather lengthy expressions for stability.

![Graphs showing metrics of mixed layer entrainment from new SMC (a), (c) $E_6 = 6$ and from the H13 SMC with (b), (d) $E_6 = 7$. (a), (c) are compared against LES results for forcing case sets identified in HD08 as $\Sigma_1$, $\Sigma_2$, $\Sigma_{3a}$, $\Sigma_{3b}$, and $\Sigma_4$. Metrics compared are (top) the $u^*$-scaled product $\min(\overline{w w}) Z_{\min}/u^3$ of the vertical buoyancy flux minimum and its position $Z_{\min}$ and (bottom) the $u^*$-scaled net rate of work against gravity $\int_{-\infty}^{z} \overline{w w} dz/u^3$. For each LES case, a gray dot gives a corresponding comparison to a SMC with $u^* = 0$ (i.e., KC94).](http://journals.ametsoc.org/doi/abs/10.1175/JPO-D-14-0046.1?journalCode=jpo)
functions. However, tuning this model to the full set of LES solutions for Reynolds stress elements did not produce significantly better agreement than the H13 ARSM because of these underlying conflicts in tuning requirements, and no new SMC was developed. Since the initial submission of this paper, Pearson et al. (2014) proposed to model the $c_{i}^{j}$ diagonal of $\Pi_{ij}^{(LC)}$ with one set of constants $[c_{i}^{j}]_{i=j}$ in Eq. (38a) and the $i\neq j$ off-diagonal elements using a different set $[c_{i}^{j}]_{i\neq j}$ in Eq. (38b) (B. Pearson 2014, personal communication). This approach does preserve the traceless, symmetric property of $\Pi_{ij}^{(LC)}$ and may serve well for certain purposes, but using different sets of constants further complicates solutions for SMC stability functions and breaks the rotational invariance built into the tensor series formalism; for coordinate rotation $\tilde{x}_{i} = R_{i}x_{j}$, the substitution of $\tilde{u}_{i} = R_{i}u_{j}$ for all fluctuating and mean velocities in Eq. (38) is only equivalent to the tensor rotation $\Pi_{ij} = R_{ik}R_{jl}\Pi_{kl}^{(LC)}$ if $[c_{i}^{j}]_{i\neq j} = [c_{i}^{j}]_{j\neq i}$.

5. A numerical thought experiment in “free-range” Langmuir turbulence

To further test the necessity for an inhomogeneous near-surface closure—as opposed to more complex quasi-homogeneous terms [e.g., Eq. (38)]—the National Center for Atmospheric Research (NCAR) LES (Sullivan et al. 1996) was used to simulate a hypothetical environment where the CL vortex force production is substantially removed from boundaries. In a horizontally periodic domain of depth $L_{z} = 128$ m and width $L_{x} = L_{y} = 256$ m centered on $z = 0$ m, and resolved isotropically with $dx = dy = dz = 1$ m, an initially unstratified layer spanning the $|z| < L_{z}/4$ central half is separated from free-slip upper and lower closed boundaries by stable layers spanning the upper and lower $L_{z}/4$, where stratification $N^{2} = 3.9 \times 10^{-5}$ s$^{-2}$ is due to $\partial_{z}\theta = 0.02$ °C s$^{-1}$, and by sharp temperature jumps of 2°C each at $|z| = L_{z}/4$ (Fig. 7a). Velocity $u$ within the $|z| < L_{z}/6$ central third of the domain is initially set to

$$u_{\text{ref}} = 5 \text{ cm s}^{-1} \left\{ \begin{array}{ll}
+1, & z \geq L_{z}/6 \\
\sin(3\pi z/L_{z}), & |z| \leq L_{z}/6 \\
-1, & z \leq -L_{z}/6
\end{array} \right.$$  

and is restored toward $u_{\text{ref}}$ by an imposed mean horizontal pressure gradient $\partial_{z}p = (10^{-4}$ s$^{-1})(\pi - u_{\text{ref}})$ aligned with steady Stokes drift forcing of $u^{*} = 0.5u_{\text{ref}}$ (Fig. 7b) and without Coriolis: $f = 0$. From a randomly perturbed
initial state, spinup to the steady-state solution in Figs. 7c–e took 10.9 model hours in 6000 time steps, and turbulence statistics were averaged over the subsequent 4000 time steps, 7.0h.

Figure 8 compares the LES-predicted $w_0 w_0$, $w_0 u_0$, and $w_0 y_0$ profiles to a bare, $P_{ij}^L(0)$ version of the low-order quasi-equilibrium ARSM [Eq. (18)] and to another version dropping all Eq. (15) terms and instead using LES to determine $P_{ij}^L(2) = P_{ij}^L - P_{ij}^L(1)$ by computing $P_{ij}$ in the LES and subtracting $P_{ij}^L(1)$ determined from Eq. (13). Also shown (wide gray dotted) in (b) is the H13 LC closure contribution $3A_1lq^{-1}P_{ij}^{L(2)} = 3A_1C_1^{-1}lq_0 w_0 u_0 w_0$ to $\overline{u'w'}$ from Eq. (15) with $C_1^{-1} = 0.08$ (note the correct sign for ARSM correction, but opposite to LES $P_{ij}^L(2)$); and in (a) and (c)–(d) are (wide gray dotted) estimates of the combined error introduced to the $(a, a)$ ARSM TKE component by the quasi-equilibrium assumptions and the neglect of turbulent transport divergence $-2\partial_2 u_0 w_0 - \partial_1 u_0 w_0 w_0$.

Figure 8 compares the LES-predicted $\overline{w'w'}$, $\overline{w'u'}$, $\overline{w'y'}$, and $\overline{u'u'}$ profiles to a bare, $P_{ij}^{L(0)} = 0$ version of the low-order quasi-equilibrium ARSM [Eq. (18)] and to another version dropping all Eq. (15) terms and instead using LES to determine $P_{ij}^{L(2)} = P_{ij}^L - P_{ij}^{L(1)}$. In the latter, $P_{ij}^{L(1)}$ is determined by evaluating Eq. (13) from LES Reynolds stress profiles, and $P_{ij}$ is computed in-line during simulations from the LES pressure variable, which differs from $p^*$ by additional contributions from the subgrid TKE and from the vorticity form of the LES momentum equation (see HD08, their appendix A). Very close agreement in vertical TKE (Fig. 8a) strongly suggests that taking $P_{ij}^{L(0)} = 0$ makes for an accurate ARSM of $\overline{w'w'}$ away from boundaries. However, this agreement is somewhat serendipitous as the pressure–strain correlation is cancelled by other terms also left out of the ARSM, chiefly the velocity transport divergence $-\partial_2 u_0 w_0$. More importantly though, the relative errors in Fig. 8 due separately to assuming $P_{ij}^{L(0)} = 0$ away from boundaries, or from assuming quasi equilibrium or neglecting transport divergence, are small compared to the near-surface errors in $\overline{w'w'}$, $\overline{u'u'}$, and $\overline{w'y'}$ in Figs. 2–3 (note that the cross-stream and downstream directions are switched in this section). Significant contributions from $P_{ij}^{L(2)}$ do appear in the downstream $\overline{u'u'}$ profile, corresponding to higher-order Eulerian-strain quasi-homogeneous terms, for example, $lq^{-1}u_0 w_0 \partial_2 u_0$, not because of the CL forcing. While it is therefore difficult to argue the unalloyed benefits of any contribution from $P_{ij}^{L(0)} \neq 0$ away from boundaries in a quasi-equilibrium ARSM that neglects transport divergence [Eq. (18)], this comparison does indicate that downstream Eulerian shear production may be more accurately treated in higher-order pressure–strain closures not included in KC94 or H13. While
predicting $\overline{w'w'}$ is more critical to the accuracy of $K_M$ and $K_H$ without Langmuir turbulence, all three new coefficients may be more sensitive to the accuracy of the ARSM for downstream TKE in intermediate regimes of Eulerian and Stokes shear forcing.

6. Conclusions

Further modifications of pressure correlation closures in the H13 SMC of Langmuir turbulence have been developed from near-wall inhomogeneous closures and applied in an improved model. The anisotropy and vertical extent of the resulting changes to the ARSM for Langmuir turbulence is relatively successful in improving the near-surface profiles of mean momentum and TKE components predicted by the model. Improvements in SMC model entrainment, relative to HD08 LES predictions, were more significantly affected by partial reversion to the constant stability function for turbulence transport used in KC94 and KC04 than they were by the modification of stability functions. Indeed the question posed recently by Kantha et al. (2014) on the benefits of increased complexity from the inclusion of Langmuir physics in H13 is relevant. On the other hand, progress in predicting upper mixed layer dynamics below ocean surface waves has a much broader range of modeling goals, such as in the dispersion of pollutants or the role of bubble clouds in gas transfer. One virtue of the new SMC of Langmuir turbulence is that the stability functions [Eq. (33)] are only marginally more complex in than KC94, even though they depend upon a new parameter from a surface-proximity function. Dependence on a direction related to the horizontal orientation of near-surface Stokes shear and stress does not enter into the stability functions or the vertical flux eddy coefficients, but may be relevant to predicting horizontal TKE and fluxes of neutral or near-neutral scalars. The relationship between Langmuir cell orientation and features of the near-surface pressure–strain closure has received limited development here and requires further evaluation over a broader range of LES upper-ocean forcing regimes. Additional near-surface treatments for shear production may be called for, but these cannot be accurately evaluated where LES subgrid dynamics are significant. Also, integration of a TKE flux from wave attenuation into the new SMC of Langmuir turbulence remains an open question, as the interaction of injected TKE, and with the parameterized CL vortex force, introduces a new source of vertical momentum flux to near-surface depths and raises questions of how best to represent the anisotropy of the injected TKE in the ARSM.

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REFERENCES


by-term analysis of near-wall second-moment closures. AIAA

turbulence in the ocean mixed layer. J. Phys. Oceanogr., 39,

Hanjalic, K., 1994: Advanced turbulence closure models: A view of
constant current and future prospects. Int. J. Heat Fluid Flow, 15,

Harcourt, R. R., 2012: Improving the representation of Langmuir
turbulence in second moment closure models. 2012 Fall
OS33D-1868.


——, and E. D’Asaro, 2008: Large eddy simulation of Langmuir


Huang, C. J., and F. Qiao, 2010: Wave-turbulence interaction and its
induced mixing in the upper ocean. J. Geophys. Res., 115,

closure turbulence models: I. Overview. J. Eng. Mech., 124,

Jenkins, A. D., 1986: A theory for steady and variable wind- and
wave-induced currents in a rotating

model for geophysical applications. J. Geophys. Res., 99,

modelling. J. Fluid Mech., 221, 641–673, doi:10.1017/
S0022112090003718.


modelling. J. Fluid Mech., 221, 641–673, doi:10.1017/
S0022112090003718.

Li, M., C. Garrett, and E. Skaggingstad, 2005: A regime diagram for
classifying turbulent large eddies in the upper ocean. Deep-Sea

McWilliams, J. C., and P. P. Sullivan, 2000: Vertical mixing by
DOI:10.1016/S1353-2561(00)00041-X.


——, E. Huckle, H.-J. Liang, and P. P. Sullivan, 2012: The wavy
Ekman layer: Langmuir circulations, breaking waves, and
JPO-D-12-071.1.


——, and A. F. Blumberg, 2004: Wave breaking and ocean surface
layer thermal response. J. Phys. Oceanogr., 34, 693–698,


